Narrow operators on vector-valued function spaces

Nonna Dzhusoeva, Eleonora Grishenko, Marat Pliev, Fedor Sukochev

Southern Mathematical Institute of the Russian Academy of Sceinces and North Caucasus Center for Mathematical Research

Vladikavkaz — 2023

1. Suppose Q is a compact topological space. The Banach space of all continuous functions on Q is denoted by $C(Q)$.

2. Suppose A is an open bounded subset of $\mathbb{R}^n$, Σ is σ-algebra of Lebesgue measurable sets and $\mu\colon \Sigma \to \mathbb{R}_+$ is the Lebesgue measure. Recall that $L_p(A, \Sigma, \mu)$ $(1 \leq p < \infty)$ are classical Lebesgue spaces of measurable functions, $f \leq g$, if $f(t) \leq g(t)$ μ-a.e.

Definition

A subset D of $L_p(\mu)$ is said to be *order bounded* if there exists $f \in L_p(\mu)_+$ such that $|g| \leq f$ for every $g \in D$.

Every such a space is additionally a vector lattice equipped with the natural partial order inherited from $L_0(A, \Sigma, \mu)$, the space of all (classes of) μ-measurable functions on A. This simple observation led to various modifications of the notion of compact operators.
In particular, a linear operator $T \colon E \to Y$ from a Banach lattice E to a Banach lattice Y is said to be *AM-compact* if T maps every order bounded subset of E to a relatively compact subset of Y. These operators were introduced by Dodds and Fremlin in their groundbreaking paper

- P. G. Dodds, D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math., **34** (1979), 287–320.

Later *AM*-compact operators were intensively studied by many authors.

- Y. A. Abramovich, C. D. Aliprantis, *An Invitation to Operator Theory*, AMS, 2002.
- C. D. Aliprantis, O. Burkinshaw, *Positive Operators*, Springer, Dordrecht, (2006).

Definition

Recall that the *support* of a μ-measurable function $f\colon A \to \mathbb{R}$ (denotation $\operatorname{supp} f$) is the μ-measurable set

$$\operatorname{supp} f := \{t \in A : \ f(t) \neq 0\}.$$

Two elements f and g of $L_p(\mu)$ is said to be *disjoint* if

$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$

Nigel Kalton
1946-2010

Haskell Rosenthal
1940-2021

Kalton obtained the analytical representation of continuous operators on $L_1([0,1],\Sigma,\mu)$. This representation is given in the terms of weak*-measurable functions from $[0,1]$ to the set of all regular Borel measures on $[0,1]$.

- N. J. Kalton, The endomorphisms of L_p $(0 \le p \le 1)$, Indiana Univ. Math. J., **27** (1978), 3, 353–381.

Theorem

For every operator $T \in \mathcal{L}(L_1)$ *there is a weak** *measurable function* $\mu_s\colon [0,1] \to \mathcal{M}[0,1]$ *taking values in the space of all regular Borel measures on* $[0,1]$ *such that for every* $f \in L_1([0,1],\Sigma,\mu)$ *the following equality*

$$Tf(s) = \int_{[0,1]} f(t)d\,\mu_s(t) \tag{1}$$

holds for almost all $s \in [0,1]$*. Conversely, every weak*-measurable function* $\mu\colon [0,1] \to \mathcal{M}[0,1]$ *defines an operator* $T \in \mathcal{L}(L_1)$ *as above.*

Definition

We say that a linear bounded operator $T\colon L_1(\mu) \to L_1(\mu)$

1. *preserves disjointness* if for all $f, g \in L_1$ the relation
$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$
implies that
$$\mu\{t \in \operatorname{supp} Tf \cap \operatorname{supp} Tg\} = 0.$$
2. *narrow* if for every $A \in \Sigma_+$ the restriction $T|_{L_1(A)}$ is not an isomorphic embedding to L_1;
3. *pseudo embedding* if for each $\varepsilon > 0$ there exists $A \in \Sigma_+$ such that the restriction $T|_{L_1(A)}$ is an into isomorphism with
   1. $\|T|_{L_1(A)}\| \geq \|T\| - \varepsilon$;
   2. there exists disjointness preserving operator $U\colon L_1(A) \to L_1$ such that $\|T|_{L_1(A)} - U\| < \varepsilon$.

