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CONVEXITY CONDITIONS FOR HOMOGENEOUS POLYNOMIALS ON QUASI-BANACH LATTICES

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Полиномы в векторных решетках обладают интересными порядковыми свойствами, а классы полиномов в банаховых решетках, определяемые в смешанных терминах нормы и порядка, имеют богатую структуру. Поэтому эти объекты вызывают растущий интерес исследователей. Классы (p,q)-выпуклых (p,q)-вогнутых линейных операторов, а также понятия типа и котипа играют важные роли в теории банаховых решеток и положительных опертоторов. Все эти конструкции и значительная часть соответствующих результатов естественно переносится на квазибанаховы пространства. Однако, в этом контексте не работают соображения, основанные на выпуклости и отделимости, в связи с чем были разработаны новые подходы и приемы. Цель настоящей работы — распространить указанный круг идей с линейных операторов на полиномы и изучить условия выпуклости для однородных полиномов в квазибанаховых решетках.

Ключевые слова: квазибанахова решетка, однородный полином, регулярный полином, (p,q)-выпуклость, (p,q)-вогнутость, вогнутизация, факторизация.

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Polynomials on vector lattices possess interesting order properties, and classes of polynomials on Banach lattices, defined in mixed terms of norm and order, have a rich structure. This is why the subject draw growing attention of researchers. The classes of (p,q)-convex and (p,q)-concave linear operators in Banach lattices, as well as the conceptions of type and cotype play an important role in the theory of Banach lattices and bounded linear operators. All these concepts and many related results may be naturally transplanted to the environment of quasi-Banach spaces. Convexity arguments do not work well in arbitrary quasi-Banach spaces and this led to developing of new approaches and techniques. The aim of this work is to extend the above circle of ideas from linear case to the polynomial setting and examine convexity conditions for homogeneous polynomials on quasi-Banach lattices.

Key words: quasi-Banach lattice, homogeneous polynomial, regular polynomial, (p,q)-convexity, (p,q)-concavity, concavification, factorization.

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CONVEXITY CONDITIONS FOR HOMOGENEOUS POLYNOMIALS ON QUASI-BANACH LATTICES

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1. Introduction

Polynomials on vector lattices possess interesting order properties, and the classes of polynomials on Banach lattices, defined in mixed terms of norm and order, have rich structure. This is why the subject draw growing attention of researchers. Historically, the study of special sets of homogeneous polynomials between Banach spaces is motivated by infinite-dimensional holomorphy (see Dineen [15] and Mujica [47]) and the theory of nonlinear ideals stemming from Pietsch's paper [49] (for historical roots see also Bernardino, Pellegrino, Seoane-Sepulveda, and Souza [4]). While the algebraic and linear-topological properties of polynomials as well as the relations between polynomials and the geometry of Banach spaces have a long history and are well covered in literature (see, for example, [15]), the study of the order properties of polynomials on vector and Banach lattices has began recently: the papers by Sundaresan [53] and Grecu and Ryan [16] should be considered as two starting points. For recent advances we refer to [2, 3, 7, 18, 34, 35, 37, 38, 48, 54] (see also recent PhD theses [36, 39, 41]) and the references therein.

The classes of (p,q)-convex and (p,q)-concave linear operators on Banach lattices introduced by Krivine [28] (the case $p = q$) and Maurey [44] (the general case), as well as the conceptions of type and cotype introduced by Maurey and Pisier [45] play an important role in the theory of Banach lattices and bounded linear operators, see Diestel, Jarchow, and Tong [14], Lindenstrauss and Tzafriri [40], Schwarz [52].

It was shown by Kalton in [21–24] that all these concepts and many related results may be naturally transplanted to the environment of quasi-Banach spaces, see also [26]. Kalton offered new approaches and invented a variety of tools, since convexity arguments do not work well in arbitrary quasi-Banach spaces because of the weaker triangle inequality.

This work is an attempt to extend the above circle of ideas from linear case to the polynomial setting and examine convexity conditions for homogeneous polynomials on quasi-Banach lattices. The paper is organized as follows.

In Section 2 we briefly sketch the needed information concerning quasi-Banach lattices and homogeneous polynomials. In Section 3 we gather some auxiliary facts concerning the concavification of quasi-Banach lattices. The main tool is the homogeneous functional calculus introduced by Krivine [28] and Lozanovskiĭ [42] which works also in quasi-Banach lattices (see also Cuartero and Triana [12], Lindenstrauss and Tzafriri [40], Popa [50], Szulga [55]).

In Section 4 we introduce (p,q)-convex homogeneous polynomials and study relations between convexities. We extend monotonicity of convexity and interpolation of distinct convexities to the context of homogeneous polynomials on quasi-Banach spaces. Some important technical tools are adopted from Cuartero and Triana [13], Kalton [21–23], and Szulga [55].

Kalton characterized in [23, Theorem 2.2] the class of quasi-Banach lattices which are p-convex for some $0 < p \in \mathbb{R}$ by means of L-convexity. He also proved in [22, Theorem 4.2] that a quasi-Banach space of Rademacher type p is p-convex. In Section 5 we prove similar results for homogeneous polynomials.

In Section 6 we give conditions under which a homogeneous disjointness preserving polynomial P between quasi-Banach lattices admits a factorization through an $L_p(\mu)$-space, either in the form $P = Q \circ T$, or in the form $P = T \circ Q$ where Q is a disjointness preserving homogeneous polynomial and T is a lattice homomorphism. Section 7 deals with the special case of homogeneous orthogonally additive polynomials. The properties of this class of polynomials resemble very much those of linear operators and, in particular, admits good factorization. In Section 8, following Raynaud and Tradacete [51], we show that a p-convex homogeneous polynomial can be factored through a p-convex quasi-Banach lattice and this fact enables us to obtain Krivine's type factorization for homogeneous polynomials. Section 9 is devoted to the question: When is the quasi-Banach lattice of regular linear operators or polynomials between quasi Banach lattices (p, q)-convex, or (p, q)-concave, or geometrically convex?

We use the standard notation and terminology of Aliprantis and Burkinshaw [1] and Meyer-Niberg [46] for the theory of vector and Banach lattices and of Dineen [15] for the theory of polynomials. In the present paper we assume that all vector spaces are defined over the field of reals and all vector lattices are Archimedean.

We let $:=$ denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

2. Homogeneous Polynomials on Quasi-Banach Lattices

In this section, we briefly sketch the needed information concerning quasi-Banach lattices and homogeneous polynomials. In the sequel we fix a natural $s \in \mathbb{N}$, and unless indicated otherwise, denote by X and Y quasi-Banach spaces and by E and F quasi-Banach lattices.

Definition 2.1. A *quasi-normed space* is a pair $(X, \|\cdot\|)$ where X is a real vector space and $\|\cdot\|$ is a *quasi-norm*, a function from X to $\mathbb{R}$ such that the following conditions hold:

(1) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.

(3) There exists a constant $C \geqslant 1$ such that $\|x + y\| \leqslant C(\|x\| + \|y\|)$ for all $x, y \in X$.

The best constant C in 2.1 (3) is called the *quasi-triangle constant*, or *quasi-norm multiplier*, or *modulus of concavity* of the quasi norm. Two quasi-norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if there is a constant $A \geqslant 1$ such that $A^{-1}\|x\| \leqslant \|x\|' \leqslant A\|x\|$ for all $x \in X$.

By the Aoki–Rolewicz theorem (see [23]), each quasi-norm is equivalent to some quasi-norm with the property that $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$ $(x, y \in X)$ for some

$0 < p \leqslant 1$. Such quasi-norm is called a *p-norm.* Thus, we may assume unless otherwise mention that a quasi-Banach space is equipped with a p-norm for some $0 < p \leqslant 1$.

A topological vector space X is said to be *locally bounded* if it has a bounded neighborhood of zero. A quasi-normed space is a locally bounded topological vector spaces if we take the sets $\{x \in X : \|x\| \leqslant \varepsilon\}$ $(0 < \varepsilon \in \mathbb{R})$ for a base of neighborhoods of zero. Moreover, this topology may be induced by metric $d(x, y) := |||x - y|||^p$ $(x, y \in X)$ where $|||\cdot|||$ is an equivalent p-norm. Conversely, Hyers [17] proved that the topology of a locally bounded topological vector space X can be deduced from a quasi-norm, which may be obtained as the Minkowski functional of a bounded balanced neighborhood B of zero:

$$\|x\| := \|x\|_B := \inf\{0 < \lambda \in \mathbb{R} : \ x \in \lambda B\} \quad (x \in X).$$

DEFINITION 2.2. A *quasi-Banach space* is a quasi-normed space which is complete in its metric uniformity. A quasi-Banach space $(X, \|\cdot\|)$ is called a *quasi-Banach lattice* if, in addition, it is a vector lattice and $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in X$.

DEFINITION 2.3. Fix any $s \in \mathbb{N}$. A mapping $P : X \to Y$ is called a *homogeneous polynomial of degree s* (or *s-homogeneous polynomial*) if there exists an s-linear operator $\varphi : X^s \to Y$ such that $P = \varphi \circ \Delta_s$, where $\Delta_s : X \to X^s$ is the *diagonal mapping* $\Delta_s : x \mapsto (x, \dots, x) \in X^s$. There exists a unique symmetric s-linear operator φ with $P = \varphi \circ \Delta_s$ which is denoted by $\check{P}$, so that $P(x) = \check{P}(x, \dots, x)$ for all $x \in X$.

An s-homogeneous polynomial P between quasi-normed spaces X and Y is continuous if and only if it is bounded, and we put, as usual,

$$\|P\| = \sup\{\|P(x)\| : \ \|x\| = 1\} = \inf\{C > 0 : \ \|P(x)\| \leqslant C\|x\|^s, \ (x \in X)\}, \quad (1)$$

so that $\|P(x)\| \leqslant \|P\|\|x\|^s$ $(x \in X)$. We denote by $\mathscr{P}(^sX, Y)$ the Banach space of all continuous s-homogeneous polynomials from X into Y endowed with the quasi-norm (1). In case $s = 1$ we put $\mathscr{L}(X, Y) := \mathscr{P}(^1X, Y)$.

The basic results of the Banach space theory such as open mapping theorem and the closed graph theorem (for linear operators) are valid also in the context of quasi-Banach spaces, see [26]. Consider now vector lattices E and F.

DEFINITION 2.4. Say that an s-linear operator $\varphi : E^s \to F$ is *positive* and write $\varphi \geqslant 0$ if $\varphi(x_1, \dots, x_n) \geqslant 0$ for all $0 \leqslant x_1, \dots, x_n \in E$. An s-linear operator $\varphi : E^s \to F$ is said to be *order bounded* if $\varphi(A^s)$ is order bounded in F for each order bounded set A in E; *orthosymmetric,* if $\varphi(x_1, \dots, x_n) = 0$ whenever $|x_k| \wedge |x_l|$ for some pair of indices $1 \leqslant k, l \leqslant s$; *lattice multimorphism* or *s-morphism* if $|\varphi(x_1, \dots, x_n)| = \varphi(|x_1|, \dots, |x_n|)$ for all $x_1, \dots, x_n \in E$, see Bu, Buskes, and Kusraev [8].

An order bounded orthosymmetric multilinear operator is symmetric [5, 6, 9, 34] and a lattice multimorphism is orthosymmetric if and only if it is symmetric [8, 9]. As usual, $\varphi \leqslant \psi$ means that $\psi - \varphi \geqslant 0$.

DEFINITION 2.5. Say that an s-homogeneous polynomial P is *positive* and write $P \geqslant 0$ if the corresponding s-linear operator $\check{P}$ is positive; P is *regular* if it is representable as a difference of two positive s-homogeneous polynomials. Denote by $\mathscr{P}^r({}^sE, F)$ the spaces of all regular s-homogeneous polynomials from E to F. The partial order on $\mathscr{P}^r({}^sE, F)$ is introduced as usual by the cone of positive polynomials: $P \leqslant Q$ if and only if $0 \leqslant Q - P$. Obviously, $\mathscr{P}^r({}^sE, F)$ is an ordered vector space. If F is Dedekind complete vector lattice then so is $\mathscr{P}^r({}^sE, F)$.

If E and F are quasi-normed lattices then $\mathscr{P}^r({}^sE, F)$ is an ordered quasi-normed space under the regular norm

$$\|P\|_r := \inf\{\|Q\| : \ \pm P \leqslant Q \in \mathscr{P}^r({}^sE, F)\}.$$

Moreover, $\mathscr{P}^r({}^sE, F)$ is a quasi-normed lattice whenever F is Dedekind complete, and in this case $\|P\|_r = \|\,|P|\,\|$, since for a positive $Q \in \mathscr{P}^r({}^sE, F)$ we have

$$\|Q\|_r = \|Q\| = \sup\{\|Q(x)\| : \ 0 \leqslant x \in E, \ \|x\| \leqslant 1\}. \tag{2}$$

Proposition 2.6. *Let E be a quasi-Banach lattice and F a quasi-normed space and $P : E \to F$ an orthogonally additive s-homogeneous polynomial. If P sends order intervals in E to norm bounded sets in F then P is continuous. In particular, every positive (and hence every regular) homogeneous polynomial from a quasi-Banach lattice to a quasi-normed lattice is continuous.*

$\lhd$ Let $P : E \to F$ be an s-homogeneous polynomial from a quasi-Banach lattice E to a quasi-normed lattice norm bounded on order intervals. Assume by way of contradiction that P is not bounded. Then there exists a sequence (x_k) of E satisfying $\|x_k\| = 1$ and $\|P(x_k)\| \geqslant (C+1)^{ks}k$ for all $k \in \mathbb{N}$ with C a quasi-triangle constatnt of E. The completeness of E and the relation $\sum_{k=1}^{\infty} C^k\|x_k\|/(C+1)^k < \infty$ implies that the sum of the series $x = \sum_{k=1}^{\infty} |x_k|/(C+1)^k$ exists in E. By hypotheses the set $P([-x, x])$ is norm bounded in Y. Clearly, $-x \leqslant x_k/(C+1)^k \leqslant x$ and thus

$$k \leqslant \|P(x_k/(C+1)^k)\| \leqslant \sup\{\|P(u)\| : \ -x \leqslant u \leqslant x\} < \infty$$

for all $k \in \mathbb{N}$, a contradiction. $\rhd$

Definition 2.7. A homogeneous polynomial P from E to F is said to be *orthogonally additive*, whenever $|x| \wedge |y| = 0$ implies $P(x+y) = P(x) + P(y)$ for all $x, y \in E$ and *orthoregular* if P can be written as a difference of two positive orthogonally additive homogeneous polynomials.

