Некоторые аспекты сходимости с регулятором в векторных решетках

ВОРКШОП ПО ФУНКЦИОНАЛЬНОМУ АНАЛИЗУ, ПОСВЯЩЕННЫЙ ЮБИЛЕЮ Д.Ф.-М.Н., ПРОФ. А.Г. КУСРАЕВА (1-3 МАРТА 2023 Г.)

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Введение

Recall that a net x_α in a vector lattice E **relatively uniformly converges**, or r-**converges** to $x \in E$ if there exists $u \in E_+$ (**a regulator of the convergence**) such that $x_\alpha \xrightarrow{\text{r}} x(u)$, i.e., for each $k \in \mathbb{N}$, there exists α_k with

$$|x_\alpha - x| \leqslant \frac{1}{k}u \quad \text{for all } \alpha \geqslant \alpha_k.$$

(cf. Definition III.11.1 in [Vulikh1961]) In this case we write $x_\alpha \xrightarrow{\text{r}} x$.

The r-convergence is **sequential** in the sense that it follows from $(x_\alpha)_{\alpha\in A} \xrightarrow{\text{r}} x$ that there exists a sequence α_{β_n} of elements of A satisfying $x_{\alpha_{\beta_n}} \xrightarrow{\text{r}} x$.

The relatively uniformly convergence is an abstraction of the classical uniform convergence of functions.

A net x_α in E is called r-**Cauchy with a regulator** $u \in E_+$ if $x_{\alpha'} - x_{\alpha''} \xrightarrow{\text{r}} 0(u)$, i.e., for each $k \in \mathbb{N}$ there exists α_k with

$$|x_{\alpha'} - x_{\alpha''}| \leqslant \frac{1}{k} u \quad \text{for all } \alpha', \alpha'' \geqslant \alpha_k.$$

A net x_α in E is called r-**Cauchy** if x_α is r-Cauchy with some regulator $u \in E_+$.

Clearly,

$$x_\alpha \xrightarrow{\text{r}} x(u) \;\Rightarrow\; x_{\alpha'} - x_{\alpha''} \xrightarrow{\text{r}} 0(u),$$

and

$$\frac{1}{n}x \xrightarrow{\text{r}} 0(x) \qquad (\forall x \in E_+).$$

A vector lattice E is called **Archimedean** if, for each $x \in E_+$,

$$\frac{1}{n}x \xrightarrow{\text{r}} y(x) \;\Rightarrow\; y = 0.$$

E is Archimedean iff every r-convergent net in E has a unique limit.

Remark 1. *Let* x_α *be an* r*-Cauchy net in a sublattice* E *of an Archimedean VL* F *with a regulator* $u \in E_+$*. If* $x_\alpha \xrightarrow{\mathrm{r}} y(w)$ *with* $y \in E$ *and* $w \in F_+$ *then* $x_\alpha \xrightarrow{\mathrm{r}} y(u)$*.*

Indeed, let $x_\alpha \xrightarrow{\mathrm{r}} y(w)$ with $y \in E$ and $w \in F_+$. For each $l \in \mathbb{N}$ we take an $\alpha(l)$ with $|x_\alpha - y| \leqslant \frac{1}{l}w$ for $\alpha \geqslant \alpha(l)$. Let $k \in \mathbb{N}$. Since $x_{\alpha'} - x_{\alpha''} \xrightarrow{\mathrm{r}} 0(u)$, there exists α_k with $|x_{\alpha'} - x_{\alpha''}| \leqslant \frac{1}{k}u$ for $\alpha', \alpha'' \geqslant \alpha_k$. Fix any $l \in \mathbb{N}$ and pick an $\alpha(k,l) \geqslant \alpha_k, \alpha(l)$. Then

$$|x_\alpha - y| \leqslant |x_\alpha - x_{\alpha(k,l)}| + |x_{\alpha(k,l)} - y| \leqslant \frac{1}{k}u + \frac{1}{l}w$$

for each $\alpha \geqslant \alpha_k$. Since $l \in \mathbb{N}$ is arbitrary and F is Archimedean then $|x_\alpha - y| \leqslant \frac{1}{k}u$ for all $\alpha \geqslant \alpha_k$, and hence $x_\alpha \xrightarrow{\mathrm{r}} y(u)$.

