About the Prehistory of Boolean valued Analysis

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March 1, 2023

Continuum Hypothesis

CH: Every infinite set of real numbers is either countable or of the same cardinality as $\mathbb{R}$ ($2^{\aleph_0} = \aleph_1$)

Theorem

*Continuum Hypothesis does not contradict the axiomatic of set theory (**ZFC**), if this axiomatic itself is consistent.*

Gödel, K. *The consistency of the axiom of choice and the generalized continuum hypothesis with the axioms of set theory.* Princeton (N.J.), 1940

Sketch of Gödel's Proof

- Let $F(x)$ be a formula of **ZFC**, where x - is a free variable. Then the class $\mathscr{F} = \{x \mid \mathbf{ZFC} \vdash F(x)\}$.
- The class $\mathscr{V}$ of all sets is defined by the formula $V(x) : x = x$
- The class $\mathscr{O}\mathscr{N}$ of all ordinals is defined by the formula $On(\alpha)$:

$$(\forall \beta \in \alpha, \gamma \in \beta \ \gamma \in \alpha) \wedge (\forall \beta, \gamma \in \alpha \ \beta \in \gamma \vee \gamma \in \beta)$$

Theorem

(Von Neumann's universe V) Consider the transfinite sequence of sets

$$\mathscr{V}_0 = \emptyset,\ \mathscr{V}_{\alpha+1} = \mathscr{P}(\mathscr{V}_\alpha),\ \mathscr{V}_\beta = \bigcup \{V_\alpha \mid \alpha < \beta\}$$

. Then $\mathscr{V} = \bigcup_{\alpha \in \mathscr{O}\mathscr{N}} \mathscr{V}_\alpha$

Definition

(Gödel's Constructible Universe $\mathscr{L}$). The constructible universe $\mathscr{L}$ is defined by the formula

$$\begin{aligned} &\mathscr{L}_0 = \emptyset,\ \mathscr{L}_{\alpha+1} = def(\mathscr{P}(\mathscr{L}_\alpha)),\ \mathscr{L}_\beta = \bigcup\{\mathscr{L}_\alpha \mid \alpha < \beta\} \\ &\mathscr{L} = \bigcup_{\alpha \in \mathscr{O}\mathscr{N}} \mathscr{L}_\alpha \end{aligned} \qquad (1)$$

Theorem

- *Axiom of Construtivity* $\forall x\, \exists \alpha \in \mathscr{O}\mathscr{N}\, \S \in \mathscr{L}_\alpha$ $(\mathscr{V} = \mathscr{L})$ *is consistent with* **ZFC**.
- *The formula of* **ZFC** $L(x) := \exists a \in \mathscr{O}\mathscr{N}\ \S \in \mathscr{L}_\alpha$ *that defines the class* $\mathscr{L}$ *is absolute.*
- *The class* $\mathscr{L}$ *is a standard tranzitive model of* **ZFC**
- $ZF + \mathscr{V} = \mathscr{L} \vdash AC + CH$

Theorem

(P. Cohen 1963) Continuum Hypothesis cannot be deduced from the axioms of **ZFC***, if* **ZFC** *is consistent itself.*

Theorem

(P. Cohen) it is impossible to construct a standard model of **ZFC** *in which the Continuum Hypothesis would be wrong.*

Cohen P. J. *Set theory and the Continuum Hypothesis*. Benjamin, New York, 1966

The second order language of Real Analysis $L_2 Real$

Scott D. A Proof of the Independence of the Continuum Hypothesis, 1967 *Mathematical Systems Theory*, v. 1 (2), - pp. 89 – 111.
Manin Yu. I. A Course in Mathematical Logic for Mathematicians. Second Edition, Springer Verlag, New York, Inc. 2010.

- $L_1 Real$ -the language of the first order theory of real closed fields. Its signature $\sigma = \langle 0, 1; +, \cdot; \leq \rangle$.

Theorem (Tarski, Seidenberg)

The subsets of $\mathbb{R}^n$ that are definable in $L_1 Real$ are semi-algebraic sets, i.e. finite unions of sets of solutions to systems of algebraic equations and inequalities

- $L_2 Real$ extends $L_1 Real$ by adding second order variables $f, g, h, ...$ that assume values in $\mathbb{R}^{\mathbb{R}}$. The first order variables $x, y, z, ...$ assume values in $\mathbb{R}$.