Let $\mathscr{P}_o^r({}^sE, F)$ denotes the space of all orthoregular s-homogeneous (continuous if E and F are quasi-normed lattices) polynomials from E to F. The *regular norm* $\|\cdot\|_r$ on $\mathscr{P}_o^r({}^sE, F)$ is defined as $\|P\|_r := \inf\{\|Q\| : \pm P \leqslant Q \in \mathscr{P}_o^r({}^sE, F)\}$.

Theorem 2.8. *Let E and F be vector lattices. An order bounded s-homogeneous polynomial P is orthogonally additive if and only if its corresponding symmetric s-linear operator $\check{P}$ is orthosymmetric.*

$\lhd$ The sufficiency is immediate. The necessity was proved by a number of authors in different situations, see [7, 18, 41, 54]. The most general form was obtained in [34, Lemma 4]. $\rhd$

It is easy to see that P is order bounded if and only if so is $\check{P}$. Let $\mathscr{P}_o^{\sim}({}^sE; F)$ denotes the space of all order bounded orthogonally additive s-homogeneous (continuous in the case that E and F are quasi-Banach lattices) polynomials from E to F. Order relation in $\mathscr{P}_o^{\sim}({}^sE,F)$ is defined as in Definition 2.5. If F is Dedekind complete then $\mathscr{P}_o^{\sim}({}^sE,F) = \mathscr{P}_o^{r}({}^sE,F)$ and this space is a Dedekind complete vector lattice.

Definition 2.9. Let $2 \leqslant s \in \mathbb{N}$ and E be a vector lattice. The pair $(E^{s\odot}, \odot_s)$ is called an *s-power* of E if the following condition are fulfilled:

(1) $E^{s\odot}$ is a vector lattice;

(2) $\odot_s : E^s \to E^{s\odot}$ is a symmetric lattice s-morphism;

(3) for every vector lattice F and every symmetric lattice s-morphism $\varphi : E^s \to F$ there is a unique lattice homomorphism $S : E^{s\odot} \to F$ such that$\varphi = S \circ \odot_s$.

This definition is introduced in Boulabier and Buskes [6, Definition 3.1] (for the case $s = 2$ see Buskes and van Rooij [11]). In [6, Theorem 3.2] the existence of a unique (up to a lattice isomorphism) s-power for every vector lattice was established. In what follows we put $E^{1\odot} = E$ and $\odot_1 = I_E$ for convenience.

The polynomial $\jmath_s := \odot_s \circ \Delta_s : E \to E^{s\odot}$ generated by $\odot_s$ is positive, orthogonally additive, and disjointness preserving, see [38]. The notation $x^{s\odot} := \jmath_s(x)$ is also used so that $\jmath_s : x \mapsto x^{s\odot}$. This polynomial called the *canonical polynomial of* E plays the role of the exponentiation missing in general vector lattices. In particular, every bounded orthogonally additive homogeneous polynomial on a vector lattice is a composition of the canonical polynomial and a bounded linear operator. The history of this representation result is reflected in [3, 6, 18, 48, 53, 54]. A general form has been found in [34] and [2]: Kusraeva [34] handled the situation when “boundedness” is understood by means of the bornology of order bounded sets of the domain vector lattice, while the range space is equipped with a separated convex bornology; Ben Amor [2, Theorem 26] improved this result showing that the convexity assumption may be omitted. This form of polynomial representation theorem stated next is applicable in setting of quasi-Banach spaces.

A mapping between bornological spaces is labeled as *bounded* if it sends bounded sets into bounded sets. A vector lattice is considered with the bornology of order bounded sets. Denote by $\mathscr{P}_o^b({}^sE,Y)$ the space of bounded s-homogeneous orthogonally additive polynomials from E to Y and put $\mathscr{L}^b(E,Y) = \mathscr{P}_o^b({}^1E,Y)$.

Theorem 2.10. *Let E be a uniformly complete vector lattice and Y be a separated bornological space. Then for any orthogonally additive bounded s-homogeneous polynomial $P : E \to Y$ there exists a unique bounded linear operator $S : E^{s\odot} \to Y$ such that the representation holds*

$$P(x) = T(x^{s\odot}) \quad (x \in E). \tag{3}$$

Moreover, the spaces $\mathscr{P}_o^b({}^sE,Y)$ and $\mathscr{L}^b(E^{s\odot},Y)$ are linearly isomorphic under the mapping $T \mapsto \circ \jmath_s$.

$\lhd$ See Kusraeva [34, Corollary 3] and Ben Amor [2, Theorem 26]. $\rhd$

Let $\mathscr{P}_o({}^sE,Y)$ stands for the part of $\mathscr{P}({}^sE,Y)$ consisting of orthogonally additive polynomials.

Corollary 2.11. *Let E be a quasi-Banach lattice and Y a quasi-normed space and $P : E \to Y$ a norm bounded orthogonally additive s-homogeneous polynomial.*

Then there exists a unique norm bounded linear operator $T : E^{s\odot} \to Y$ *such that the representation* (3) *holds. Moreover, the correspondence* $T \mapsto T \circ \jmath_s$ *is an isometric isomorphism of quasi-normed spaces* $\mathscr{L}(E^{s\odot}, Y)$ *and* $\mathscr{P}_o(^sE, Y)$.

$\lhd$ Follows from Theorem 2.10 as $\mathscr{P}_o(^sE, Y) = \mathscr{P}_o^b(^sE, Y)$ by Proposition 2.6. $\rhd$

Corollary 2.12. *Let* E *be a quasi-Banach lattice,* F *a quasi-normed lattice, and* $P : E \to F$ *a regular orthogonally additive* s*-homogeneous polynomial. Then there exists a unique regular linear operator* $T : E^{s\odot} \to F$ *such that the representation* (3) *holds. Moreover, the correspondence* $T \mapsto T \circ \jmath_s$ *is an isomertric isomorphism of ordered quasi-normed spaces* $\mathscr{L}^r(E^{s\odot}, F)$ *and* $\mathscr{P}_o^r(^sE, F)$. *If* F *is Dedekind complete then* $\mathscr{L}^r(E^{s\odot}, F)$ *and* $\mathscr{P}_o^r(^sE, F)$ *are Dedekind complete quasi-normed lattices.*

$\lhd$ This is immediate from Theorem 2.10 and formula (2). $\rhd$

Remark 2.13. Corollaries 2.11 and 2.12 in case of Banach lattices E and F and a Banach space Y are proved in Bu and Buskes [7], see Theorems 4.3 and 5.4; however these theorems are covered by an earlier general result due to Kusraeva [34, Theorem 4].

3. Concavification of Quasi-Banach Lattices

In this section we gather some auxiliary facts about the concavification of quasi-Banach lattices. The main tool is homogeneous functional calculus.

Proposition 3.1. *Every quasi-Banach lattice is uniformly complete.*

$\lhd$ See Szulga [55, Proposition 2.2]. $\rhd$

Thus every quasi-Banach lattices admits a *homogeneous functional calculus*, see [28, 40, 42, 55]. Let $\mathscr{H}_n := \mathscr{H}(\mathbb{R}^n)$ be the vector lattice of positively homogeneous continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ equipped with a lattice norm $\|\varphi\| = \sup\{|\varphi(t)| : t \in \mathbb{R}^n, \|t\|_\infty = 1\}$. If E is a uniformly complete vector lattice, $n \in \mathbb{N}$, and $\mathbf{x} = (x_1, \dots, x_n) \in E^n$ then there exists a unique lattice homomorphism $\widehat{\mathbf{x}} : \mathscr{H}_n \to E$ such that $\widehat{\mathbf{x}}(dt_k) = x_k$ with $dt_k : t \mapsto t_k$, $t = (t_1, \dots, t_n)$ $(k = 1, \dots, n)$. Moreover, $|\widehat{\mathbf{x}}(\varphi)| \leqslant \|\varphi\|_\infty |x_1| \vee \dots \vee |x_n|$ and $\|\widehat{\mathbf{x}}(\varphi)\| \leqslant \|\varphi\|_\infty \|\,|x_1| \vee \dots \vee |x_n|\,\|$ whenever E is a quasi-Banach lattice. The element $\widehat{\mathbf{x}}(\varphi) \in E$ is usually denoted by $\varphi(x_1, \dots, x_n)$.

Proposition 3.2. *Let* $\varphi \in \mathscr{H}(\mathbb{R}^n)$ *and* $\langle (u_1, \dots, u_n) \rangle$ *stands for* $|u_1| \vee \dots \vee |u_N|$. *Then for every* $\varepsilon > 0$ *there exists a number* $R_\varepsilon > 0$ *such that*

$$|\widehat{\varphi}(\mathbf{y}) - \widehat{\varphi}(\mathbf{x})| \leqslant \varepsilon \langle \mathbf{x} \rangle + R_\varepsilon \langle \mathbf{y} - \mathbf{x} \rangle$$

for all $\mathbf{x} = (x_1, \dots, x_n) \in E^n$ *and* $\mathbf{y} = (y_1, \dots, y_n) \in E^n$. *In particular, the mapping* $\mathbf{x} \mapsto \varphi(\mathbf{x})$ $(\mathbf{x} \in E^n)$ *is continuous relative to the topology on* E *generated by the quasi-norm.*

$\lhd$ See Buskes and van Rooj [11, Theorem 7]. $\rhd$

Proposition 3.3. *Let* E *be a uniformly complete vector lattice and* $x_1, \dots, x_n \in E$. *If a function* $\varphi \in \mathscr{H}(\mathbb{R}^n)$ *is convex, then the representation holds*

$$\varphi(x_1, \dots, x_n) = \sup \left\{ \sum_{k=1}^{n} \alpha_k x_k : \ (\alpha_1, \dots, \alpha_n) \in \partial\varphi \right\}, \tag{4}$$

where

$$\partial\varphi := \left\{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \sum_{k=1}^{n} \alpha_k t_k \leqslant \varphi(t_1, \ldots, t_n),\ (t_1, \ldots, t_n) \in \mathbb{R}^n\right\}.$$

Moreover, $\varphi(x_1, \ldots, x_N)$ is a uniform limit of an increasing sequence which is comprised of the finite suprema of sums $\sum_{k=1}^{n} \alpha_k x_k$ with $(\alpha_1, \ldots \alpha) \in \partial\varphi$.

$\lhd$ See Kusraev [30, Theorem 5.5]. $\rhd$

Proposition 3.4. *If $\varphi \in \mathscr{H}(\mathbb{R}^n)$, $x_1, \ldots, x_n \in E$ and $h : E \to F$ is a lattice homomorphism then $h(\varphi(x_1, \ldots, x_n)) = \varphi(h(x_1), \ldots, h(x_n))$.*

$\lhd$ See Kusraev [30, Proposition 3.6]. $\rhd$

Definition 3.5. Take a positive real number p. Using the homogeneous functional calculus, we can introduce new vector operations on E by putting $x \oplus y = (x^p + y^p)^{1/p}$ and $\lambda \circledast x = \lambda^{1/p} x$, where $x, y \in E$ and $\lambda \in \mathbb{R}$. Endowed with these new operations, the original order and lattice structures, E becomes a vector lattice. Define a function $\|\cdot\|_{(p)} : E \to \mathbb{R}$ by $\|x\|_{(p)} := \|x\|^p$ $(x \in E)$ and note that $\|x \oplus y\|_{(p)} \leqslant 2^{|1-p|} C^p(\|x\|_{(p)} + \|y\|_{(p)})$. This new vector lattice together with the function $\|\cdot\|_{(p)}$ is called the *p-concavification* of E and is denoted by $E_{(p)}$. If $s \in \mathbb{N}$ then $E_{(s)} = E^{s\odot}$, see Boulabier and Buskes [6].

Proposition 3.6. *For every fixed $0 < p < \infty$, $(E_{(p)}, \|\cdot\|_{(p)})$ is a quasi-Banach lattice if and only if $(E, \|\cdot\|)$ is a quasi-Banach lattice. In particular, E and $E_{(p)}$ are relatively uniformly complete whenever E is a quasi-Banach lattice.*

$\lhd$ See [13, Proposition 1.2]. $\rhd$

Denote by ι_p the identity mapping of $(E, \leqslant)$ considered as an operator from E onto $E_{(p)}$. Clearly, ι_p is order isomorphism of E onto $E_{(p)}$, since the vector lattices E and $E_{(p)}$ have the same underlying ordered set $(E, \leqslant)$.

Proposition 3.7. *The nonlinear order isomorphism ι_p from E onto $E_{(p)}$ is modulus preserving $\big(|\iota_p(x)| = \iota_p(|x|)\big)$ and odd $\big(\iota_p(-x) = -\iota_p(x)\big)$. Moreover, for all $x, y \in E$ and $\lambda \in \mathbb{R}$ we have*

$$\iota_p\big((x^p + y^p)^{\frac{1}{p}}\big) = \iota_p(x) \oplus \iota_p(y),$$
$$\iota_p(\lambda^{\frac{1}{p}} x) = \lambda \circledast \iota_p(x).$$

In particular, ι_p is disjointness preserving and orthogonally additive. If $p \in \mathbb{N}$ then we also have $x^{p\odot} = \iota_p(x^+) + (-1)^p \iota_p(x^-)$ for all $x \in E$.