This is no longer true in every non-Archimedean VL F. Indeed, WLOG assume $F = \mathbb{R}^2_{lex}$. Then, for $E := \{< 0, t >: t \in \mathbb{R}\}$:

$$< 0, 1/n > \stackrel{\mathrm{r}}{\to} < 0, 0 > (< 0, 1 >) \text{ and}$$

$$< 0, 1/n > \stackrel{\mathrm{r}}{\to} < 0, 1 > (< 1, 0 >),$$

$$\text{yet } < 0, 1/n > \stackrel{\mathrm{r}}{\not\to} < 0, 1 > (< 0, 1 >).$$

Remark 2. *For a sublattice E of an Archimedean VL F,*

if $E \ni x_\alpha \xrightarrow{\mathsf{r}} y(u)$ and $x_\alpha \xrightarrow{\mathsf{r}} z(w)$ with $y, z \in F, u, w \in F_+$ then $y = z$.

Indeed, under the assumption of Remark 2, $x_\alpha \xrightarrow{\mathsf{r}} y(u+w)$ and $x_\alpha \xrightarrow{\mathsf{r}} z(u+w)$. Since F is Archimedean, it follows $y = z$.

As above, it is no longer true in every non-Archimedean F. Indeed, WLOG assume $F = \mathbb{R}^2_{lex}$. Then, for $E := \{< 0, t >: t \in \mathbb{R}\}$:

$$< 0, 1/n > \xrightarrow{\mathsf{r}} < 0, 0 > (< 0, 1 >) \text{ and}$$

$$< 0, 1/n > \xrightarrow{\mathsf{r}} < 0, 1 > (< 1, 0 >).$$

Архимедизация векторной решетки

The **Archimedeanization of an ordered vector space with a (strong) order unit** was constructed in [PT2009] (V.I. Paulsen, M. Tomforde: Vector spaces with an order unit. Indiana Univ. Math. J. (2009)).

The extension to arbitrary ordered vector space was obtained in [E2017] (E.Y. Emelyanov: Archimedean Cones in Vector Spaces. Journal of Convex Analysis (2017)).

Here, we discuss the **Archimedeanization of a vector lattice**.

Given a vector lattice E, denote by

$$I_E := \{x \in E | (\exists y \in E)(\forall n \in \mathbb{N}) |x| \leqslant \frac{1}{n} y\}$$

the set of all **infinitesimals** of E. Then I_E is an order ideal in E. A VL E is Archimedean iff $I_E = \{0\}$.

If E has a strong order unit $u \in E$ then $u \notin I_E$. However, in the absence of strong order units it may happened $I_E = E$ (e.g., for the nonstandard extension $^*\mathbb{R}$ of $\mathbb{R}$).

Denote

$$D_E := \{x \in E | (\exists y \in E_+)(\forall \varepsilon > 0) \ x + \varepsilon y \geqslant 0\}.$$

Then $E_+ \subseteq D_E$ and

$$I_E = D_E \cap (-D_E).$$

The set D_E is a **wedge**, i.e.:

$$D_E + D_E \subseteq D_E \quad \text{and} \quad rW \subseteq W \quad \text{for all} \quad r \geqslant 0.$$

Consider the sets

$$E_+ + I_E = [E_+]_{I_E}$$

and

$$D_E + I_E = [D_E]_{I_E}$$

in the quotient VL E/I_E. Both sets are cones since

$$(D_E + I_E) \cap (-D_E + I_E) = D_E \cap (-D_E) = I_E$$

and

$$(E_+ + I_E) \cap (-E_+ + I_E) = I_E.$$

If $A \subseteq E$ be an order ideal then, by the Veksler theorem (A.I. Veksler: Archimedean principle in homomorphic images of l-groups and of vector lattices. Izv. Vyshs. Ucebn. Zaved. Matematika, (1966)),

$$E/A \text{ is Archimedean} \Leftrightarrow A \text{ is r-closed}.$$

In general, I_E need not to be r-closed in E.

To see this, consider the following example that is due to T. Nakayama (see, [LZ1971] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces, I, (1971).