- H. P. Rosenthal, *Embeddings of L^1 in L^1*, Contemp. Math., **26** (1984), 335–349.

Theorem

Every operator $T \in \mathcal{L}(L_1(\mu))$ *has a unique representation*

$$T = T_{\mathcal{N}} + T_{\mathcal{H}},$$

where $T_{\mathcal{N}}$ *is a narrow operator,* $T_{\mathcal{H}}$ *is a pseudo-embedding and* $T_{\mathcal{N}}, T_{\mathcal{H}} \in \mathcal{L}(L_1)$.

Theorem

Let $T \in \mathcal{L}(L_1)$. Then the following statements are equivalent:

1. *T is a narrow operator;*
2. *if for every $f \in L_1(\mu)$ and $\varepsilon > 0$ there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $\|T(f_1 - f_2)\|_Y < \varepsilon$*

Definition

Let Y be a normed space. A bounded linear operator $T\colon L_p(\mu) \to Y$ is said to be

1. *narrow* if for every $f \in L_p(\mu)$ and $\varepsilon > 0$ there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $\|T(f_1 - f_2)\|_Y < \varepsilon$;
2. *strictly narrow* if for every $f \in L_p(\mu)$ there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $Tf_1 = Tf_2$.

Narrow operators were explicitly articulated by Plichko and Popov in

- Plichko A., M.Popov, *Symmetric function spaces on atomless probability spaces,* Dissertationes Math. (Rozprawy Mat.), **306** (1990), 1–85.

For a detailed historical account, we refer to

- M. Popov, B. Randrianantoanina, *Narrow operators on function spaces and vector lattices*, 45, De Gruyter Studies in Mathematics, 2013 Walter de Gruyter & Co., Berlin/Boston.

On the other hand, in

- O. Maslyuchenko, V. Mykhaylyuk, M. Popov, *A lattice approach to narrow operators*, Positivity, **13** (2009), 459–495.

Maslyuchenko, Mykhaylyuk and Popov demonstrated that the notion of narrow operators admits a natural extension in the setting of vector lattices and that this general approach has a serious advantage.

Namely, they proved the following remarkable theorem which shows that narrow operators can be regarded as a generalization of *AM*-compact operators.

Theorem

Let Y be a Banach space. Then every AM-compact order-to-norm continuous linear operator $T\colon L_p(\mu) \to Y$ is narrow.

Let (A, Σ, μ) be a finite measure space and X be a Banach space. We say that a function $f\colon A \to X$ is *strongly μ-measurable* if there is a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to\infty} \|f(\cdot) - f_n(\cdot)\|_X = 0$ μ-almost everywhere. By $L_0(\mu, X)$ we denote the space of all (equivalence classes of) strongly μ-measurable X-valued functions defined on A. A Lebesgue-Bochner space $L_p(\mu, X)$ on (A, Σ, μ) as defined as

$$L_p(\mu, X) := \{f \in L_0(\mu, X) : \ \|f(\cdot)\|_X \in L_p\}.$$

We observe that the space $L_p(\mu, X)$ is equipped with a "mixed"norm

$$\|f\|_{L_p(\mu,X)} := \|\|f(\cdot)\|_X\|_E, \ f \in L_p(\mu, X).$$

it is turned out to be a Banach space.
For the detailed exposition of the theory of more general Köthe-Bochner spaces we to monograph

- P. K. Lin, *Köthe-Bochner function spaces*, Birkhäuser, Boston, (2004).

Definition

Recall that the *support* of a strongly μ-measurable X-valued function $f : A \to X$ (denotation $\operatorname{supp} f$) is the μ-measurable set

$$\operatorname{supp} f := \{t \in A : \ f(t) \neq 0\}.$$

Two elements f and g of $L_p(\mu, X)$ is said to be *disjoint* if

$$\mu\{t \in \operatorname{supp} f \cap \operatorname{supp} g\} = 0$$

Definition

Let X be a Banach space and Y be a vector space. An operator $T\colon L_p(\mu, X) \to Y$ is said to be *orthogonally additive* if

$$T(x+y) = Tx + Ty \text{ for all disjoint } x, y \in L_p(\mu, X).$$

Clearly $T(0) = 0$.