Proposition 3.8. *Given $\varphi \in \mathscr{H}(\mathbb{R}^n)$ and $0 < s \in \mathbb{R}$, define $\varphi_s \in \mathscr{H}(\mathbb{R}^n)$ by putting $\varphi_s(t_1, \ldots, t_n) := \varphi(t_1^s, \ldots, t_n^s)^{\frac{1}{s}}$ for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$. Then for every uniformly complete vector lattice E and any finite collection $x_1, \ldots, x_n \in E$ the representation holds:*

$$\varphi(\iota_s(x_1), \ldots, \iota_s(x_n)) = \iota_s(\varphi_s(x_1, \ldots, x_n)).$$

$\lhd$ Denote $y = \iota_s^{-1}(\varphi(\iota_s(x_1), \ldots, \iota_s(x_n)))$ and prove that $y = \varphi_s(x_1, \ldots, x_n)$. Denote by L the uniformly closed vector sublattice of E generated by $\{x_1, \ldots, x_n, y\}$ and $\mathrm{Hom}(L)$ the set of all $\mathbb{R}$-valued lattice homomorphisms on L. Then $e :=$

$|x_1|+\ldots+|x_N|+|y|$ is a strong order unit in L and $\mathrm{Hom}(L)$ separates the points of L. Observe that $L_{(s)} = \iota_s(L)$, since L is uniformly complete, and the set of $\mathbb{R}_{(s)}$-valued lattice homomorphisms $\mathrm{Hom}(L_{(s)})$ separates the points of $L_{(s)}$ by Proposition 3.4. By Buskes, de Pagter, and van Rooij [10, Corollary 3.4] $y = \varphi_s(x_1, \ldots, x_n)$ if and only if $\omega(y) = \varphi_s(\omega(x_1), \ldots, \omega(x_n))$ for all $\omega \in \mathrm{Hom}(L)$ and $u = \varphi(u_1, \ldots, u_n)$ with $u_k = \iota_s(x_k)$ if and only if $\omega(u) = \varphi(\omega(u_1), \ldots, \omega(u_n))$ for all $\omega \in \mathrm{Hom}(L_{(s)})$. Making use of Proposition 3.4 we deduce

$$\begin{aligned}\omega(y) = \omega_s(\varphi(\iota_s(x_1), \ldots, \iota_s(x_n)))^{1/s} &= \varphi(\omega_s(\iota_s(x_1)), \ldots, \omega_s(\iota_s(x_n)))^{1/s} \\ &= \varphi(\omega(x_1)^s, \ldots, \omega(x_n)^s)^{1/s} = \omega(\varphi_s(x_1, \ldots, x_n)),\end{aligned}$$

which completes the proof. $\rhd$

Corollary 3.9. *Let E be a uniformly complete vector lattice, $s \in \mathbb{N}$, $1 \leqslant r \in \mathbb{R}$, and $0 \leqslant \alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\sum_{k=1}^n \alpha_k = 1$. Then for any finite collection $x_1, \ldots, x_s \in E$ the representations hold:*

$$\left(\sum_{k=1}^n |x_k^{s\odot}|^r\right)^{\frac{1}{r}} = \left[\left(\sum_{k=1}^n |x_k|^{rs}\right)^{\frac{1}{rs}}\right]^{s\odot},$$

$$\prod_{k=1}^n |x_k^{s\odot}|^{\alpha_k} = \left(\prod_{k=1}^n |x_k|^{\alpha_k}\right)^{s\odot}.$$

Proposition 3.10. *Let E and F be uniformly complete vector lattices and $h : E \to F$ a lattice homomorphism. Then $h_p := \iota_p \circ h \circ \iota_p^{-1}$ is a lattice homomorphism from $E_{(p)}$ to $F_{(p)}$. Moreover, $\|h_p\| = \|h\|^p$ if E and F are quasi-Banach lattices.*

$\lhd$ Using Propositions 3.4 and 3.7, for $u = \iota_p(x)$ and $v = \iota_p(y)$ with $x, y \in E$, we have:

$$\begin{aligned}h_p(u \oplus v) = \iota_p h\big((x^p + y^p)^{1/p}\big) &= \iota_p\big((h(x)^p + h(y)^p)^{1/p}\big) = h_p(u) \oplus h_p(v); \\ h_p(\lambda \circledast u) = h_p(\iota_p(\lambda^{1/p} x) &= \iota_p h(\lambda^{1/p} x) = \iota_p(\lambda^{1/p} h(x)) = \lambda \circledast h_p(u).\end{aligned}$$

Thus, h_p is linear and h_p also preserves lattice operations according to the definition of order relation on $E_{(p)}$. The equation $\|h_p\| = \|h\|^p$ is straightforward. $\rhd$

Definition 3.11. A quasi-Banach lattice E is said to be (p, q)-*convex* with $0 < p \leqslant q \leqslant \infty$ and $p < \infty$, respectively (p, q)-*concave* if there exists a constant C such that

$$\left\|\left(\sum_{k=1}^n |x_k|^q\right)^{1/q}\right\| \leqslant C\left(\sum_{k=1}^m \|x_k\|^p\right)^{1/p},$$

respectively,

$$\left(\sum_{k=1}^m \|x_k\|^q\right)^{1/q} \leqslant C\left\|\left(\sum_{k=1}^n |x_k|^p\right)^{1/p}\right\|$$

for every finite collection $\{x_1, \ldots, x_m\}$ in E, see [13]. The smallest possible constant C is called the (p, q)-*convexity constant* (respectively (p, q)-*concavity constant*) and is denote by $M^{(p,q)}(C)$ (respectively, by $M_{(p,q)}(C)$). For $p = \infty$ put

$$\left(\sum_{k=1}^n |x_k|^p\right)^{1/p} = \bigvee_{k=1}^m |x_k|.$$

In the case when $p = q$ we speak of p-convexity (respectively, p-concavity) and write $M^{(p)} := M^{(p,p)}$ (respectively, $M_{(p)} := M_{(p,p)}$).

Corollary 3.12. *The concavification $E_{(s)}$ of a quasi-Banach lattice E is (p,q)-convex (resp. (p,q)-concave) if and only if E is (ps,qs)-convex (resp. (ps,qs)-concave). Moreover, $M^{(p,q)}(E_{(s)}) = M^{(ps,qs)}(E)$ and $M_{(p,q)}(E_{(s)}) = M_{(ps,qs)}(E)$.*

◁ If E is (ps,qs)-convex then, for arbitrary $x_1,\dots,x_n \in E$, we estimate by using Proposition 3.9

$$\left\|\left(\sum_{k=1}^{n}|x_k^{s\odot}|^q\right)^{1/q}\right\|_{(s)} = \left\|\left(\sum_{k=1}^{n}|x_k|^{qs}\right)^{1/(qs)}\right\|^s$$
$$\leqslant M^{(ps,qs)}(E)\left(\sum_{k=1}^{n}\|x_k\|^{ps}\right)^{1/p} = M^{(ps,ps)}(E)\left(\sum_{k=1}^{n}\|x_k^{s\odot}\|_{(s)}^{p}\right)^{1/p},$$

so that $E_{(s)}$ is (p,q)-convex and $M^{(p,q)}(E_{(s)}) \leqslant M^{(ps,qs)}(E)$. Conversely, if $E_{(s)}$ is (p,q)-convex then again making use of Proposition 3.9 we get

$$\left\|\left(\sum_{k=1}^{n}|x_k|^{qs}\right)^{1/(qs)}\right\| = \left\|\left(\sum_{k=1}^{n}|x_k^{s\odot}|^q\right)^{1/q}\right\|_{(s)}^{1/s}$$
$$\leqslant M^{(p,q)}(E_{(s)})\left(\sum_{k=1}^{n}\|x_k^{s\odot}\|_{(s)}^{p}\right)^{1/(ps)} \leqslant M^{(p,q)}(E_{(s)})\left(\sum_{k=1}^{n}\|x_k\|^{ps}\right)^{1/(ps)},$$

so that E is (ps,qs)-convex and $M^{(ps,qs)}(E) \leqslant M^{(p,q)}(E_{(s)})$. The argument for concavity is similar. ▷

If the convexity constant of a quasi-Banach lattice is finite, then one can always find an equivalent quasi-norm whose convexity constant is equal to one.

Proposition 3.13. *If a quasi-Banach lattice $(E,\|\cdot\|)$ is (p,q)-convex, $0 < q < p \leqslant \infty$, then $M^{(p,q)}(E,|||\cdot|||) = 1$ and $1/M^{(p,r)}(E)\|x\| \leqslant |||x||| \leqslant \|x\|$, where*

$$|||x||| := \inf\left\{\left(\sum_{k=1}^{n}\|x_k\|^p\right)^{1/p} :\ n\in\mathbb{N},\ x_1,\dots,x_n\in E;\ |x| = \left(\sum_{k=1}^{n}|x_k|^q\right)^{1/q}\right\}.$$

◁ See Szulga [55, p. 211]. ▷

Proposition 3.14. *Let E be a quasi-Banach lattice with the quasi-triangle constant C. If $x_1,\dots,x_n \in E$, $0 < \alpha_1,\dots,\alpha_n \in \mathbb{R}$, and $\alpha_1 + \dots + \alpha_n = 1$, then*

$$\|x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}\| \leqslant C^{n-1}\|x_1\|^{\alpha_1}\cdot\ldots\cdot\|x_n\|^{\alpha_n}.$$

◁ In the case $n = 2$ the proof is similar to that of Proposition 1.d.2 (i) of Lindenstrauss and Tzafriri [40]. The general case is handled by induction, see Kusraev [29, Proposition 5.2]. ▷

Definition 3.15. A quasi-Banach lattice is said to be 0^+-*convex* (Szulga [55]) or *geometrically convex* (Kalton and Montgomery-Smit [27]) if there is a constant

$M > 0$ such that

$$\left\| \left(\prod_{k=1}^{n} |x_k| \right)^{1/n} \right\| \leqslant M \left(\prod_{k=1}^{n} \|x_k\| \right)^{1/n}.$$

for every finite collection $\{x_1, \dots, x_n\}$ in E. The best constant is denoted by $M^{(0^+)}$.

Proposition 3.16. *A quasi-Banach lattice E is 0^+-convex if and only if there exists $C > 0$ such that*

$$\||x_1|^{\alpha_1} \cdot \ldots \cdot |x_n|^{\alpha_n}\| \leqslant C\|x_1\|^{\alpha_1} \cdot \ldots \cdot \|x_n\|^{\alpha_n}$$

for all finite collections $x_1, \dots, x_m \in E$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$ with $\alpha_1 + \dots + \alpha_n = 1$.

$\lhd$ In the if part one can take $C := (M^{(0^+)})^2$, see Szulga [55, Lemma 4.2]. $\rhd$

Proposition 3.17. *A quasi-Banach lattice E is 0^+-convex if and only if all its concavifications are 0^+-convex, i. e., $E_{(p)}$ is 0^+-convex for every $0 < p < \infty$.*

$\lhd$ This is immediate from Corollary 3.9. $\rhd$

4. (p, q)-Convex Homogeneous Polynomials

In this section we define (p, q)-convex homogeneous polynomials and study some of their properties. Some important technical tools are adopted from Cuartero and Triana [13], Kalton [21], and Szulga [55].

Definition 4.1. Let X be a quasi-Banach space, F a quasi-Banach lattice, and $0 < p \leqslant q \leqslant \infty$. A continuous s-homogeneous polynomial $P : E \to F$ is said to be (p, q)-*convex* if there exists a constant $C \in \mathbb{R}_+$ such that

$$\left\| \left(\sum_{k=1}^{m} |P(x_k)|^{q/s} \right)^{s/q} \right\| \leqslant C \left(\sum_{k=1}^{m} \|x_k\|^p \right)^{s/p} \tag{5}$$

for any finite collection $x_1, \dots, x_m \in E$. The best constant C in the inequality (5) is denoted by $M^{(p,q)}(P)$.

Definition 4.2. Let E be a quasi-Banach lattice, Y a quasi-Banach space, and $0 < p \leqslant q \leqslant \infty$. A continuous s-homogeneous polynomial $P : E \to F$ is said to be (p, q)-*concave* if there exists a constant $C \in \mathbb{R}_+$ such that

$$\left(\sum_{k=1}^{m} \|P(x_k)\|^{q/s} \right)^{1/q} \leqslant C \left\| \left(\sum_{k=1}^{m} |x_k|^p \right)^{1/p} \right\| \tag{6}$$

for any finite collection $x_1, \dots, x_m \in E$. The best constant C in the inequality (6) is denoted by $M_{(p,q)}(P)$. For $p = \infty$ we put in both definitions

$$\left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} = \bigvee_{k=1}^{m} |x_k|.$$

In the case when $p = q$ we speak of p-*convexity* and p-*concavity* and write $M^{(p)} := M^{(p,p)}$ and $M_{(p)} := M_{(p,p)}$. Putting $s = 1$ and $P = I_E$ we arrive at Definition 3.11.

Proposition 4.3. *Assume that an s-homogeneous polynomial P from a quasi-Banach space X to a quasi-Banach lattice F is (p,q)-convex with $0 < p \leqslant q \leqslant \infty$. Then P is also (p_1,q_1)-convex with $M^{(p_1,q_1)}(P) \leqslant M^{(p,q)}(P)$ whenever $q \leqslant q_1 \leqslant \infty$ and $0 < p_1 \leqslant p$ or $0 < q_1 < q \leqslant \infty$, $0 < p < p_1$ and*

$$\frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q} \quad \textit{if } q < \infty \quad \textit{and} \quad \frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} \quad \textit{if } q = \infty.$$

$\lhd$ The proof uses essentially the same line of an argument as in the proofs of Cuartero and Triana [13, Proposition 1.3] and Szulga [55, Theorem 4.1] for (p,q)-convexity of homogeneous functions. The case $q \leqslant q_1 \leqslant \infty$ and $0 < p_1 \leqslant p$ is obvious. Consider the other case, i. e. $q > q_1$ and $p_1 < p$. Observe first that for any choice of $f_1, \dots, f_n$ in F, positive scalars $\lambda_1, \dots, \lambda_n$, $r > 0$, and $1 \leqslant \bar{p}, \bar{q} \leqslant \infty$ with $1/\bar{p} + 1/\bar{q} = 1$ a Holder's inequality holds:

$$\left(\sum_{k=1}^{n} \lambda_k |f_k|^r \right)^{1/r} \leqslant \left(\sum_{k=1}^{n} \lambda_k^{\bar{q}} \right)^{1/\bar{q}r} \left(\sum_{k=1}^{n} |f_k|^{\bar{p}r} \right)^{1/\bar{p}r}.$$

Let $0 < \alpha < 1$ and take a finite collection of nonzero $x_1, \dots, x_n \in X$. Assume that $q < \infty$. Putting $\bar{p} := q/q_1$, $\bar{q} := q/(q-q_1)$, $r := q_1/s$, $f_k := P(x_k)$ and $\lambda_k := \|x_k\|^{(q_1/s)-\alpha}$ $(k = 1, \dots, n)$ and making use of (p,q)-convexity of P we deduce

$$\begin{aligned}
\left\| \left(\sum_{k=1}^{n} |P(x_k)|^{q_1/s} \right)^{s/q_1} \right\| &= \left\| \left(\sum_{k=1}^{n} \lambda_k |\lambda_k^{-s/q_1} P(x_k)|^{q_1/s} \right)^{s/q_1} \right\| \\
\leqslant \left(\sum_{k=1}^{n} \lambda_k^{q/(q-q_1)} \right)^{s(q-q_1)/(qq_1)} & \left\| \left(\sum_{k=1}^{n} |P(\lambda_k^{-1/q_1} x_k)|^{q/s} \right)^{s/q} \right\| \\
\leqslant M^{(p,q)}(P) \left(\sum_{k=1}^{n} \lambda_k^{q/(q-q_1)} \right)^{s(q-q_1)/(qq_1)} & \left(\sum_{k=1}^{n} \lambda_k^{p/q_1} \|x_k\|^p \right)^{s/p} \\
\leqslant M^{(p,q)}(P) \left(\sum_{k=1}^{n} \|x_k\|^{q(q_1-\alpha)/(q-q_1)} \right)^{s(q-q_1)/(qq_1)} & \left(\sum_{k=1}^{n} \|x_k\|^{\alpha p/q_1} \right)^{s/p} =: A.
\end{aligned}$$

The choice $\alpha = q_1^2 q/(p(q-q_1) + qq_1)$ yields

$$\frac{q(q_1-\alpha)}{q-q_1} = \frac{\alpha p}{q_1} = p_1, \qquad \frac{s(q-q_1)}{qq_1} + \frac{s}{p} = \frac{s}{q_1} - \frac{s}{q} + \frac{s}{p} = \frac{s}{p_1},$$

so that $A = M^{(p,q)}(P)\big(\sum_{k=1}^{n} \|x_k\|^{p_1}\big)^{s/p_1}$.