Example 1. *Consider the vector lattice*

$$E = \{a = (a_k^1, a_k^2)_k |\ (a_k^1, a_k^2) \in (\mathbb{R}^2, \leqslant_{lex}), a_k^1 \neq 0 \textit{ for finitely many } k\}$$

with respect to the pointwise ordering and operations. Then I_E is not r*-closed in E and hence the VL E/I_E still has nonzero infinitesimals by the Veksler theorem.*

Definition 1. *Let E be a VL and $\mathcal{R}_{Arch}(E)$ be the category whose objects are pairs $\langle F, \phi \rangle$, where F is an Archimedean VL and $\phi : E \to F$ a lattice homomorphism, and morphisms $\langle F_1, \phi_1 \rangle \to \langle F_2, \phi_2 \rangle$ are lattice homomorphisms $q_{12} : F_1 \to F_2$ such that $q_{12} \circ \phi_1 = \phi_2$.*

If $\mathcal{R}_{Arch}(E)$ possesses an initial object $\langle F_0, \phi_0 \rangle$, then F_0 is called an **Archimedization** *of E.*

Denote by $Arch_{VL}(E)$ the Archimedization of a VL E, if exists.

Theorem 1. *Any VL has an Archimedeanization.*

The idea of a proof: Let E be a VL. Denote $I_0 := \{0\}$,

$$I_1 := I_E = \{x \in E | [x]_{I_0} \text{ is an infinitesimal in } E/I_0 = E\},$$

$$I_{n+1} := \{x \in E | [x]_{I_n} \text{ is an infinitesimal in } E/I_n\},$$

and, more generally, for an arbitrary ordinal $\alpha > 0$:

$$I_\alpha = I_\alpha(E) = \{x \in E | [x]_{\cup_{\beta<\alpha} I_\beta} \text{ is an infinitesimal in } E/_{\cup_{\beta<\alpha} I_\beta}\}.$$

All I_α are order ideals in E and $I_{\alpha_1} \subseteq I_{\alpha_2}$ for $\alpha_1 \leqslant \alpha_2$.

Take the first ordinal, say λ_E, such that $I_{\lambda_E+1} = I_{\lambda_E}$. Then the VL E/I_{λ_E} has no nonzero infinitesimals and hence is Archimedean.

The quotient map $p_E : E \to E/I_{\lambda_E}$ is a lattice homomorphism. For any other pair $\langle F, \phi \rangle$, where F is an Archimedean VL and $\phi : E \to F$ is a lattice homomorphism, we have $\phi(I_\alpha) \subseteq I_F$ for each ordinal α.

Since F is Archimedean, $I_F = \{0\}$ and hence $I_{\lambda_E} \subseteq \ker(\phi)$. So, the map $\tilde{\phi} : E/I_{\lambda_E} \to F$ is well defined by $\tilde{\phi}([x]_{I_{\lambda_E}}) = \phi(x)$ and satisfies $\tilde{\phi} \circ p_E = \phi$. Moreover, $\tilde{\phi}$ is a lattice homomorphism.

In order to show that $\tilde{\phi}$ is unique, take any $\psi : E/I_{\lambda_E} \to F$, that satisfies $\psi \circ p_E = \phi$. Then

$$\psi([y]_{I_{\lambda_E}}) = \psi(p_E(y)) = \phi(y) = \tilde{\phi}(p_E(y)) = \tilde{\phi}([y]_{I_{\lambda_E}}) \qquad (\forall y \in E)\,,$$

and hence $\psi = \tilde{\phi}$. Thus, $(E/I_{\lambda_E}, \tilde{\phi})$ is an initial object of $\mathcal{R}_{Arch}(E)$, and hence the VL E/I_{λ_V} is an Archimedization of the VL E. ■

Let E be a VL. Denote by $\alpha_{VL}(E)$ the first ordinal α such that $I_{\alpha+1}(E) = I_\alpha(E)$.

Conjecture 1. *For each VL E, $\alpha_{VL}(E) < \omega_1$, where ω_1 is the first uncountable ordinal. Moreover, for each countable ordinal α there exists a VL E such that $\alpha_{VL}(E) = \alpha$.*

Критерий топологичности сходимости с регулятором

Recall that $x_\alpha \xrightarrow{\text{o}} 0$ in a VL E if there exists a net y_β in E with $y_\beta \downarrow 0$ such that, for every β there is an α_β satisfying $|x_\alpha| \leqslant y_\beta$ for all $\alpha \geqslant \alpha_\beta$.