Definition

Let X be a Banach space and Y be a vector space. An operator $T\colon L_p(\mu, X) \to Y$ is said to be *orthogonally additive* if

$$T(x+y) = Tx + Ty \text{ for all disjoint } x, y \in L_p(\mu, X).$$

Clearly $T(0) = 0$.

Definition

An orthogonally additive operator $T : L_p(\mu) \to L_q(\nu)$ is called:

1. *positive* if $T(L_p(\mu)) \subset L_q(\nu)_+$;
2. *regular* if $T = S_1 - S_2$, where S_1, S_2 are positive orthogonally additive operators from $L_p(\mu)$ to $L_q(\nu)$;
3. *order bounded* if T maps order bounded subsets of $L_p(\mu)$ to order bounded subsets of $L_q(\nu)$;
4. *C-bounded* if $T(\mathcal{F}_x)$ is an order bounded subset of $L_q(\nu)$ for every $x \in L_p(\mu)$.

Theorem

$\mathcal{OA}_r(L_p(\mu), L_q(\nu))$ is a Dedekind complete vector lattice. Moreover, for every $S, T \in \mathcal{OA}_r(L_p(\mu), L_q(\nu))$ and every $x \in L_p(\mu)$ the following hold:

1. $(T \vee S)x = \sup\{Ty + Sz : x = y \sqcup z\}$;
2. $(T \wedge S)x = \inf\{Ty + Sz : x = y \sqcup z\}$;
3. $T^+x = \sup\{Ty : y \sqsubseteq x\}$;
4. $T^-x = -\inf\{Ty : y \sqsubseteq x\}$;
5. $|Tx| \leq |T|x$.

M. Pliev, K. Ramdane, *Order unbounded orthogonally additive operators in vector lattices*, Mediter. J. Math., **15** (2018), 2, article 55.

Definition

Let (A, Σ, μ) and (B, Ξ, ν) be finite measure spaces. By $(A \times B, \mu \otimes \nu)$ we denote the completion of their product measure space. A map $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if it satisfies the following conditions:

(C_1) $K(\cdot, \cdot, r)$ is $\mu \otimes \nu$-measurable for all $r \in \mathbb{R}$;

(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

We say that a Carathéodory function K is *normalized* if $K(s, t, 0) = 0$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

Let $K\colon A \times B \times \mathbb{R} \to \mathbb{R}$ be a normalized Carathéodory function and for every $f \in L_{p_1}(\mu)$ the function

$$s \mapsto \int_B K(s,t,f(t))\, d\nu(t)$$

belongs to $L_{p_2}(\nu)$. Then an orthogonally additive operator $T\colon L_{p_1}(\mu) \to L_{p_2}(\nu)$ is defined by the following setting

$$Tf(s) = \int_B K(s,t,f(t))\, d\nu(t).$$

M. A. Krasnosel'skii, P. P. Zabrejko, E. I. Pustil'nikov, P. E. Sobolevski, *Integral operators in spaces of summable functions*, Noordhoff, Leiden (1976).

A map $f\colon E \to \mathbb{R}$ is called valuation if

$$F(f \vee g) + F(\wedge g) = F(f) + F(g), \quad f, g \in E$$

P. Tradacete, and I. Villanueva, *Valuations on Banach lattices*, Int. Math. Res. Not., **2020** (2020), 1, 287–319.

Definition

Let X be a Banach space and Y be a normed space. An orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ is said to be

1. *narrow* if for every $f \in L_p(\mu, X)$ and $\varepsilon > 0$ there exist there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $\| Tf_1 - Tf_2)\|_Y < \varepsilon$;
2. *strictly narrow* if for every $f \in L_p(\mu, X)$ there exist there exist a disjoint decomposition $f = f_1 \sqcup f_2$ such that $Tf_1 = Tf_2$.

Definition

Let $f \in L_p(\mu, X)$. By $\mathfrak{F}_f$ we denote the following set:

$$\mathfrak{F}_f := \{g \in L_p(\mu, X) : (f - g) \perp g\}$$

Elements of $\mathfrak{F}_f$ are said to be fragments of f (notations $g \sqsubseteq f$).

Definition

We say that a net $(f_\alpha)_{\alpha \in A}$ in $L_p(\mu, X)$ *order* converges to $f \in L_p(\mu, X)$ (notation $f_\alpha \xrightarrow{o} f$) if the net $(\|f - f_\alpha\|_X)_{\alpha \in A}$ order converges to 0.