Assume now that $q = \infty$. Observe first that

$$\left(\sum_{k=1}^{n} \lambda_k |f_k|^r \right)^{1/r} \leqslant \left(\sum_{k=1}^{n} \lambda_k \right)^{1/r} \left(\bigvee_{k=1}^{n} |f_k| \right).$$

Taking $r := q_1/s$, $\lambda_k = \|x_k\|^\alpha$, $f_k = P(x_k)$ and using (p,∞)-convexity of P we get

$$\begin{aligned}\left\|\left(\sum_{k=1}^{n}|P(x_k)|^{q_1/s}\right)^{s/q_1}\right\| &= \left\|\left(\sum_{k=1}^{n}\lambda_k|\lambda_k^{-s/q_1}P(x_k)|^{q_1/s}\right)^{s/q_1}\right\| \\ &\leqslant \left(\sum_{k=1}^{n}\lambda_k\right)^{s/q_1}\left\|\bigvee_{k=1}^{n}|P(\lambda_k^{-1/q_1}x_k)|\right\| \\ &\leqslant M^{(p,\infty)}(P)\left(\sum_{k=1}^{n}\|x_k\|^\alpha\right)^{s/q_1}\left(\sum_{k=1}^{n}\|x_k\|^{p(q_1-\alpha)/q_1}\right)^{s/p} =: A.\end{aligned}$$

Putting $\alpha := q_1p/(q_1+p)$ yields $\alpha = p_1 = p(q_1-\alpha)/q_1$ and again we get $A = \big(\sum_{k=1}^n \|x_k\|^{p_1}\big)^{s/p_1}$. $\rhd$

Definition 4.4. Let $D \subset E$ be a *conic segment*, that is $\lambda D \subset D$ for all $0 \leqslant \lambda \leqslant 1$. Denote by $H_q(D)$ the collection of all $(\sum_{k=1}^n |x_k|^q)^{1/q}$ with $x_k \in D$ and $\|x_k\| \leqslant 1$ for all $k = 1,\dots,n$. For $n \in \mathbb{N}$ and $0 < q \in \mathbb{R}$ define $0 < a_n^{(q)} := a_n^{(q)}(D) \in \mathbb{R}$ by

$$a_n^{(q)}(D) := \sup\left\{\left\|\left(\sum_{k=1}^{n}|x_k|^q\right)^{1/q}\right\| :\ x_k \in H_q(D),\ \|x_k\| \leqslant 1\ (k=1,\dots,n)\right\}.$$

Lemma 4.5. *If D is a conic segment in E then $a_{mn}^{(q)}(D) \leqslant a_m^{(q)}(D)a_n^{(q)}(D)$ for all $m,n \in \mathbb{N}$.*

$\lhd$ Take an arbitrary double sequence $\{x_{kl} : 1 \leqslant k \leqslant m;\ 1 \leqslant l \leqslant n\}$ in D with $\|x_{kl}\| \leqslant 1$ for all k,l. Put $x_k := (\sum_{l=1}^n |x_{kl}|^q)^{1/q}$ and note that $\|x_k/a_n^{(q)}\| \leqslant 1$ for all $k = 1,\dots,m$. Thus,

$$\frac{1}{a_n^{(q)}}\left\|\left(\sum_{k=1}^{m}\sum_{l=1}^{n}|x_{kl}|^q\right)^{1/q}\right\| = \left\|\left(\sum_{k=1}^{m}\left|\frac{x_k}{a_n^{(q)}}\right|^q\right)^{1/q}\right\| \leqslant a_m^{(q)},$$

whence $a_{mn}^{(q)}/a_n^{(q)} \leqslant a_m^{(q)}$. $\rhd$

Lemma 4.6. *If $0 < p,q \in \mathbb{R}$ and $0 < pq < 1$, then $\lim\limits_{n\to\infty} n^{-1/(pq)}a_n^{(q)}(D) = 0$ if and only if D is (r,q)-convex for some $r > pq$.*

$\lhd$ The proof is similar to that of Kalton [23, Proposition 2.2 (ii)]. If D is (r,q)-convex for some $r > pq$ then evidently $n^{-1/(pq)}a_n^{(q)} \leqslant M^{(p,q)}(D)n^{1/r-1/(pq)}$ and thus $\lim_{n\to\infty} n^{-1/(pq)}a_n^{(q)}(D) = 0$. To prove the converse, assume that $\lim_{n\to\infty} n^{-1/(pq)}a_n^{(q)}(D) = 0$ and ensure first that $\iota_q(D)$ is $(\bar r,1)$-convex for some $\bar r > p$. Observe that

$$\begin{aligned}a_n^{(q)}(D)^q = \sup\big\{\|\iota_q(x_1)\oplus\dots\oplus\iota_q(x_n)\|_q :\ &\iota_q(x_k) \in H_1(\iota_q(D)),\\ &\|x_k\| \leqslant 1\ (k=1,\dots,n)\big\} = a_n^{(1)}(\iota_q(D))\end{aligned}$$

and thus $\lim_{n\to\infty} n^{-1/p}a_n^{(1)}(\iota_q(D)) = \lim_{n\to\infty}\big[n^{-1/(pq)}a_n^{(q)}(D)\big]^q = 0$. Just as in [23, Proposition 2.2 (ii)] we can prove by using Lemma 4.5 that there exist $p < \bar r \in \mathbb{R}$ and $N \in \mathbb{N}$ such that

$$n^{-1/\bar r}a_n^{(1)}(\iota_q(D)) < 1/2 \text{ for all } n \geqslant N. \tag{7}$$

Moreover, by Aoki–Rolewicz theorem (applied to $E_{(q)}$) there exist constants $t > 0$ and $A \geqslant 1$ such that

$$\|\iota_q(x_1) \oplus \ldots \oplus \iota_q(x_n)\|_{(q)}^t \leqslant A(\|\iota_q(x_1)\|_{(q)}^t + \cdots + \|\iota_q(x_n)\|_{(q)}^t) \quad (x_1, \ldots, x_n \in E). \tag{8}$$

Next, by arguing by induction on l and using (7) and (8) as in [23, Proposition 2.2 (ii)], one can show that

$$\big\|\alpha_1 \circledast |u_1| \oplus \ldots \oplus \alpha_l \circledast |u_l|\big\|_{(q)} \leqslant N^{1/t} A\big(1 - (1/2)^t\big)^{-1/t} \tag{9}$$

for all $u_1, \ldots, u_l \in E_{(q)}$ and $\alpha_1, \ldots, \alpha_l \in \mathbb{R}_+$ with $\|u_k\|_{(q)} \leqslant 1$ $(1 \leqslant k \leqslant l)$ and $\sum_{k=1}^n \alpha_k^{\bar r} = 1$. Taking arbitrary $u_1, \ldots, u_l \in E_{(q)}$ and substituting $\|u_k\|_{(q)}/\beta$ with $\beta := (\sum_{k=1}^l \|u_k\|_{(q)}^{\bar r})^{1/\bar r}$ for α_k and $\|u_k\|_{(q)}^{-1} \circledast \|u_k\|$ for $\|u_k\|$ in (9) yields the $(\bar r, 1)$-convexity of $E_{(q)}$. Coming back to $E = (E_{(q)})_{(1/q)}$, putting $r := \bar r q > pq$, and using Lemma 3.12 we see that E is (r, q)-convex. $\rhd$

Theorem 4.7. *Assume that an s-homogeneous polynomial P from a quasi-Banach space X to a quasi-Banach lattice F is (p_0, q_0)-convex and (p_1, q_1)-convex with $q_0 < q_1$. If*

$$\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1} \quad \text{for some} \quad 0 < \theta < 1,$$

then P is (p, q)-convex for every p satisfying

$$\frac{1}{p} > \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

$\lhd$ The Hölder inequality [40, 1.d.2 (ii)] is true in a quasi-Banach lattice, since it is uniformly complete. Moreover, by Proposition 3.15 we have $\||x|^\theta |y|^{1-\theta}\| \leqslant C\|x\|^\theta \|y\|^{1-\theta}$ where C is a quasi-norm multiplier. Take $x_1, \ldots, x_n \in E$ with $\|x_k\| \leqslant 1$ $(k = 1, \ldots, n)$. Using this two Hölder type inequalities and taking into account (p_0, q_0)- and (p_1, q_1)-convexity of P and the the monotonicity property of the quasi-norm we deduce:

$$\begin{aligned}
\left\|\left(\sum_{k=1}^n |P(x_k)|^{q/s}\right)^{s/q}\right\| &\leqslant \left\|\left(\sum_{k=1}^n |P(x_k)|^{q_0/s}\right)^{s\theta/q_0} \left(\sum_{k=1}^n |P(x_k)|^{q_1/s}\right)^{s(1-\theta)/q_1}\right\| \\
&\leqslant C\left\|\left(\sum_{k=1}^n |P(x_k)|^{q_0/s}\right)^{s/q_0}\right\|^\theta \left\|\left(\sum_{k=1}^n |P(x_k)|^{q_1/s}\right)^{s/q_1}\right\|^{1-\theta} \\
&\leqslant B\left(\sum_{k=1}^n \|x_k\|^{p_0}\right)^{s\theta/p_0} \left(\sum_{k=1}^n \|x_k\|^{p_1}\right)^{s(1-\theta)/p_1} \\
&\leqslant B n^{s\theta/p_0 + s(1-\theta)/p_1},
\end{aligned}$$

where $B = CM^{(p_0,q_0)}(P)M^{(p_1,q_1)}(P)$. Putting $\bar p := p/q$ and $\bar q := q/s$ we see that $\lim n^{-1/(\bar p\bar q)} a_n^{(\bar q)} = \lim n^{-s/p} a_n^{(q/s)} = 0$. By Lemma 4.6 $D = P(X)$ is $(r, q/s)$-convex for some $r > \bar p\bar q = p/s$ with some convexity constant M. Therefore,

$$\left\|\left(\sum_{k=1}^n |P(x_k)|^{q/s}\right)^{s/q}\right\| \leqslant M\left(\sum_{k=1}^n \|P(x_k)\|^r\right)^{1/r} \leqslant M\|P\|\left(\sum_{k=1}^n \|x_k\|^{rs}\right)^{s/(rs)}.$$

Thus P is (rs, q)-convex and also (p, q)-convex, since $p < rs$. $\rhd$

5. L-Convexity and Type

It can be easily seen that a quasi-Banach lattice is p-convex for some $0 < p \in \mathbb{R}$ if and only if its p-concavification is a Banach lattice. At the same time there exist quasi-Banach lattices which are not p-convex for any $0 < p \in \mathbb{R}$; the corresponding examples can be found in Cuartero and Triana [13] and Kalton [23]. In [23] Kalton also discovered an intrinsic characterization of the class of concavifications of Banach lattices in terms of L-convexity.

Definition 5.1. A quasi-Banach lattice E is said to be *L-convex* if there exists $C > 0$ such that if $u, x_1, \dots, x_n \in E$ with $\max_{k\leqslant n}|x_k| \leqslant |u|$ but $1/n\sum_{k=1}^n |x_k| \geqslant |u|$ then the inequality holds $|u| \leqslant C\max_{k\leqslant n}|x_k|$.

Theorem 5.2. *Let E be a quasi-Banach lattice. Then the following are equivalent:*

(1) *E is L-convex.*

(2) *E is 0^+-convex.*

(3) *There exists $C > 0$ such that for any finite collection $x_1, \dots, x_n \in E$ we have*

$$\left\| \left(\prod_{k=1}^n |x_k| \right)^{1/n} \right\| \leqslant C \max_{k\leqslant n} \|x_k\|.$$

(4) *There exists $0 < p \in \mathbb{R}$ such that E is p-convex.*

◁ The equivalences (1) $\Longleftrightarrow$ (4) and (1) $\Longleftrightarrow$ (3) are due to Kalton, see [23, Theorem 2.2] and [23, Theorem 4.4], respectively. The equivalence (1) $\Longleftrightarrow$ (4) was proved by Szulga [55, Theorem 4.5]. Both authors used technique of random variables. ▷

Now, we are going to establish a polynomial version of Theorem 5.2. For this purpose we need some inequalities obtained in Szulga [55].

Definition 5.3. For $0 < \alpha < \infty$, let X_α denote a positive *α-Pareto random variable*, i. e., with the density $f(x) = \alpha/x^{1+\alpha}$ if $x \geqslant 1$ and $f(x) = 0$ if $x < 1$. We can choose $X_\alpha = U^{-1/\alpha}$ where U is a random variable *uniformly distributed* on $[0, 1]$, i. e., the characteristic function of $[0, 1]$ is the density of U.