It was proved in Theorem 1 of [DEM2017] (Y.A. Dabboorasad, E.Y. Emelyanov, M.A.A. Marabeh: Order convergence in infinite-dimensional vector lattices is not topological, arXiv:1705.09883v1 (2017)) that, for a topological VL (E, τ) the following statements are equivalent.

(1) For every net x_α of E: $x_\alpha \to 0$ in τ iff $x_\alpha \xrightarrow{\text{o}} 0$.

(2) $\dim(E) < \infty$.

In particular, in an Archimedean VL E the order convergence is topological iff $\dim(E) < \infty$.

It is well known that in $c_{00}(\Omega)$: $x_\alpha \xrightarrow{\mathrm{r}} 0$ iff $x_\alpha \xrightarrow{\mathrm{o}} 0$. This can be extended as follows.

The next fact is Proposition 4 of [DEM2018] (Y.A. Dabboorasad, E.Y. Emelyanov, M.A.A. Marabeh: $u\tau$-Convergence in locally solid vector lattices. Positivity (2018)):

Proposition 1. *The following conditions are equivalent*:

(1) $f_\alpha \xrightarrow{\mathrm{r}} 0$ *iff* $f_\alpha \xrightarrow{\mathrm{o}} 0$ *for any net* f_α *in the VL* $\mathbb{R}^\Omega$;

(2) Ω *is countable.*

Since order ideals are regular, it follows from Proposition 1 that, for any order ideal E of an atomic universally complete VL, the following conditions are equivalent.

(1) $f_\alpha \xrightarrow{\mathrm{r}} 0$ iff $f_\alpha \xrightarrow{\mathrm{o}} 0$ for each net f_α in E.

(2) E has at most countably many pairwise disjoint atoms.

Since in purely nonatomic universally complete VL the o-convergence is properly weaker than the r-convergence, it follows:

Proposition 2. *Let E be an order ideal of a universally complete VL. Then the following conditions are equivalent.*

(1) $f_\alpha \xrightarrow{\text{r}} 0$ *iff* $f_\alpha \xrightarrow{\text{o}} 0$ *for any net* f_α *in* E.

(2) E *is discrete with at most countably many pairwise disjoint atoms.*

The following is an r-version of Theorem 1 in [DEM2017].

Theorem 2. (Theorem 5 in [DEM2018]) *Let E be an Archimedean VL. Then the following conditions are equivalent.*

(1) *There exists a linear topology τ on E such that, for any net x_α in E: $x_\alpha \xrightarrow{\mathrm{r}} 0$ iff $x_\alpha \xrightarrow{\tau} 0$.*

(2) *There exists a norm $\|\cdot\|$ on X such that, for any net x_α in E: $x_\alpha \xrightarrow{\mathrm{r}} 0$ iff $\|x_\alpha\| \to 0$.*

(3) *E has a strong order unit.*

In other words, in an Archimedean VL E the r*-convergence is topological iff E has a strong order unit. Clearly, in any non-Archimedean VL the* r*-convergence is not topological.*

Конструкция свободной r-полной векторной решетки над непустым множеством

The existence of a free vector lattice $FVL(A)$ over a set A is the long established fact going back to Birkhoff [Birk1942], where more general result was established for algebraic systems. A concrete representation of $FVL(A)$ as a vector lattice of real-valued functions was given by Weinberg [Wein1963] and Baker [Baker1968]

Following the approach of de Pagter and Wickstead [PW2015], a **free vector lattice over a non-empty set** A is a pair (F, i), where F is a vector lattice and $i : A \to F$ is a map such that, for any vector lattice E and for any map $q : A \to E$, there exists a unique lattice homomorphism $T : F \to E$ satisfying $q = T \circ i$. If (F, i) is a free vector lattice over A, then F is generated by $i(A)$ as a vector lattice.

Here, we discuss a **free uniformly complete vector lattice over a non-empty set** A and give some of its representations ([EG2022] E. Emelyanov, S.G. Gorokhova: Free uniformly complete vector lattices. arxiv.org/abs/2109.03895).

A r-complete vector lattice will be abbreviated as a UCVL.

A UCVL F is called an r-**completion** of a VL E if there is a lattice embedding $i : E \to F$ such that, for each UCVL G and each lattice homomorphism $T : E \to G$, there exists a unique lattice homomorphism $S : F \to G$ satisfying $T = S \circ i$. If an r-completion F of a vector lattice E exists, it must be unique up to a lattice isomorphism.