Definition

We say that a net $(f_\alpha)_{\alpha\in A}$ in $L_p(\mu, X)$ *laterally* converges to $f \in L_p(\mu, X)$ (notation $f_\alpha \xrightarrow{l} f$) if the net $(f_\alpha)_{\alpha\in A}$ order converges to f and $f_\alpha \sqsubseteq f_\beta \sqsubseteq f$ for all $\alpha, \beta \in A$ with $\alpha \leq \beta$.

Definition

Let Y be a normed space. A mapping $T\colon L_p(\mu, X) \to Y$ is said to be:

1. *laterally-to-norm continuous* if for a net $(f_\alpha)_{\alpha\in\Lambda}$ in $L_p(\mu, X)$ laterally convergent to $f \in L_p(\mu, X)$ a net $(Tf_\alpha)_{\alpha\in\Lambda}$ norm converges to Tf;
2. *laterally-to-norm σ-continuous* if for every sequence $(f_n)_{n\in\mathbb{N}}$ in $L_p(\mu, X)$ laterally convergent to $f \in L_p(\mu, X)$ the sequence $(Tf_n)_{n\in\mathbb{N}}$ norm converges to Tf;
3. *order-to-norm continuous* if for every net $(f_\alpha)_{\alpha\in\Lambda}$ in $L_p(\mu, X)$ order convergent to $f \in L_p(\mu, X)$ the net $(Tf_\alpha)_{\alpha\in\Lambda}$ norm converges to Tf;
4. *AM-compact* if T maps order bounded subsets in $L_p(\mu, X)$ to a relatively compact subsets of Y;
5. *C-compact* if $T(\mathfrak{F}_f)$ is a relatively compact subset of Y for all $x \in L_p(\mu, X)$;
6. *disjointness preserving* if Y is additionally a lattice-normed space and $Tx \perp Ty$ for all disjoint $x, y \in L_p(\mu, X)$.

Definition

We observe that laterally-to-norm continuity of an orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ implies its σ-laterally-to-norm continuity. Below we consider some examples of laterally-to-norm continuous and C-compact OAOs.

Example

Since $\mathfrak{F}_f$ is an order bounded subset of $L_p(\mu, X)$ for all $f \in L_p(\mu, X)$ we have that every AM-compact OAO is C-compact.

Definition

Example

Let (A, Σ, μ), (B, Ξ, ν) be finite measure spaces and $p, q \in [1, \infty)$. Then every order bounded Uryson integral operator $T \colon L_p(\mu) \to L_q(\nu)$ is AM-compact.

Let (A, Σ, μ) a finite measure space, and X, Y be Banach spaces. Then every bounded linear operator $T \colon L_p(\mu, X) \to Y$ is laterally-to-norm σ-continuous.

The next example shows that the class of C-compact OAOs strictly includes the class of AM-compact orthogonally additive operators even in the setting of the one-dimensional vector lattice $\mathbb{R}$.

Example

There is a C-compact OAO $T: \mathbb{R} \to \mathbb{R}$ which is not AM-compact. Indeed, Consider a mapping $T: \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Theorem

Let X, Y be Banach spaces. Then every laterally-to-norm continuous C-compact orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ is narrow.

First, we need to introduce nonlinear superposition operators on $E(X)$. Suppose that (A, Σ, μ) is a finite measure space and X is a Banach space. Recall that the *support* of a strongly μ-measurable X-valued function $f\colon A \to X$ (denotation $\operatorname{supp} f$) is the μ-measurable set

$$\operatorname{supp} f := \{t \in A:\ f(t) \neq 0\}.$$

Definition

Let (A, Σ, μ) be a finite measure space and X be a Banach space. A function $N : A \times X \to X$ is said to be:

1. *superpositionally measurable* (or *super-measurable* for brevity) if $N(t, f(\cdot)) \in L_0(\mu, X)$ for every $f \in L_0(\mu, X)$
2. $L_p(\mu, X)$*-super-measurable* if $N(t, f(\cdot)) \in L_p(\mu, X)$ for every $f \in L_p(\mu, X)$;
3. normalised if $N(\cdot, 0) = 0$ for μ-almost all $t \in A$.