Lemma 5.4. *Let $X_j = X_{\alpha,j}$ be independent copies of an α-Pareto random variable X_α, $0 < a < \infty$, and $(E, \|\cdot\|)$ be a quasi-Banach lattice. The following assertions hold:*

(1) *If $x_1, \dots, x_n \in E_+$ then*

$$\exp\left(\int \ln \left\| \sum_{j=1}^n x_j X_{\alpha,j} \right\| d\mu \right) \geqslant \left(\sum_{j=1}^n \|x_j\|^\alpha \right)^{1/\alpha}. \tag{10}$$

(2) *For every $\alpha < r \in \mathbb{R}$ and $t_1, \dots, t_n \in \mathbb{R}_+$ there exists $B = B(\alpha, r)$ such that*

$$\exp\left(\int \ln \left(\sum_{j=1}^n t_j^r X_{\alpha,j}^r \right)^{1/r} d\mu \right) \leqslant B \left(\sum_{j=1}^n \|t_j\|^\alpha \right)^{1/\alpha}. \tag{11}$$

◁ See Szulga [55, Theorem 3.2]. ▷

Corollary 5.5. *Under the hypotheses of Lemma 5.3 we have*

$$\exp\left(\int \ln\left\|\left(\sum_{j=1}^{n}|x_jX_{\alpha,j}|^r\right)^{1/r}\right\|d\mu\right) \geqslant \left(\sum_{j=1}^{n}\|x_j\|^\alpha\right)^{1/\alpha}. \tag{12}$$

◁ This is immediate from 5.3 (1) by concavification in the left-hand side. ▷

Lemma 5.6. *If $(E, \|\cdot\|)$ is an L-convex quasi-Banach lattice, then the inequality holds:*

$$\left\|\exp\left(\int \ln\left(\sum_{j=1}^{n}|x_jX_j|^r\right)^{1/r}d\mu\right)\right\| \leqslant C^{(0^+)}\exp\left(\int \ln\left(\sum_{j=1}^{n}\|x_jX_j\|^r\right)^{1/r}d\mu\right).$$

◁ See Szulga [55, Proposition 4.4]. ▷

Theorem 5.7. *For quasi-Banach lattices E and F the following are equivalent:*

(1) *F is L-convex.*

(2) *If an s-homogeneous polynomial $P : E \to F$ is (p, q)-convex for some $p, q \in \mathbb{R}$ with $0 < p \leqslant q \leqslant \infty$, then P is r-convex for every $0 < r < p$.*

◁ (2) $\Longrightarrow$ (1) By Aoki–Rolewicz Theorem F is $(p, 1)$-convex for some $0 < p \leqslant 1$ and putting in 5.6 (2) $F = E$ and $P = I_F$ we get that F is r-convex for some $0 < r < p$. Thus F is L-convex by Theorem 5.2.

(1) $\Longrightarrow$ (2) Now assume that 5.7 (1) is fulfilled and consider a (p, q)-convex s-homogeneous polynomial $P : E \to F$ with $0 < p \leqslant q \leqslant \infty$. Take arbitrary $0 < r < p$, $n \in \mathbb{N}$, and $x_1, \dots, x_n \in E$. Consecutive application of Corollary 5.5, Lemma 5.6, and Lemma 5.4 (2) yields

$$\begin{aligned}\left\|\left(\sum_{j=1}^{n}|P(x_j)|^{r/s}\right)^{s/r}\right\| &\leqslant \left\|\exp\left(\int \ln\left(\sum_{j=1}^{n}|P(x_j)X_{r/s,j}|^{q/s}\right)^{s/q}d\mu\right)\right\| \\ &\leqslant C^{(0^+)}\exp\left(\int \ln\left\|\left(\sum_{j=1}^{n}|P(x_jX_{r,j})|^{q/s}\right)^{s/q}\right\|d\mu\right) \\ &\leqslant M^{(p,q)}C^{(0^+)}\exp\left(\int \ln\left(\sum_{j=1}^{n}\|x_jX_{r,j}\|^p\right)^{s/p}d\mu\right) \\ &\leqslant B(p,r)M^{(p,q)}C^{(0^+)}\left(\sum_{j=1}^{n}\|x_j\|^r\right)^{s/r},\end{aligned}$$

so that P is r-convex. ▷

Definition 5.8. Let r_n be the nth *Rademacher function*, i. e., $r_n : [0, 1] \to \mathbb{R}$ is defined as $r_n(t) = \operatorname{sign}(\sin 2^n\pi t)$ $(t \in [0, 1])$. An s-homogeneous polynomial $P : X \to Y$ is said to have (*Rademacher*) *type* p $(0 < p \leqslant 2)$ if there exists a constant $C > 0$ such that, for any choice of $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$, the inequality holds:

$$\int_0^1 \left\|\sum_{k=1}^{n}r_k(t)P(x_k)\right\|dt \leqslant C\left(\sum_{k=1}^{n}\|x_k\|^{sp}\right)^{1/p}.$$

The least C with the above property is denoted by $\tau_p(P)$.

The L_1-average in Definition 5.8 can be replaced by any L_r-average with $0 < r < \infty$ without altering the definition. Sometimes, it is convenient to use the same exponents on both sides:

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)P(x_k)\right\|^p dt\right)^{1/p} \leqslant C\left(\sum_{k=1}^n \|x_k\|^{sp}\right)^{1/p}.$$

This follows immediately from the Kahane inequality, the vector-valued version of the classical Khintchine inequality.

Theorem 5.9 (Kahane Inequality). *If X is a quasi-Banach space and $0 < p < q < \infty$, then there exists a constant $K = K(p,q) \geqslant 1$ such that for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$ the inequalities hold:*

$$\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\|^p dt\right)^{1/p} \leqslant \left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\|^q dt\right)^{1/q}$$
$$\leqslant K\left(\int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\|^p dt\right)^{1/p}.$$

$\lhd$ This was proved by Kahane for Banach spaces [19]; the generalization for quasi-Banach spaces is due to Kalton [22]. $\rhd$

Theorem 5.10. *Let X and Y be quasi-Banach spaces and $0 < p < 1$. An s-homogeneous polynomial $P : X \to F$ is of type p if and only if it is (sp, s)-convex.*

$\lhd$ The if part is trivial: the inequality $\int_0^1 \|\sum_{k=1}^n r_k(t)P(x_k)\|\,dt \leqslant \|\sum_{k=1}^n |P(x_k)|\|$ implies that P is of type p whenever P is (sp, s)-convex. To prove the converse assume that P is of type $0 < p < 1$ and note that by Aoki–Rolewicz theorem we may assume that Y is r-normed for some $0 < r < p$. Let $P : X \to Y$ be an s-homogeneous polynomial of type p and, for any $n \in \mathbb{N}$, define $D_n \in \mathbb{R}_+$ by

$$D_n := \sup\left\{\left\|\sum_{k=1}^n P(x_k)\right\| : \quad x_1, \dots, x_n \in X;\ \sum_{k=1}^n \|x_k\|^{sp} \leqslant 1\right\}. \tag{13}$$

Then $D_n < \infty$, since for any finite collection $x_1, \dots, x_n \in X$ we have

$$\left\|\sum_{k=1}^n P(x_k)\right\| \leqslant C^{n-1}\|P\|\sum_{k=1}^n \|x_k\|^s \leqslant C^{n-1}\|P\|\left(\sum_{k=1}^n \|x_k\|^{sp}\right)^{1/p}.$$

It is sufficient to prove that the increasing sequence $(D_n)_{n\in\mathbb{N}}$ is bounded. Since P is of type p, the estimate holds:

$$\int_0^1 \left\|\sum_{k=1}^n r_k(t)P(x_k)\right\| dt \leqslant C\left(\sum_{k=1}^n \|x_k\|^{sp}\right)^{1/p}. \tag{14}$$

For any collection $x_1, \dots, x_n \in X$ there exist $\sigma_k = \pm 1$ $(k = 1, \dots, n)$ such that

$$\|\sigma_1 P(x_1) + \dots + \sigma_n P(x_n)\| \leqslant C\left(\sum_{k=1}^n \|x_k\|^{sp}\right)^{1/p}.$$

We can assume that if $S := \{k : \sigma_k = -1\}$ then $\sum_{k\in S} \|x_k\|^{sp} \leqslant (1/2)\sum_{k=1}^n \|x_k\|^{sp}$ and hence

$$\Big\|\sum_{k\in S} P(x_k)\Big\| \leqslant D_n\Big(\sum_{k\in S}\|x_k\|^{sp}\Big)^{1/p} \leqslant 2^{-1/p} D_n\Big(\sum_{k=1}^n \|x_k\|^{sp}\Big)^{1/p}. \qquad (15)$$

Now, using the representation $\sum_{k=1}^n P(x_k) = \sum_{k=1}^n \sigma_k P(x_k) + 2\sum_{k\in S} P(x_k)$ and taking into account inequalities (14) and (15) we deduce

$$\Big\|\sum_{k=1}^n P(x_k)\Big\|^r \leqslant \Big\|\sum_{k=1}^n \sigma_k P(x_k)\Big\|^r + 2^r\Big\|\sum_{k\in S} P(x_k)\Big\|^r$$
$$\leqslant (C^r + 2^{r(1-1/p)} D_n^r)\Big(\sum_{k=1}^n \|x_k\|^{sp}\Big)^{r/p},$$

so that $D_n^r \leqslant C^r + 2^{r(1-1/p)} D_n^r$. It follows that $D_n \leqslant C(1-2^{r(1-1/p)})^{-1/r}$ and the sequence $(D_n)_{n\in\mathbb{N}}$ is bounded. $\rhd$

Remark 5.11. Putting $X = Y$ and $P = I_X$ in Theorem 5.10 we arrive at the following assertion: *If a quasi-Banach space X is of type p for some $0 < p < 1$ then X is $(p, 1)$-convex.* This fact was obtained by Kalton in [22, Theorem 4.2].

6. Factorization of Disjointness Preserving Polynomials

In this section we give conditions under which a homogeneous disjointness preserving polynomial P between quasi-Banach lattices admits a factorization through an $L_p(\mu)$-space, either in the form $P = Q \circ T$ or in the form $P = T \circ Q$, where Q is a homogeneous disjointness preserving polynomial and T is a lattice homomorphism.

Definition 6.1. Let E and F be vector lattices. An order bounded homogeneous polynomial $P : E \to F$ is said to be *disjointness preserving* (resp., *lattice polymorphism*) if its corresponding symmetric s-linear operator $\check{P}$ from E^s to F is disjointness preserving in each variable (resp., lattice s-morphism).

Much of the structure of order bounded homogeneous disjointness preserving polynomials is analogous to that of order bounded disjointness preserving linear operators. In particular, a Meyer type theorem is valid for such polynomials: An order bounded disjointness preserving s-homogeneous polynomial $P : E \to F$ has the modulus $|P|$, the positive part P^+, and the negative part P^- which are s-polymorphisms. Moreover, $P^+(x) = (Px)^+$, $P^-(x) = (Px)^-$, and $|P|(x) = |P(x)|$ for all $x \in E_+$, see [38, Theorem 2.12].

Lemma 6.2. *An order bounded s-homogeneous polynomial $P : E \to F$ is disjointness preserving if and only if there exists an order bounded disjointness preserving linear operator $T : E_{(s)} \to F$ such that $Px = T(x^{s\odot})$ for all $x \in E$. In particular, any order bounded homogeneous disjointness preserving polynomial is orthogonally additive.*

$\lhd$ See Kusraeva [38, Theorem 3.9]. $\rhd$

Lemma 6.3. *An s-homogeneous polynomial $P : E \to F$ is a polymorphism if and only if there exist a vector lattice G and a lattice homomorphism $S : E \to G$ such that $G_{(s)}$ is a sublattice of F and $Px = (Sx)^{s\odot}$ for all $x \in E$.*

$\lhd$ See Kusraeva [38, Corollary 3.10]. $\rhd$

Lemma 6.4. *Let $P : E \to F$ be an s-homogeneous polynomorphism, $0 < q \leqslant p \leqslant \infty$, and $x_1, \dots, x_n \in E$. Then*

$$\Bigg(\sum_{k=1}^{n} |P(x_k)|^{q/s}\Bigg)^{s/q} = P\Bigg(\Bigg(\sum_{k=1}^{n} |x_k|^q\Bigg)^{1/q}\Bigg).$$

$\lhd$ Clearly, P in Lemma 6.2 is a polymorphism if and only if T is a lattice homomorphism. Thus, the claim follows from Proposition 3.4 and Corollary 3.9. $\rhd$

Proposition 6.5. *If $P : E \to F$ is a homogeneous polymorphism between quasi-Banach lattices, then $M^{(p,q)}(P) \leqslant M^{(p,q)}(E)\|P\|$ and $M_{(p,q)}(P) \leqslant M_{(p,q)}(E)\|P\|$. If E is (p,q)-convex (F is (p,q)-concave), then P is (p,q)-convex ((p,q)-concave).*

$\lhd$ This is immediate from Lemma 6.4 and Definitions 3.11, 4.1, and 4.2. $\rhd$

Two factorization results proved by Raynaud and Tradacete in [51, Theorems 1 and 3] enables one to reduce famous Krivin's factorization theorem [28] to factorization of lattice homomorphisms between quasi-Banach lattice, see [51, Lemma 17]. An easy generalization of the latter to disjointness preserving linear operators is given in the following result. The proof runs along the lines of the paper [51] and is provided for the convenience of the reader.

Theorem 6.6. *Let E be a p-convex quasi-Banach lattice and F a p-concave quasi-Banach lattice with $0 < p \in \mathbb{R}$. Then each disjointness preserving linear operator H from E to F factors through some $L_p(\mu)$ and the two factors are disjointness preserving linear operators any of which can be chosen to be a lattice homomorphism. If H is a lattice homomorphism then both factors may be chosen to be lattice homomorphisms.*

$\lhd$ Assume first that $H : E \to F$ is a lattice homomorphism. By Lemma 3.13 we may assume $E_{(p)}$ is a Banach lattice and by Lemma 3.10 H_p is a lattice homomorphism from $E_{(p)}$ to $F_{(p)}$. Observe that the function $\varphi : E_{(p)} \to \mathbb{R}$ defined by $\varphi(u) = \|H(u)\|_{(p)}/\|H\|$ is superlinear. Indeed, φ is obviously homogeneous, and taking into account Lemma 3.4 and p-concavity of F, for $u_k = \iota_p(x_k)$ with $x_k \in E_+$ and $k = 1, 2$ we have

$$\begin{aligned}\|H_p(u_1 \oplus u_2)\|_{(p)} = \|H_p(u_1) \oplus H_p(u_2)\|_{(p)} &= \|(H(x_1)^p + H(x_2)^p)^{1/p}\|^p \\ &\geqslant \|H(x_1)\|^p + \|H(x_2)\|^p = \|H_p(u_1)\|_{(p)} + \|H_p(u_2)\|_{(p)}\end{aligned}$$

so that $\varphi(u_1 \oplus u_2) \geqslant \varphi(u_1) + \varphi(u_2)$. It follows that the set $\{\varphi \geqslant \|H\|\}^+$ consisting of all $u \in E_{(p)}$ with $H_p(u) \geqslant 0$ and $\|\varphi\| \geqslant 1\}$ is convex and disjoint from the interior of the unit ball B in $E(p)$. Using the Hahn-Banach theorem, we can find a nonzero continuous linear functional $f \in E'_{(p)}$ such that $\sup f(B) \leqslant \inf f(\{\varphi \geqslant 1\}^+)$. This inequality implies that f is positive and $\varphi(u) \leqslant f(|u|) \leqslant \|u\|$ for all $u \in E_{(p)}$. In particular, $\ker(f) \subset \ker(H_p)$.