Examples of mathematical statements formalized in $L_2 Real$

- **Agreement**: The subsets of $\mathbb{R}$ are defined by their characteristic functions: $Set(f) := \ f^2 = f$. Second order variables satisfying *Set* are denoted by capital Latin letters. We write $x \in A$ for $A(x)$. Standard notations $\mathbb{N}, \mathbb{Z}, \mathcal{Q}, \mathbb{R}$ are second order set constants
- **Natural numbers**: $\mathbb{N}(x) := x$ - is an integer: $(x \geq 0 \wedge \forall f\,(f(0) = 0 \wedge \forall y\,(f(y) = f(y+1))) \longrightarrow f(x) = 0$ Any real function that vanishes at 0 and is periodic with the period 1 vanishes at x as well.
- **Ordered pair**
- $\langle x, y \rangle := f(z) = \begin{cases} f(1) = x \wedge f(0) = y \\ f(z) = 0, z \notin \{0, 1\} \end{cases}$

- **Continuum Hypothesis**: For any set $A \subseteq \mathbb{R}$ there exists either a surjection $g : A \to \mathbb{R}$ or a surjection $f : \mathbb{N} \to A$.
- **Completeness Axiom** Every bounded above function f has a least upper bound.
- **Axiom of Choice** $\forall x \, \exists y \, P(x, y) \longrightarrow \exists f \, \forall x \, P(x, f(x))$. Here P is a formula in the language $L_2 Real$ that does not contain free variables, except for x, y and the variable f.

Boolean valued model for Real Analysis

1. **Algebra of random variables.** The algebra of random variables $\mathscr{R}$ on the probability space $\Omega, \sum, \ \mu$ is a model for the field of real numbers $\mathbb{R}$. Here $\omega = [0,1]^I$, $|I| > \aleph_1$, $\sum$ is the σ-algebra of μ-measurable sets, where μ is the product measure induced by Lebesgue measure on $[0,1]$.
2. **Theorem** Let $\xi_1, \ldots, \xi_n, \eta_1, ...\eta_m : \Omega \to \mathbb{R}$ are random variables and $f_1, ..., f_m \in \mathbb{R}^{\mathbb{R}}$ are variables of type 2 that are Borel measurable functions. Let $F(f_1, \ldots f_m, y_1, \ldots y_m, x_1, \ldots, x_n)$ be a formula in $L_2 Real$, $\omega \in \Omega$. Then $L_2 Real \models F(f_1 \circ \eta_1, \ldots, f_m \circ \eta_m, \xi_1, ..., \xi_n)$ iff $F(\omega) = F(f_1 \circ \eta_1(\omega), \ldots, f_m \circ \eta_m(\omega), \xi_1(\omega), ..., \xi_n(\omega))$ is true almost surely.
3. $\|F\| := \mu\left(\{\omega \mid F(\omega) \textit{ is true}\}\right)/mod0$, $\Omega/mod0 = \mathbb{B}$ is a complete boolean algebra.
4. Obiously the algebra $\mathscr{R}/\mathrm{mod}\prime$ is a universally complete vektor lattice (K-space) with the base $\mathbb{B}$.

Boolen-valued approach

Scott D. and Solovay R. Boolean valued models for set theory. Preprint (1965)

- Boolean Valued Universe: $\mathbb{B}$ - complete Boolean algebra (c.b.a) - the set of truth values of formulas **ZFC**
- $V_0^{\mathbb{B}} = \emptyset,\ V_{\alpha+1}^{\mathbb{B}} = \{\varphi : V_\alpha^{\mathbb{B}} \to \mathbb{B}\},$
  $V_\beta^{\mathbb{B}} = \bigcup\{V_\alpha^{\mathbb{B}} \mid \alpha < \beta\},\ V^{\mathbb{B}} = \bigcup_{\alpha \in On} V_\alpha^{\mathbb{B}}$
- F - formula **ZFC**, $\|F\|$ - truth value of F. $\|F\| = \mathbf{1}$ if **ZFC** $\vdash F$
- If $\|F\| < \mathbf{1}$, for some $\mathbb{B}$, then F cannot be proved in **ZFC.**