Let (A, Σ, μ) be a finite measure space X be a Banach space and $N : A \times X \to X$ be a normalised $L_p(\mu, X)$-super-measurable function. Then there is a map $T_N \colon L_p(\mu, X) \to L_p(\mu, X)$ defined by the setting

$$T_N(f)(\cdot) = N(\cdot, f(\cdot)), \;\; f \in L_p(\mu, X).$$

We note that T_N is one of the most important operators of a nonlinear analysis. It is known in literature as a nonlinear superposition operator or Nemyskij operator.

- J. Appell, P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge University Press, Cambridge, 1990.
- T. Runst, W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, De Gruyter Series of Nonlinear Analysis and Applications, 1996.
- J. Appell, J. Banas, N. Merentes *Bounded variation and around*, De Gruyter, 2014.

Theorem

Let (A, Σ, μ) *be a finite atomless measure space,* E *be a Köthe-Banach space on* (A, Σ, μ) *with an order continuous norm,* X *be a Banach space and* $N : A \times X \to X$ *be a normalised* $L_p(\mu, X)$*-super-measurable function. Then for the a nonlinear superposition operator* $T_N : L_p(\mu, X) \to L_p(\mu, X)$ *the following assertions are equivalent:*

1. T_N *is a* C*-compact operator;*
2. $N(\cdot, f(\cdot)) = 0$ *for all* $f \in L_p(\mu, X)$.

Definition

Let X, Y be Banach spaces. An orthogonally additive operator $T\colon L_p(\mu,X)\to L_q(\nu,Y)$ is said to be *dominated* if there exists $S\in\mathcal{OA}_+(L_p(\mu),L_q(\nu))$ such that

$$\|Tf(\cdot)\|\leq S\|f(\cdot)\| \text{for all } f\in L_p(\mu,X).$$

The operator S is called a *dominant* of T. The set of all dominants of a dominated orthogonally additive operator T is denoted by $\mathfrak{D}(T)$. We note that $\mathfrak{D}(T)$ is a partially ordered set with respect to the partial order induced by $\mathcal{OA}_r(L_p(\mu),L_q(\nu))$. If $\mathfrak{D}(T)$ has a minimal element, then this element is called the *exact dominant* of T and is denoted by $|||T|||$. The vector space of all dominated OAOs between $L_p(\mu,X)$ and $L_q(\nu,Y)$ we denote by $\mathcal{OA}_D(L_p(\mu,X),L_q(\nu,Y))$.

Suppose that X and Y are coincide with the field $\mathbb{R}$. Then the space $\mathcal{OA}_D(L_p(\mu), L_q(\nu))$ coincides with the space $\mathcal{OA}_r(L_p(\mu), L_q(\nu))$ of all regular orthogonally additive operators from $L_p(\mu)$ to $L_q(\nu)$. Moreover the exact dominant of a dominated OAO T exists and $|||T||| = |T|$, where $|T|$ is the modulus of T.

Definition

Let (A, Σ, μ) be a finite measure space and X, Y be Banach spaces. A function $K\colon A \times X \to Y$ is said to be a *weakly μ-super measurable* if $\langle z, K(t, f(t))\rangle \in L_0(\mu)$ for all $z \in Y^*$ and $f \in L_0(\mu, X)$. We recall that K is a normalised function if $K(t, 0) = 0$ for μ-almost all $t \in A$. We say that $K\colon A \times X \to Y$ belongs to the class $\mathfrak{A}$ if it satisfies the following conditions:

1. K is a weakly μ-super measurable function;
2. K is a normalised function;
3. the following inequality

$$\langle z, K(\cdot, f(\cdot))\rangle \leq w(\cdot)\varphi(\cdot), \ \ \varphi \in L_0(\mu)$$

holds for every $z \in B_{Y^*}$, $f \in L_0(\mu, X)$ such that $\|f(\cdot)\|_X = \varphi$ and some $w \in L_0(\mu)$.