Define an AL-space L as the completion of the quotient space $E_{(p)}/\ker(f)$ endowed with a lattice norm induced by a seminorm $u \mapsto f(|u|)$ $(u \in E_{(p)})$. By Kakutani representation theorem L is isometrically lattice isomorphic to $L_1(\mu) := L_1(\Omega, \Sigma, \mu)$ for some localizable measure space (Ω, Σ, μ). Denote by $\bar{T}$ the lattice homomorphism from $E_{(p)}$ to $L_1(\mu)$ induced by the quotient mapping $E_{(p)} \to E_{(p)}/\ker(f)$ and observe that $\varphi(u) \leqslant \|\bar{T}(u)\| \leqslant \|u\|$ for all $u \in E_{(p)}$. This inequality implies $\ker(\bar{T}) \subset \ker(H_p)$ and thus there is a linear operator $S_0 : \bar{T}(E_{(p)}) \to F_{(p)}$ such that $H_p = S_0 \circ \bar{T}$. Clearly S_0 is a norm bounded lattice homomorphism. Since $\bar{T}(E_{(p)})$ is dense in $L_p(\mu)$, S_0 admits a norm bounded extension $\bar{S}$ to the whole $L_1(\mu)$ which is also a lattice homomorphism satisfying $H_p = \bar{S} \circ \bar{T}$.

Finally, we apply a q-convexification procedure with $q = 1/p$ to $E_{(p)}$ and $F_{(p)}$ and observe that $(E_{(p)})_{(q)} = E$, $(F_{(p)})_{(q)} = F$, and $(H_p)_q = H$. Moreover, $(L_1(\mu))_{(q)} = L_p(\mu)$ and $H = S \circ T$ with $S = \bar{S}_p$ and $T = \bar{T}_q$.

Consider now the case of an order bounded disjointness preserving operator H. By Meyer theorem $H = H_1 - H_2$ with H_1 and H_2 lattice homomorphisms and $H_1(E) \perp H_2(E)$, see [46, Theorem 3.1.4]. On the basis of the result just proved we can find localizable measure spaces $(\Omega_k, \Sigma_k, \mu_k)$ and lattice homomorphisms $S_k \in L(L_p(\mu_k), F)$ and $T_k \in L(E, L_p(\mu_k))$ $(k = 1, 2)$ such that $H_1 = S_1 \circ T_1$ and $H_2 = S_2 \circ T_2$. Denote by (Ω, Σ, μ) the direct sum of $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ and identify $L_p(\mu_1)$ and $L_p(\mu_2)$ with the complementary band in $L_p(\mu)$. Then $H = S \circ T$ whenever $S = S_1 - S_2$ and $T = T_1 + T_2$ or $S = S_1 + S_2$ and $T = T_1 - T_2$. Clearly, $S_1 + S_2$ and $T_1 + T_2$ are lattice homomorphisms and $S_1 - S_2$ and $T_1 - T_2$ are disjointness preserving. $\rhd$

Theorem 6.7. *Let E and F be quasi-Banach lattices and $0 < p \in \mathbb{R}$. For an arbitrary order bounded disjointness preserving s-homogeneous polynomial P from E to F the following hold:*

(1) *If E is ps-convex and F is p-concave then there exist a localizable measure space (Ω, Σ, μ), a disjointness preserving s-homogeneous polynomial $Q : E \to L_p(\mu)$ and a disjointness preserving linear operator $S : L_p(\mu) \to F$ such that $P = S \circ Q$. Moreover, any of S and Q may be chosen positive.*

(2) *If E is p-convex and F is ps-concave then there exist a localizable measure space (Ω, Σ, μ), an s-polymorphism $Q : L_p(\mu) \to F$, and a lattice homomorphism $T : E \to L_p(\mu)$, such that $P = Q \circ T$.*

$\lhd$ (1): Consider an order bounded disjointness preserving s-homogeneous polynomial P from a ps-convex quasi-Banach lattice E to a p-concave quasi-Banach lattice F. By Lemma 6.2 $P = H \circ \jmath_s$ for some order bounded disjointness preserving linear operator $H : E_{(s)} \to F$ and $E_{(s)}$ is a p-convex quasi-Banach lattice according to Corollary 3.12. In view of Theorem 6.6 there exist a localizable measure space (Ω, Σ, μ) and order bounded disjointness preserving linear operators $T : E_{(s)} \to L_p(\mu)$ and $S : L_p(\mu) \to F$ such that $H = S \circ T$ and thus $P = S \circ T \circ \jmath_s = T \circ Q$ with S and $Q = T \circ \jmath_s : E \to L_p(\mu)$ disjointness preserving. Any of S and Q may be chosen positive, since this is true for S and T by Theorem 6.6.

(2): Assume now that E is p-convex and F is ps-concave. By Lemma 6.2 there exists a vector lattice G such that $G_{(s)}$ is a sublattice of F and $P = \jmath_s \circ H$ for some disjointness preserving linear operator $H : E \to G$. Observe that the closure $\bar{G}$ of $G_{(s)}$ in F is a ps-concave quasi-Banach lattice, while $\bar{G}_{(1/s)}$ is a p-concave quasi-Banach lattice by Corollary 3.12. Moreover, G is embedded into $\bar{G}_{(1/s)}$ and we may assume that H acts from E to $\bar{G}_{(1/s)}$. In view of Theorem 6.6 there exist a localizable measure space (Ω, Σ, μ), a disjointness preserving linear operator $T : E \to L_p(\mu)$, and lattice homomorphism $S : L_p(\mu) \to \bar{G}_{(1/s)}$ such that $H = S \circ T$ and thus $P = \jmath_s \circ T = Q \circ T$ with $Q = \jmath_s \circ S : L_p(\mu) \to F$ being an s-polymorphism. It remains to note that the range space of Q is contained in $\jmath_s(\bar{G}_{(1/s)}) = \bar{G}$ which in turn is a sublattice in F. $\rhd$

7. (p, q)-Convex Orthogonally Additive Polynomials

This Section deals with the special case of homogeneous orthogonally additive polynomials. The properties of this class of polynomials resemble very much those of linear operators. Linearization results (Theorem 2.10 and Corollaries 2.11 and 2.12) enables one to transfer theorems about linear operators to results about orthogonally additive polynomials. We restrict our consideration to a few remarks concerning convexity, concavity, and factorization of orthogonally additive polynomials.

Proposition 7.1. *Let E and F be quasi-Banach lattices, $s \in \mathbb{N}$, and $p, q \in \mathbb{R}$ with $0 < p \leqslant q \leqslant \infty$ and $p < \infty$. Let $T : E_{(s)} \to F$ be a positive linear operator. The polynomial $T \circ \jmath_s : E \to F$ is (p, q)-convex if and only if T is $(p/s, q/s)$-convex. Moreover, $M^{(p,q)}(P) = M^{(p/s,q/s)}(T)$.*

$\lhd$ If T is $(p/s, q/s)$-convex then using the representation of Corollary 2.12, Definition of $(E_{(s)}, \|\cdot\|_{(s)})$, and by Corollary 3.9 we have

$$\left\|\left(\sum_{k=1}^{m} |P(x_k)|^{q/s}\right)^{s/q}\right\| = \left\|\left(\sum_{k=1}^{m} |T(x_k^{s\odot})|^{q/s}\right)^{s/q}\right\|$$

$$\leqslant M^{(p/s,q/s)}(T)\|\left(\sum_{k=1}^{m} \|x_k^{s\odot}\|_{(s)}^{p/s}\right)^{s/p} = M^{(p/s,q/s)}(T)\left(\sum_{k=1}^{m} \|x_k\|^{p}\right)^{s/p}.$$

Thus P is (p, q)-convex and $M^{(p,q)}(P) \leqslant M^{(p/s,q/s)}(T)$.

Now, assume that P is (p, q)-convex and take $u_1, \dots, u_n \in E_{(s)}$. If $x_k := \iota^{-1}(u_k)$ then $|u_k| = |x_k^{s\odot}| = |x_k|^{s\odot}$ and using the same argument we deduce

$$\left\|\left(\sum_{k=1}^{m} |T(u_k)|^{q/s}\right)^{s/q}\right\| \leqslant \left\|\left(\sum_{k=1}^{m} (T|x_k|^{s\odot})^{q/s}\right)^{s/q}\right\|$$

$$= \left\|\left(\sum_{k=1}^{m} P(|x_k|)^{q/s}\right)^{s/q}\right\|_{(s)} \leqslant M^{(p,q)}(P)\left(\sum_{k=1}^{m} \|x_k\|^{p}\right)^{s/p} = M^{(p,q)}(P)\left(\sum_{k=1}^{m} \|u_k\|_{(s)}^{p/s}\right)^{s/p}.$$

It follows that T is $(p/s, q/s)$-convex and $M^{(p,q)}(P) \geqslant M^{(p/s,q/s)}(T)$. $\rhd$

Proposition 7.2. *Let E be a quasi-Banach lattices and Y a quasi-Banach space, $s \in \mathbb{N}$, and $p, q \in \mathbb{R}$ with $0 < p \leqslant q \leqslant \infty$ and $p < \infty$. Let $T : E_{(s)} \to Y$ be a positive* *linear operator. The polynomial* $T \circ \jmath_s : E \to F$ *is* (p,q)*-concave if and only if* T *is* $(p/s, q/s)$*-concave. Moreover,* $M_{(p,q)}(P) = M_{(p/s,q/s)}(T)$.

$\lhd$ The proof is similar to that of Proposition 7.1. $\rhd$

Proposition 7.3. *Let* E *and* F *be quasi-Banach lattices and* T *a positive linear operator from* E *to* F*. Then for every* $1 \leqslant p \leqslant \infty$ *and every finite collection* $x_1, \dots, x_m \in E$ *we have*

$$\left\| \left(\sum_{k=1}^m |T(x_k)|^p \right)^{1/p} \right\| \leqslant \|T\| \left\| \left(\sum_{k=1}^m |x_k|^p \right)^{1/p} \right\|. \tag{16}$$

$\lhd$ The proof in [40, Proposition 1.d.9] works by using the monotonicity of the quasi-norm in F. $\rhd$

Proposition 7.4. *Let* E *and* F *be quasi-Banach lattices and* P *a positive* s*-homogeneous orthogonally additive polynomial from* E *to* F*. Then for every* $1 \leqslant p \in \mathbb{R}$ *and every finite collection* $x_1, \dots, x_m \in E$ *we have*

$$\left\| \left(\sum_{k=1}^m |P(x_k)|^p \right)^{1/p} \right\| \leqslant \|P\| \left\| \left(\sum_{k=1}^m |x_k|^{ps} \right)^{1/(ps)} \right\|^s. \tag{17}$$

$\lhd$ From Proposition 7.3 and Corollary 3.9 we deduce

$$\left\| \left(\sum_{k=1}^m |P(x_k)|^p \right)^{1/p} \right\| = \left\| \left(\sum_{k=1}^m |T(x_k^{s\odot})|^p \right)^{1/p} \right\|$$
$$\leqslant \|T\| \left\| \left(\sum_{k=1}^m |x_k^{s\odot}|^p \right)^{1/p} \right\|_{(s)} = \|P\| \left\| \left(\sum_{k=1}^m |x_k|^{ps} \right)^{1/(ps)} \right\|^s. \ \rhd$$

Proposition 7.5. *Let* E *be a quasi-Banach lattice,* $s \in \mathbb{N}$ *and* $0 < p, q \in \mathbb{R}$*. Then the following assertions are equivalent:*

(1) *E is (p,q)-convex.*

(2) *$E^{s\odot}$ is $(p/s, q/s)$-convex.*

(3) *The canonical polynomial $x \mapsto x^{s\odot}$ from E to $E^{s\odot}$ is (p,q)-convex.*

(4) *For every quasi-Banach lattice F, each positive orthogonally additive s-homogeneous polynomial P from E to F is (p,q)-convex.*

(5) *For every quasi-Banach lattice F, each positive linear operator T from E to F is (p,q)-convex.*

$\lhd$ The equivalence (1) $\iff$ (2) follows from Corollary 3.12 (see also [55, Proposition 4.8(iii)], while (2) $\iff$ (3) is immediate from the Definitions 3.11 and 4.1 and Corollary 3.9. The implications (4) $\Longrightarrow$ (3) and (5) $\Longrightarrow$ (1) are easily seen by putting $P = \jmath_s$ and $T = I_E$ and (5) is the particular case of (4) with $s = 1$. It remains to ensure (1) $\Longrightarrow$ (4).