As every r-complete VL E coincides with its r-completion, a VL that has an r-completion need not to be Archimedean (e.g., $\mathbb{R}^2_{lex}$ is r-complete).

It is long known that if E is Archimedean, then the intersection of all uniformly complete sublattices containing E of the Dedekind completion E^δ of E is the r-completion of E (see, for example [Veksler1969] A.I. Veksler: A new construction of Dedekind completion of vector lattices and of l-groups with division. Siberian Math. J. (1969)).

We recall some details of the construction of the r-completion in a slightly more general case.

As the r-convergence is sequential, we can restrict ourselves to r-convergent (and r-Cauchy) sequences. In particular, a VL U is UCVL iff every r-Cauchy sequence in U is r-convergent.

Furthermore, Remark 1 tells us that in the definition of a r-complete sublattice E of an Archimedean VL F we may always take from E the regulators of r-convergence.

Suppose E is a sublattice of some Archimedean UCVL U. Then the intersection F of all r-complete sublattices of U containing E is a UCVL.

Indeed, let x_n be r-Cauchy in F with a regulator $u \in F_+$. Take any r-complete sublattice V of U containing E. Then $x_n \xrightarrow{\mathrm{r}} x(v)$ for some $v \in V_+$ and hence $x_n \xrightarrow{\mathrm{r}} x(u)$ by Remark 1.

Define a transfinite sequence $(F_\beta)_{\beta\in\mathsf{Ord}}$ of sublattices of U by:

$F_1 := E$;

$F_{\beta+1} := \{x \in U : x_n \xrightarrow{\mathsf{r}} x(x_1),$ for a sequence x_n in $F_\beta\}$;

$F_\beta := \bigcup_{\gamma\in\mathsf{Ord};\ \gamma<\beta} F_\gamma$ for a limit ordinal β.

Then $F_{\beta_1} \subseteq F_{\beta_2}$ if $\beta_1 \leqslant \beta_2$.

Lemma 1. *Let E be a sublattice of an Archimedean UCVL U. Then the intersection F of all* r*-complete sublattices of U containing E satisfies $F = \bigcup_{\gamma\in\mathsf{Ord}} F_\gamma$.*

Since r-convergence is sequential,

$$F = \bigcup_{\gamma \in \mathrm{Ord}; \gamma < \omega_1} F_\gamma,$$

where ω_1 is the first uncountable ordinal.

Lemma 1 leads to the following proposition that was already stated in indirect form in [Veksler1969]

Proposition 3. *Let E be a sublattice of an Archimedean UCVL U. Then $\bigcup_{\gamma<\omega_1} F_\gamma$ is the* r-*completion of E.*

Definition 2. *If A is a non-empty set, then a* **free UCVL over** A *is a pair (F, i), where F is a UCVL and $i : A \to F$ is a map with the property that, for any UCVL E and for any map $q : A \to E$, there exists a unique lattice homomorphism $T : F \to E$ such that $q = T \circ i$.*

We denote a free UCVL over A by $FUCVL(A)$. It is an initial object in the category, whose objects are pairs (E, q) with a UCVL E and $q : A \to E$, and whose morphisms $T : (E_1, q_1) \to (E_2, q_2)$ are lattice homomorphisms from E_1 to E_2 satisfying $q_2 = T \circ q_1$. Thus $FUCVL(A)$ is defined similarly to $FVL(A)$ in a proper subcategory.

Routine arguments show that if $FUCVL(A)$ exists it is unique up to a lattice isomorphism and the map $i : A \to FUCVL(A)$ above is injective.

By Theorem 2.4 of [Baker1968], $FVL(A)$ is a vector sublattice of $\mathbb{R}^{\mathbb{R}^A}$ generated by the evaluation functionals δ_a on $\mathbb{R}^A$, $\delta_a(\xi) = \xi(a)$.