With every $K \in \mathfrak{A}$ is associated a μ-measurable function $|K|\colon A \times \mathbb{R} \to \mathbb{R}$ defined by the setting

$$|\mathcal{K}|(t, r) := \sup\{\langle z, K(t, x)\rangle : x \in S_X^r,\ z \in B_{Y^*}\}$$

where the supremum is taken in $L_0(\mu)$ (the existence of the supremum is guaranteed by the assumption (3) above). We say that $T\colon L_p(\mu, X) \to Y$ is a *weakly integral* operator if there exists $K \in \mathfrak{A}$ such that the equality

$$\langle z, Tf\rangle = \int_A \langle z, K(t, f(t))\rangle d\mu(t), \quad f \in L_p(\mu, X).$$

holds for every $z \in Y^*$. The function K is called the *kernel* of an integral operator T.

Lemma

Let (A, Σ, μ) *be a finite measure space,* X, Y *be Banach spaces and* $T\colon L_p(\mu, X) \to Y$ *be a weakly integral operator with the kernel* K*. Then* T *is an orthogonally additive operator.*

Definition

A weakly integral operator $T\colon L_p(\mu, X) \to Y$ with the kernel K is called *regular* if there exists an integral functional $S\colon L_p(\mu) \to \mathbb{R}$ with the kernel $|K|$. We note that S is a positive orthogonally additive integral functional on E.

The following theorem is first main result of this section. It states necessary and sufficient conditions for a weakly integral operator from $L_p(\mu, X)$ to Y to be dominated.

Theorem

Let (A, Σ, μ) be a finite measure space, X, Y be Banach spaces and $T\colon L_p(\mu, X) \to Y$ be a weakly integral operator with the kernel K. Then the following statements are equivalent:

1. *T is a regular operator;*
2. *T is a dominated operator.*

Moreover the exact dominant $|T|$ of T is the integral functional $S\colon L_p(\mu) \to \mathbb{R}$ with the kernel $|\mathcal{K}|$.

Let Y be a vector lattice of real-valued functions $f\colon \mathbb{N} \to \mathbb{R}$ equipped with the pointwise order and a lattice norm $\|\cdot\|_Y$. We say that Y is a *Banach sequence space with an order compatible basis* if the the following conditions hold:

1. Y is a Banach lattice with respect to the norm $\|\cdot\|_Y$;
2. unit vectors $(e_i)_{i=1}^{\infty}$, where $e_i = (\delta_{ij})_{j=1}^{\infty}$, form a Schauder basis in Y;
3. projections $\pi_n\colon Y \to Y_n$, where Y_n is the linear span of $e_1, \dots, e_n$, are positive linear operators with $0 \leq \pi_n \leq Id$ for all $n \in \mathbb{N}$.

Theorem

Let X be a Banach space and Y be a Banach sequence space with an order compatible basis. Then every laterally-to-norm continuous dominated orthogonally additive operator $T\colon L_p(\mu, X) \to Y$ is narrow.

By $E(M,\tau)_{sa}$ we denote a real vector space of self-adjoint elements of $E(M,\tau)$. We note that $E(M,\tau)_{sa}$ is an ordered Banach space with respect to the partial order defined by: $x \leq y \Leftrightarrow (y-x) \in E(M,\tau)_+$. For every $h \in E(\mathcal{M},\tau)$ there exists the modulus $|h| := (h^*h)^{\frac{1}{2}}$.

Suppose that $a \in (0, \infty]$, $I = (0, a)$ and Σ is the σ-algebra of Lebesgue measurable subsets of I. By (I, m) we denote the measure space (I, Σ, m) equipped with Lebesgue measure m. Let $L_0(I, m)$ (or $L_0(I)$ for brevity) be the space of all equivalence classes of almost everywhere finite Lebesgue measurable real-valued functions on I. For $x \in L_0(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t; x) = \inf \{s \geq 0 : \ m(\{|x| > s\}) \leq t\}, \quad t > 0.$$

Definition

We say that $(E(I), \|\cdot\|_E)$ (or $(E, \|\cdot\|_E)$) is a symmetric Banach function on I if the following hold:

1. $E(I)$ is a subspace of $L_0(I)$;
2. $(E, \|\cdot\|_E)$ is a Banach space;
3. If $x \in E$ and if $y \in L_0(I)$ are such that $|y| \leq |x|$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$;
4. If $x \in E$ and if $y \in L_0(I)$ are such that $\mu(y) = \mu(x)$, then $y \in E$ and $\|y\|_E = \|x\|_E$.