Assume that E is (p,q)-convex and take a positive orthogonally additive s-homogeneous polynomial P from E to a quasi-Banach lattice F. By Corollary 2.12

the representation $P(x) = T(x^{s\odot})$ $(x \in E)$ holds with a positive linear operator T from $E^{s\odot}$ to F. Making use of Proposition 7.4, Definitions 4.1 and 3.11 we estimate

$$\left\| \left(\sum_{k=1}^{m} |P(x_k)|^{q/s} \right)^{s/q} \right\| \leqslant \|P\| \left\| \left(\left(\sum_{k=1}^{m} |x_k|^q \right)^{1/q} \right)^{s\odot} \right\|_{(s)}$$

$$= \|P\| \left\| \left(\sum_{k=1}^{m} |x_k|^q \right)^{1/q} \right\|^s \leqslant \|P\| M^{(p,q)}(E) \left(\sum_{k=1}^{m} \|x_k\|^p \right)^{s/p},$$

ensuring that P is (p,q)-convex. ▷

Theorem 7.6. *Let E be a quasi-Banach lattice, X a quasi-Banach spaces, $s \in \mathbb{N}$, and $0 < p \leqslant \infty$. A linear operator $T : E \to X$ is p-concave if and only if there exist a p-concave quasi-Banach lattice F, a bounded linear operator $S : F \to X$, and an order interval preserving lattice homomorphism $R : E \to F$ with dense image such that $T = S \circ R$.*

◁ The proof given in Raynaud and Tradacete [51, Theorem 1] for the case of Banach lattices and $p \geqslant 1$ works with minor modifications, see [51, Remark 6]. ▷

Theorem 7.7. *Let E be a quasi-Banach lattice, Y a quasi-Banach space, $s \in \mathbb{N}$, and $0 < p \leqslant \infty$. An s-homogeneous orthogonally additive polynomial $P : E \to Y$ is p-concave if and only if there exist a p/s-concave quasi-Banach lattice F, a bounded linear operator $S : F \to Y$, and an order preserving lattice polymorphism $Q : E \to F$ with dense image such that $P = S \circ Q$.*

◁ According to Corollary 2.11 the representation $P = T \circ \jmath_s$ holds with a linear operator $T : E_{(s)} \to Y$ which is p/s-concave by Proposition 7.1. In view of Theorem 7.6 here exist a p/s-concave quasi-Banach lattice F, a bounded linear operator $S : F \to X$, and an order preserving lattice homomorphism $R : E \to F$ with dense image such that $P = S \circ R \circ \jmath_s$. Putting $Q := R \circ \jmath_s$ and taking into account Lemma 6.2 we arrive at the required conclusion. ▷

Theorem 7.8. *Let F be an L-convex quasi-Banach lattice. Then there exists a constant A depending only on F such that whenever E is a quasi-Banach lattice and $P : E \to F$ is a bounded orthogonally additive s-homogeneous polynomial then for any finite collection $x_1, \dots, x_n \in E$ the inequality holds:*

$$\left\| \left(\sum_{k=1}^{n} |P(x_k)|^2 \right)^{1/2} \right\| \leqslant A\|P\| \left\| \left(\sum_{k=1}^{n} |x_k|^{2s} \right)^{1/(2s)} \right\|^s.$$

◁ Write P as $P = T \circ \jmath_s$ with a bounded linear operator from a quasi-Banach lattice $E_{(s)}$ to an L-space F. By Kalton's result (a generalization of the Krivine's version of Grothendieck's theorem) [23, Theorem 3.3] for any finite collection $x_1, \dots, x_n$ we have

$$\left\| \left(\sum_{k=1}^{n} |T(x_k^{s\odot})|^2 \right)^{1/2} \right\| \leqslant A\|P\| \left\| \left(\sum_{k=1}^{n} |x_k^{s\odot}|^2 \right)^{1/2} \right\|.$$

It remains to apply Proposition 3.9. ▷

Corollary 7.9. *Let E and F be quasi-Banach lattices and $P : E \to F$ an orthogonally additive bounded s-homogeneous polynomial. Then the following hold:*

(1) *If E is $2s$-convex then P is $2s$-convex.*

(2) *If F is 2-concave then P is $2s$-concave.*

8. Factorization of p-Convex Polynomials

In this section we show that a p-convex homogeneous polynomial can be factored through a p-convex quasi-Banach lattice. This fact together with Theorem 6.7 enables us to obtain Krivine's type factorization for homogeneous polynomials.

Definition 8.1. A subset U of a vector space is called *balanced* if $\lambda U \subset U$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant 1$. A subset U of a vector lattice is called *solid* whenever $|x| \leqslant |y|$ and $y \in U$ imply $x \in U$.

Lemma 8.2. *Let U be a solid balanced subset of a vector lattice E containing no lines. Let $\|\cdot\|_U$ be the Minkowski functional of U and $E_0 := \{x \in E : \|x\|_U < \infty\}$. Then $(E_0, \|\cdot\|_U)$ is quasi-Banach lattice if and only if there exists $C > 0$ such that the following hold:*

(1) *$U + U \subset C \cdot U$.*

(2) *For any pair of sequences (λ_k) in $\mathbb{R}_+$ and (x_k) in E with $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $C^k x_k \in \lambda_k U$ for all $k \in \mathbb{N}$ there exist $x \in E$ and a sequence (ν_k) in $\mathbb{R}$ such that $\lim \nu_k = 0$ and $x - \sum_{k=1}^{n} x_k \in \nu_n U$ for all $n \in \mathbb{N}$.*

$\lhd$ Evidently, 8.2 (1) is equivalent to 3.1 (3), while 8.2 (2) is a rephrased version of the criterion of completeness: A quasi-normed space E_0 is complete if and only if $\sum_{k=1}^{\infty} C^k \|x_k\|_U < \infty$ implies that the series $\sum_{k=1}^{\infty} x_k$ is converent in E_0. $\rhd$

Lemma 8.3. *$(E_0, \|\cdot\|_U)$ is p-convex if and only if, there exists $M > 0$ such that*

$$\left(\sum_{k=1}^{n} |\alpha_k x_k|^p \right)^{1/p} \in MU$$

for all finite sequences $x_1, \dots, x_n \in U$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$ with $\sum_{k=1}^{n} \alpha_k^p = 1$.

$\lhd$ The only if part is obvious with $M = M^{(p)}(F)$. To ensure the if part pick arbitrary $x_1, \dots, x_n \in F$ and $0 < \varepsilon \in \mathbb{R}$ and put $\sigma(\varepsilon) := (\sum_{k=1}^{n} (\|x_k\| + \varepsilon)^p)^{1/p}$, $\alpha_k := (\|x_k\| + \varepsilon)/\sigma(\varepsilon)$. Then $x_k/(\|x_k\| + \varepsilon) \in U$, $\sum_{k=1}^{n} \alpha_k^p = 1$ and

$$\left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} = \sigma(\varepsilon) \left(\sum_{k=1}^{n} \left| \alpha_k \frac{x_k}{\|x_k\| + \varepsilon} \right|^p \right)^{1/p} \in M\sigma(\varepsilon)U.$$

It follows that $\|(\sum_{k=1}^{n} |x_k|^p)^{1/p}\| \leqslant M\sigma(\varepsilon)$ and sending ε to zero yields p-convexity of F. $\rhd$

Theorem 8.4. *Let X be a quasi-Banach space, E a quasi-Banach lattice, $s \in \mathbb{N}$, and $0 < p \leqslant \infty$. A bounded s-homogeneous polynomial $P : X \to E$ is p-convex if and only if there exist a ps-convex Banach lattice G, an injective interval preserving lattice homomorphism $S : G \to E$ and a bounded s-homogeneous polynomial $Q : X \to G$ such that $P = S \circ Q$.*

$\lhd$ Assume that $P : X \to E$ is a p-convex s-homogeneous polynomial. Put $\bar{P} := \iota_{1/s} \circ P$ where $\iota_{1/s} : E \to E_{(1/s)}$ is a nonlinear order isomorphism in Lemma 3.4. Define the set $U \subset E_{(1/s)}$ by

$$U := \Bigg\{ u \in E_{(1/s)} : \ (\exists\, n \in \mathbb{N})\, (\exists\, x_1, \dots, x_n \in X) \qquad |u| \leqslant \left(\sum_{k=1}^{n} |\bar{P}(x_k)|^p \right)^{1/p} \text{ and } \sum_{k=1}^{n} \|x_k\|^p = 1 \Bigg\}.$$

Let $\overline{U}$ and $\|\cdot\|_{\overline{U}}$ stand for the closure of U in $E_{(1/s)}$ and the Minkowski functional of $\overline{U}$, respectively. If $|u_j| \leqslant (\sum_{k=1}^{n_j} |\bar{P}(x_{jk})|^p)^{1/p}$ and $\sum_{k=1}^{n} \|x_{jk}\|^p = 1$ for $j = 1, 2$, then we estimate

$$|u_1| \oplus |u_2| \leqslant 2 \left(\sum_{k=1}^{n} |\bar{P}(x_{1k})|^p \vee \sum_{k=1}^{m} |\bar{P}(x_{2k})|^p \right)^{1/p} \leqslant 2^{1+1/p} \left(\sum_{k=1}^{n} |\bar{P}(x_{1k}/2^{1/p})|^p + \sum_{k=1}^{m} |\bar{P}(x_{2k}/2^{1/p})|^p \right)^{1/p};$$

$$\sum_{k=1}^{n} \|x_{1k}/2^{1/p}\|^p + \sum_{k=1}^{m} \|x_{2k}/2^{1/p}\|^p = 1.$$

It follows that $U \oplus_{1/s} U \subset 2^{1+1/p} \overline{U}$ and $\overline{U} \oplus_{1/s} \overline{U} \subset \overline{U \oplus_{1/s} U} \subset 2^{1+1/p} \overline{U}$, so that 8.2 (1) is fulfilled with $C = 2^{1+1/p}$. Denote $F = \bigcup_{k=1}^{\infty} k\overline{U}$ and note that F is an order ideal in $E_{(1/s)}$ and $\|\cdot\|_{\overline{U}}$ is a monotone quasi-norm. Let $\bar{C}$ stands for the quasi-triangle constant of F.

Observe that if $u \in U$ then by Corollary 3.9 we have

$$\|u\|_E \leqslant \left\| \left(\sum_{k=1}^{n} |\bar{P}(x_k)|^p \right)^{1/p} \right\|_{E_{(1/s)}} = \left\| \left(\sum_{k=1}^{n} |P(x_k)|^{p/s} \right)^{s/p} \right\|_E^{1/s} \leqslant M^{(p)}(P) \left(\sum_{k=1}^{n} \|x_k\|^p \right)^{1/p} = M^{(p)}(P),$$

whence $U \subset M^{(p)}(P) V$ where $V = \{\|\cdot\|_E \leqslant 1\}$. It follows that $\overline{U} \subset M^{(p)}(P) V$ and hence $M^{(p)}(P)\|u\|_E \leqslant \|u\|_{\overline{U}}$ for all $u \in E_0$. In particular, $\overline{U}$ is closed in the topology of $\|\cdot\|_{\overline{U}}$ and $\|u\|_{\overline{U}} = 0$ implies $u = 0$.

In order to ensure that F is complete consider the sequences (λ_k) in $\mathbb{R}_+$ and (x_k) in F with $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $\bar{C}^k x_k \in \lambda_k \overline{U}$ for all $k \in \mathbb{N}$. Then $\bar{C}^k y_k \in \lambda_k V$ where $y_k = x_k / M^{(p)}(P)$ and, since $E_{(1/s)}$ is complete, there exist $y \in E$ and a sequence (ν_k) in $\mathbb{R}$ such that $\lim \nu_k = 0$ and $y - \sum_{k=1}^{n} y_k \in \nu_n U$ for all $n \in \mathbb{N}$. Clearly, $x = y M^{(p)}(P) = \sum_{k=1}^{\infty} x_k$ in $E_{(1/s)}$. Denote $\sigma_n := \sum_{k=1}^{n} x_k$ and estimate

$$\|\sigma_{n+m} - \sigma_n\|_{\overline{U}} \leqslant \sum_{k=n+1}^{n+m} \bar{C}^{k-n} \|x_k\|_{\overline{U}} \leqslant \sum_{k=n+1}^{n+m} \bar{C}^k \|x_k\|_{\overline{U}} \leqslant \sum_{k=n+1}^{n+m} \lambda_k \to 0.$$

It follows that for an arbitrary $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $\|\sigma_{n+m} - \sigma_n\|_{\overline{U}} < \varepsilon$ and consequently $\sigma_{n+m} - \sigma_n \in \varepsilon \overline{U}$ for all $n \geqslant N$ and $m \in \mathbb{N}$. Since $\overline{U}$ is closed in E,

$x - \sigma_n = \lim_{m\to\infty}(\sigma_{n+m} - \sigma_n) \in \varepsilon\overline{U}$ for all $n \geqslant \mathbb{N}$, which implies that $x = \sum_{k=1}^{\infty} x_k$ in F.

Prove that F is p-convex. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$ and $\sum_{k=1}^{n} \alpha_k^p = 1$. Given $u_1, \dots, u_n \in U$, there are finite sequences $x_{1l}, \dots, x_{nl} \in X$ such that $|u_k| \leqslant (\sum_{j=1}^{n_k} |\bar{P}(x_{kj})|^p)^{1/p}$ and $\sum_{j=1}^{n_k} \|x_{kj}\|^p = 1$ for all $k = 1, \dots, n$. Using Lemma 3.8 and p-convexity of P we deduce:

$$\left\|\left(\sum_{k=1}^{n} |\alpha_k u_k|^p\right)^{1/p}\right\| \leqslant \left\|\left(\sum_{k=1}^{n} \alpha_k^p \sum_{j=1}^{n_k} |\bar{P}(x_{kj})|^p\right)^{1/p}\right\| \leqslant \left\|\left(\sum_{k=1}^{n}\sum_{j=1}^{n_k} |P(\alpha_k x_k)|^{p/s}\right)^{s/p}\right\|$$
$$\leqslant M^{(p)}(P)\left(\sum_{k=1}^{n} \alpha_k^p \sum_{j=1}^{n_k} \|x_k\|^p\right)^{1/p} = M^{(p)}(P).$$

It follows that $(\sum_{k=1}^{n} \alpha_k |u_k|^p)^{1/p} \in M^{(p)}\overline{U}$. Take now $v_1, \dots, v_n \in \overline{U}$ and choose the sequences $(u_{kj})_{j\in\mathbb{N}}$ in U such that $v_k = \lim_{j\to\infty} u_{kj}$ for all $k = 1, \dots, n$. Put $\mathbf{v} = (\alpha_1 v_1, \dots, \alpha_n v_n)$ and $\mathbf{u}_j = (\alpha_1 u_{1j}, \dots, \alpha_n u_{nj})$ $(j \in \mathbb{N})$. Define $\varphi(t) = (\sum_{k=1}^{n} |t|^p)^{1/p}$ for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. By what we have proved $\varphi(\mathbf{u}_j) \in M^{(p)}\overline{U}$ and $\varphi(\mathbf{v}) = \lim \varphi(\mathbf{u}_j) \in M^{(p)}\overline{U}$ by Lemma 3.6. Thus, according to Lemma 8.3 E_0 is p-convex. $G := F_{(s)}$ is ps-convex and $P(X) \subset G \subset E$. If $S : G \to E$ is a formal inclusion and $Q := S^{-1} \circ P$, then $P = S \circ P$. $\triangleright$

The particular case $s = 1$ of this last result extends Krivin's theorem [28], see also [40, Theorem 1.d.11].