Theorem 3. (Theorem 1 in [EG2022]) *Let A be a non-empty set, and assume $FVL(A)$ to be a vector sublattice of $\mathbb{R}^{\mathbb{R}^A}$. The intersection F of all* r*-complete sublattices of $\mathbb{R}^{\mathbb{R}^A}$ containing $FVL(A)$ together with the embedding $a \xrightarrow{i} \delta_a$ is a $FUCVL(A)$.*

Following the tradition, for $B \subseteq A$, we identify $\mathbb{R}^{\mathbb{R}^B}$ with a sublattice of $\mathbb{R}^{\mathbb{R}^A}$ by assigning $\xi \in \mathbb{R}^{\mathbb{R}^B}$ to $\widehat{\xi} \in \mathbb{R}^{\mathbb{R}^A}$ such that $\widehat{\xi}(f) = \xi(f|_B)$.

By Proposition 3.5(2) of [PW2015], there exists a unique order projection P_B of $FVL(A)$ onto $FVL(B)$ satisfying

$$P_B(\delta_a) = \begin{cases} \delta_a & \text{if } \; a \in B \\ 0 & \text{if } \; a \in A \setminus B \,. \end{cases}$$

In particular, $FVL(A) = \bigcup_{B \in \mathcal{P}_{fin}(A)} FVL(B)$, where $\mathcal{P}_{fin}(A)$ is the set of all finite subsets of A (Proposition 3.7 in [PW2015]).

Denote by $H(\mathbb{R}^A)$ (by $H(\Delta_A)$) the space of all positively homogeneous real-valued functions on $\mathbb{R}^A$ (on $\Delta_A := [-1,1]^A$) which are continuous in the product topology of $\mathbb{R}^A$ (of Δ_A).

Then $H(\Delta_A)$ is a closed in $\|.\|_\infty$-norm vector sublattice of $C(\Delta_A)$, and hence $H(\Delta_A)$ is itself a Banach lattice.

The following notion was introduced by de Pagter and Wickstead.

Definition 3. (Definition 4.1 in [PW2015]) *If A is a non-empty set, then a* **free Banach lattice over** *A (shortly, $FBL(A)$) is a pair (F, i), where F is a Banach lattice and $i : A \to F$ is a bounded map with the property that for any Banach lattice E and any bounded map $\kappa : A \to E$ there exists a unique vector lattice homomorphism $T : F \to E$ such that $\kappa = T \circ i$ and $\|T\| = \sup\{\kappa(a) : a \in A\}$.*

The existence of $FBL(A)$ over a non-empty set A was established in Theorem 4.7 of [PW2015].

It is well known that $FVL(A)$ may be identified with a sublattice of $H(\mathbb{R}^A)$ and hence with a sublattice of $H(\Delta_A)$ in view of Lemma 5.1 of [PW2015].

By Corollary 5.7 of [PW2015], $FBL(A)$ is embedded into $H(\Delta_A)$ as an order ideal $J(FBL(A))$.

Furthermore, $J(FBL(A)) = H(\Delta_A)$ iff A is finite and, in this case, $FBL(A)$ is isomorphic to $H(\Delta_A)$ under the supremum norm by Theorem 8.2 of [PW2015].

Theorem 4. (Theorem 2 in [EG2022]) *Let B be a non-empty finite set. Then $FUCVL(B)$ is lattice isomorphic to $FBL(B)$, to $H(\Delta_B)$, and to $H(\mathbb{R}^B)$.*

Since any Banach lattice is UCL, it follows from Proposition 4 that $FBL(A)$ contains an r-completion of $FVL(A)$.

$FUCVL(A)$ is a proper sublattice of $FBL(A)$ unless A is finite.

Corollary 1. *Let A be a non-empty set. Then*

$$\bigcup_{B\in\mathcal{P}_{fin}(A)} FBL(B) \subseteq FUCVL(A) \subseteq FBL(A).$$

Furthermore, both inclusions are proper unless A is finite.

Proposition 4. (Proposition 2 in [EG2022]) *Let A be a non-empty set, and let $x \in FBL(A)$. Then there exists a sequence x_n in $FVL(A)$ which* r-*converges to x with a regulator $u \in FBL(A)$.*

Proposition 5. (Proposition 3 in [EG2022]) *If a sequence g_n of $H(\mathbb{R}^A)$* r-*converges with a regulator $u \in H(\mathbb{R}^A)$ to some $g \in \mathbb{R}^{\mathbb{R}^A}$ then $g \in H(\mathbb{R}^A)$.*

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THANK YOU FOR THE ATTENTION!