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ. Let $\mathbf{1}$ be the identity. A closed and densely defined operator A affiliated with $\mathcal{M}$ is called τ-measurable if $\tau(E_{|x|}(s,\infty)) < \infty$ for sufficiently large s, where $E_{|x|}$ stands for the spectral measure of $|x|$. We denote the set of all τ-measurable operators by $L_0(\mathcal{M},\tau)$. Let $\mathcal{P}(\mathcal{M})$ denote the lattice of all projections in $\mathcal{M}$. We denote by $L_0(M,\tau)_+$ the collection of all non-negative operators in $L_0(M,\tau)$.

For every $x \in L_0(\mathcal{M}, \tau)$, we define its singular value function $\mu(x)$ by setting

$$\mu(t; x) = \inf\{\|x(\mathbf{1} - p)\|_{\mathcal{M}} : p \in P(\mathcal{M}), \quad \tau(p) \leq t\}, \quad t > 0.$$

For more details on generalised singular value functions, we refer the reader to

- S. Lord, F. Sukochev, D. Zanin, *Singular traces: Theory and applications*, De Gruyter, Berlin, 2013.

If $\mathcal{M}$ is the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ and τ is the standard trace, then $L_0(M, \tau) = \mathcal{M}$, the measure topology coincides with the operator norm topology.

Definition

Let $\mathfrak{A}$ be a linear subspace of $L_0(\mathcal{M}, \tau)$ equipped with a complete norm $\|\cdot\|_{\mathfrak{A}}$. We say that $\mathfrak{A}$ is a symmetric operator space if for $x \in \mathfrak{A}$ and for every $y \in L_0(\mathcal{M}, \tau)$ with $\mu(x) \leq \mu(y)$, we have $x \in \mathfrak{A}$ and $\|x\|_{\mathfrak{A}} \leq \|y\|_{\mathfrak{A}}$.

Recall the following construction of a symmetric Banach operator space (or non-commutative symmetric Banach space) $E(\mathcal{M},\tau)$. Set

$$E(\mathcal{M},\tau)=\Big\{x\in L_0(\mathcal{M},\tau):\ \mu(x)\in E\Big\}$$

(respectively, $E(\mathcal{M},\tau)=\Big\{x\in L_0(\mathcal{M},\tau):\ \{\mu(n;x)\}_{n\geq 0}\in E\Big\}$).

There exists the natural norm on $E(\mathcal{M},\tau)$ defined by

$$\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E, \quad x \in E(\mathcal{M},\tau).$$

We note that $E(\mathcal{M},\tau)$ is a Banach space with respect to $\|\cdot\|_{E(\mathcal{M},\tau)}$ and it is called the (non-commutative) symmetric operator space associated with $(\mathcal{M},\tau)$ corresponding to $(E,\|\cdot\|_E)$. If $M = B(H)$, then we denote $E(M,\tau)$ by C_E. If E is the Lebesgue space L_p, $1 \leq p < \infty$, the space C_E is familiar Schatten-von Neumann ideal denoted by C_p for brevity.

Definition

Let $E(M,\tau) \subset L_0(M,\tau)$ be is a symmetric operator space. The norm $\|\cdot\|_{E(M,\tau)}$ is called *order continuous* if $\|a_\beta\|_{E(M,\tau)} \to 0$ whenever $\{a_\beta\}$ is a downwards directed net in $E(M,\tau)_+ := E(M,\tau) \cap L_0(M,\tau)_+$ satisfying $a_\beta \downarrow 0$.

Definition

Let X be a Banach space and $E(M,\tau)$ be a symmetric operator space. An orthogonally additive operator $T\colon L_p(\mu,X)\to E(M,\tau)_{sa}$ is called *dominated* if there exists a positive orthogonally additive operator $S\colon L_p(\mu)\to E(M,\tau)_+$ such that

$$|Tf|\leq S\|f(\cdot)\|_X;\ f\in L_p(\mu,X).$$

An operator S is called a *dominant* of T.

Lemma

Suppose that M is a von Neumann algebra and X is a Banach space. Then every bounded linear operator $T\colon L_p(\mu, X) \to E(M,\tau)_{sa}$ is laterally-to-norm continuous.

Theorem

Let X be a Banach space. Then every laterally-to-norm continuous dominated orthogonally additive operator $T: L_p(\mu, X) \to (C_E)_{sa}$ is narrow, whenever C_E is separable.