Theorem 8.5. *Let X and Y be quasi-Banach spaces, E a quasi-Banach lattice, $s \in \mathbb{N}$, and $0 < p \leqslant \infty$. If $P : X \to E$ is a p/s-convex s-homogeneous polynomial and $S : E \to Y$ is p-concave linear operator then there exist a localizable measure space (Ω, Σ, μ), an s-homogeneous polynomial $Q : X \to L_p(\mu)$ and a bounded linear operator $\bar{P} : L_p(\mu) \to Y$, such that $S \circ P = T \circ Q$.*

$\triangleleft$ Put together Theorem 6.7, Theorem 7.7, and Theorem 8.4. $\triangleright$

9. p-Convex Lattices of Linear Operators and Polynomials

In this section we study the following natural questions: When is the quasi-Banach lattice of regular linear operators or regular polynomials between quasi Banach lattices (p, q)-convex? (p, q)-concave? geometrically convex?

Definition 9.1. A *gauge* is a sublinear function $\varphi : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$. For $s, t \in \mathbb{R}$, $s = (s_1, \dots, s_n)$, and $t = (t_1, \dots, t_n)$, denote $\langle s, t\rangle := \sum_{k=1}^{n} s_k t_k$. The *polar* function φ° of a gauge φ is defined by (with the conventions $\inf \varnothing = +\infty$ and $0(+\infty) = 0$)

$$\varphi^\circ(t) := \inf\{\lambda \geqslant 0 : \ (\forall\, s \in \mathbb{R}^n)\, \langle s, t\rangle \leqslant \lambda\varphi(s)\} \quad (t \in \mathbb{R}^n).$$

Denote by $\mathscr{G}_\vee(\mathbb{R}^n)$ the set of all continuous gauges $\varphi : \mathbb{R}^n \to \mathbb{R}_+$.

Thus, $\varphi^\circ : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is also a gauge and the inequality holds

$$\langle s, t\rangle \leqslant \varphi(s)\varphi^\circ(t) \quad (s, t \in \mathbb{R}^n).$$

Moreover, the polar function φ° can be also calculated by by the formula

$$\varphi^\circ(t) = \sup_{s\in\mathbb{R}^n} \frac{\langle s,t\rangle}{\varphi(s)} = \sup\{\langle s,t\rangle : \ s\in\mathbb{R}^n, \ \varphi(s)\leqslant 1\} \quad (t\in\mathbb{R}^n),$$

(with the conventions $\alpha/0=+\infty$ for $\alpha>0$ and $\alpha/0=0$ for $\alpha\leqslant 0$).

We need some auxiliary results concerning homogeneous functions of regular linear and bilinear operators (see Lemmas 9.2, 9.7 and 9.9 below).

Lemma 9.2. *Let E and F be uniformly complete vector lattices with F Dedekind complete and $\varphi\in\mathscr{G}_\vee(\mathbb{R}^N)$. Then for $x_1,\dots,x_N\in E$ and $T_1,\dots,T_N\in L^\sim(E,F)$ the inequality holds:*

$$\sum_{k=1}^n T_k x_k \leqslant \varphi(T_1,\dots,T_N)(\varphi^\circ(x_1,\dots,x_N)).$$

$\lhd$ See Kusraev [32, Corollary 4.5]. $\rhd$

Theorem 9.3. *Let E and F be quasi-Banach lattices with F Dedekind complete. Then $\mathscr{L}^r(E,F)$ is a (p,q)-concave quasi-Banach lattice for some $1\leqslant q\leqslant p<\infty$ whenever E is (q',p')-convex with $p'=p/(p-1)$ and $q'=q/(q-1)$. Moreover, $M_{(p,q)}(\mathscr{L}^r(E,F))\leqslant M^{(q',p')}(E)$.*

$\lhd$ Take a finite collection of regular operators $T_1,\dots,T_n\in\mathscr{L}(E,F)$. Denote $B(E)=\{x\in E:\ \|x_k\|\leqslant 1\})$. Using Proposition 3.3, Lemma 9.2 with $\varphi(t)=(\sum_{k=1}^n|t_k|^p)^{1/p}$ and $\varphi^\circ(t)=(\sum_{k=1}^n|t_k|^{p'})^{1/p'}$, and the (q',p')-concavity of E, we have

$$\begin{aligned}
\left(\sum_{k=1}^n\|T_k\|^q\right)^{1/q} &= \sup\left\{\sum_{k=1}^n\alpha_k\|T_k\| :\ \alpha_k\in\mathbb{R}_+\ (k\leqslant n\in\mathbb{N}),\ \sum_{k=1}^n\alpha_k^{q'}=1\right\}\\
&= \sup\left\{\sum_{k=1}^n\|T_k(\alpha_k x_k)\| :\ \alpha_k\in\mathbb{R}_+,\ x_k\in B(E)_+,\ \sum_{k=1}^n\alpha_k^{q'}=1\right\}\\
&\leqslant \sup_{x_k\in B(E)_+}\left\|\sup\left\{\sum_{k=1}^n|T_k|(\alpha_k|x_k|) :\ \alpha_k\in\mathbb{R}_+,\ \sum_{k=1}^n\alpha_k^{q'}=1\right\}\right\|\\
&\leqslant \sup_{x_k\in B(E)_+}\left\|\sup_{\alpha_1^{q'}+\dots+\alpha_n^{q'}=1}\left(\sum_{k=1}^n|T_k|^p\right)^{1/p}\left(\sum_{k=1}^n|\alpha_k x_k|^{p'}\right)^{1/p'}\right\|\\
&\leqslant \sup_{x_k\in B(E)_+}\left\|\left(\sum_{k=1}^n|T_k|^p\right)^{1/p}\right\|\cdot\left\|\sup_{\alpha_1^{q'}+\dots+\alpha_n^{q'}=1}\left(\sum_{k=1}^n|\alpha_k x_k|^{p'}\right)^{1/p'}\right\|\\
&\leqslant M^{(q',p')}(E)\left\|\left(\sum_{k=1}^n|T_k|^p\right)^{1/p}\right\|.
\end{aligned}$$

The last inequality follows from the fact that if $\|x_k\|\leqslant 1$ and $\alpha_1^{q'}+\dots+\alpha_n^{q'}=1$ then $\left\|\left(\sum_{k=1}^n|\alpha_k x_k|^{p'}\right)^{1/p'}\right\|\leqslant M^{(q',p')}(E)$ due to (q',p')-convexity of E. $\rhd$

Corollary 9.4. *Let E and F be quasi-Banach lattices with F Dedekind complete. Then $\mathscr{P}_o^r({}^sE,F)$ is a (p,q)-concave quasi-Banach lattice for some $1\leqslant p,q<\infty$ whenever E is (sp',sq')-convex. Moreover, $M_{(p,q)}(\mathscr{P}_o^r({}^sE,F))\leqslant M^{(sq',sp')}(E)$.*

$\lhd$ This is immediate from Theorem 9.3, Corollary 2.12, and Lemma 3.12. $\rhd$

Definition 9.5. A quasi-Banach lattice E is said to be *quasi-AM-space* whenever it is ∞-*convex*, i. e., there exists a constant C such that

$$\left\| \bigvee_{k=1}^{n} |x_k| \right\| \leqslant C \bigvee_{k=1}^{n} \|x_k\|,$$

for every finite collection $\{x_1, \dots, x_n\}$ in E, see Definition 4.11. The smallest possible constant C is called the ∞-*convexity constant* and is denoted by $M^{(\infty)}(E)$.

Proposition 9.6. *For a quasi-AM-space $(E, \|\cdot\|)$ there is an equivalent quasi-norm $|||\cdot|||$ such that $|||x \vee y||| = |||x||| \vee |||y|||$ for all $x, y \in E_+$.*

$\lhd$ If $\|\cdot\|$ is a monotone quasi-norm on E then $1/M^{(\infty)}\|x\| \leqslant |||x||| \leqslant \|x\|$ for all $x \in E$ whenever a quasi-norm $|||\cdot||| : E \to \mathbb{R}_+$ is defined as

$$|||x||| := \inf \left\{ \bigvee_{k=1}^{m} \|x_k\| : \ |x| = \bigvee_{k=1}^{m} |x_k|; \ x_1, \dots, x_m \in E; \ m \in \mathbb{N} \right\},$$

see Proposition 3.13 with $p = q = \infty$. $\rhd$

Lemma 9.7. *Let E and F be vector lattices with E uniformly complete and F Dedekind complete. Assume that $\varphi \in \mathscr{G}_\vee(\mathbb{R}^n)$ is increasing and $T_1, \dots, T_n \in L^+(E, F)$. Then for every $x \in E_+$ the representation holds*

$$\varphi(T_1, \dots, T_n)x = \sup \left\{ \sum_{k=1}^{n} T_k x_k : \ \varphi^\circ(x_1, \dots, x_n) \leqslant x \right\}.$$

$\lhd$ See Kusraev [31, Theorem 3.4]. $\rhd$

Definition 9.8. A quasi-Banach lattice $(E, \|\cdot\|$ is said to have the *Fatou property* (or its norm is *Fatou*) if $0 \leqslant x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$ for all $x \in E$ and $(x_\alpha) \subset E$.

Theorem 9.9. *Let E be a quasi-Banach lattices and F be a Dedekind complete quasi-AM-space having the Fatou property. Then $\mathscr{L}^r(E, F)$ is a (p, q)-convex quasi-Banach lattice for some $1 \leqslant p, q < \infty$ whenever E is (q', p')-concave with $p' = p/(p-1)$ and $q' = q/(q-1)$. Moreover, $M^{(p,q)}(\mathscr{L}^r(E, F)) \leqslant M^{(\infty)}(F) M_{(p',q')}(E)$.*

$\lhd$ Take a finite collection of regular operators $T_1, \dots, T_n \in \mathscr{L}(E, F)$. Put $M := M^{(\infty)}(F)$ and $M' := M_{(p',q')}(E)$. Using Lemma 9.7, Definition 9.6, Fatou property in F, and (q',p')-concavity of E, we deduce the estimates

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{n}|T_k|^q\Big)^{1/q}\Big\| &= \sup\Big\{\Big\|\Big(\sum_{k=1}^{n}|T_k|^q\Big)^{1/q}(x)\Big\|:\ 0\leqslant x\in E,\ \|x\|\leqslant 1\Big\}\\ &= \sup_{x\in B(E)}\Big\|\sup\Big\{\sum_{k=1}^{n}T_kx_k:\ \Big(\sum_{k=1}^{n}|x_k|^{q'}\Big)^{1/q'}\leqslant x\Big\}\Big\|\\ &\leqslant M\sup\Big\{\sum_{k=1}^{n}\|T_k\|\|x_k\|:\ \Big\|\Big(\sum_{k=1}^{n}|x_k|^{q'}\Big)^{1/q'}\Big\|\leqslant 1\Big\}\\ &\leqslant M\sup\Big\{\sum_{k=1}^{n}\|T_k\|\|x_k\|:\ \Big(\sum_{k=1}^{n}\|x_k\|^{p'}\Big)^{1/p'}\leqslant M'\Big\}\\ &\leqslant MM'\Big(\sum_{k=1}^{n}\|T_k\|^p\Big)^{1/p}\end{aligned}$$

from which (p,q)-convexity of $\mathscr{L}^r(E,F)$ and the required estimate follows. ▷

Corollary 9.10. *Let E and F be as in Theorem 9.8. Then $\mathscr{P}_o^r({}^sE,F)$ is a (p,q)-convex quasi-Banach lattice for some $1\leqslant p,q<\infty$ whenever E is (sq',sp')-concave with $p'=p/(p-1)$ and $q'=q/(q-1)$. Moreover, $M^{(p,q)}(\mathscr{P}_o^r({}^sE,F))\leqslant M^{(\infty)}(F)M_{(sp',sq')}(E)$.*

◁ This is immediate from Theorem 9.9, Corollary 2.12, and Lemma 3.12. ▷

Lemma 9.11. *Let E and F be vector lattices with E uniformly complete and F Dedekind complete. Then for $T_1,\dots,T_n\in L_+^{\sim}(E,F)$, $x_1,\dots,x_n\in E_+$, and $\alpha_1,\dots,\alpha_n\in\mathbb{R}_+$ with $\alpha_1+\dots+\alpha_n=1$ we have*

$$(T_1^{\alpha_1}\dots T_n^{\alpha_n})(x_1^{\alpha_1}\dots x_n^{\alpha_n})\leqslant(T_1x_1)^{\alpha_1}\dots(T_nx_n)^{\alpha_n}.$$

◁ See Kusraev [31, Proposition 3.3]. ▷

Theorem 9.12. *Let E and F be quasi-Banach lattices with F Dedekind complete. If F is geometrically convex then $\mathscr{L}^r(E,F)$ is also geometrically convex. Moreover, $M^{(0^+)}(\mathscr{L}^r(E,F))\leqslant M^{(0^+)}(F)$.*

◁ Take a finite collection of positive operators $T_1,\dots,T_n\in\mathscr{L}(E,F)$ and apply formula (2), 0^+-convexity of F, and Lemma 9.11:

$$\begin{aligned}\big\|(T_1\cdot\ldots\cdot T_n)^{1/n}\big\| &= \sup_{\|x\|\leqslant 1,\,x\geqslant 0}\big\|(T_1\cdot\ldots\cdot T_n)^{1/n}\big((x\cdot\ldots\cdot x)^{1/n}\big)\big\|\\ &\leqslant \sup_{\|x\|\leqslant 1,\,x\geqslant 0}\big\|\big(T_1(x)\cdot\ldots\cdot T_n(x)\big)^{1/n}\big\|\\ &\leqslant M^{(0^+)}(F)\sup_{\|x\|\leqslant 1,\,x\geqslant 0}(\|T_1(x)\|\cdot\ldots\cdot\|T_n(x)\|)^{1/n}\\ &\leqslant M^{(0^+)}(F)(\|T_1\|\cdot\ldots\cdot\|T_n\|)^{1/n}.\ \triangleright\end{aligned}$$

Corollary 9.13. *Let E and F be quasi-Banach lattices with F Dedekind complete. If F is geometrically convex then $\mathscr{P}_o^r({}^sE,F)$ is also geometrically convex. Moreover, $M^{(0^+)}(\mathscr{P}_o^r({}^sE,F))\leqslant M^{(0^+)}(F)$.*

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