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Басаева Е. К., Кусраев А. Г., Кусраева З. А., Плиев М. А., Табуев С. Н., Тасоев Б. Б. Операторы в векторных решетках и пространствах сечений.—Владикавказ, 2011.—40 с. —(Препринт / ЮМИ ВНЦ РАН; № 2).

Работа посвящена исследованию порядковых свойств и аналитическому представлению некоторых классов линейных и нелинейных (полилинейных, сублинейных, полиномиальных) операторов в векторных решетках и банаховых пространствах сечений. Основное внимание сосредоточено на трех темах: развитие метода квазилинеаризации и общие неравенства в равномерно полных векторных решетках; порядковое строение и аналитическое представление ограниченных полилинейных операторов и полиномов в векторных решетках; мажорируемые операторы в банаховых решетках непрерывных и измеримых сечений, ассоциированных с непрерывными или измеримыми расслоениями банаховых решеток.

Цель публикации — дать обзор основных результатов, полученных авторским коллективом в последние три-четыре года в рамках проекта «Операторы в пространствах сечений измеримых банаховых расслоений». Исследование выполнено при финансовой поддержке Российского фонда фундаментальных исследований, проект № 09-01-00442-а.

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The work is devoted to order properties and analytic representation of some classes of linear and nonlinear (multilinear, polynomial, sublinear) operators in vector lattices and Banach section spaces. The presentation is concentrated mainly around three topics: envelope representation method and general inequalities in uniformly complete vector lattices; order structure and analytic representation of multilinear operators and polynomials; dominated operators in Banach lattices of continuous and measurable sections associated with the continuous or measurable bundles of Banach lattices. The aim of this collective work is to give an overview of main results obtained last three years within the framework of the project "Operators in spaces of sections of measurable Banach bundles". The material in the paper is based upon work supported by Russian Foundation of Basic Researches under Grant № 09-01-00442-a.

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CONTENTS

Part I. Envelopes and Inequalities in Vector Lattices

Continuous functional calculus on uniformly complete vector lattices is constructed and the abstract form of envelope representation method in vector lattices is developed. Some general convexity inequalities involving operators in vector lattices are proved. A transfer principle from inequalities with inner products to inequalities containing positive semidefinite symmetric bilinear operators with values in a vector lattices is also presented.

1. Introduction

The expression $\widehat{\varphi}(x_1, \dots, x_N)$ can naturally be defined for elements $x_1, \dots, x_N$ of a uniformly complete vector lattice whenever a positively homogeneous function φ is defined and continuous on a conic subset of $\mathbb{R}^N$. Such extension of the *homogeneous functional calculus* (see [21, 60, 89, 92]) makes it possible to translate the Minkowski duality to the vector lattice setting and to obtain *envelope representations* [70]. As was shown in [79] the homogeneous functional calculus can be extended further so as to make it the *continuous functional calculus* and to produce some new envelope representation results in vector lattices. This machinery, often called *quasilinearization* (see [4, 7, 99]), yields the validity of the classical inequalities in every uniformly complete vector lattice [69, 70]. The aim of this part to give a brief overview of the method and some general inequalities in vector lattices recently obtained in [69, 70, 73, 74, 76, 79].

The unexplained terms of use below can be found in [1, 64, 78, 98]. All vector lattices in this paper are assumed real and Archimedean.

2. Homogeneous Functional Calculus

Let E be a uniformly complete vector lattice and let a finite collection $x_1, \dots, x_N \in E$ be given. Assume that $\langle x_1, \dots, x_N \rangle$ stands for the vector sublattice of E generated by $\{x_1, \dots, x_N\}$ and $H(L)$ denotes the set of all $\mathbb{R}$-valued lattice homomorphisms on a vector lattice L. Put

$$[x_1, \dots, x_N] := \big\{(\omega(x_1), \dots, \omega(x_N)) \in \mathbb{R}^N : \ \omega \in H(\langle x_1, \dots, x_N \rangle)\big\}.$$

Then $[x_1, \dots, x_N]$ is a closed conic set in $\mathbb{R}^N$ uniquely determined by each point separating subset of $H(\langle x_1, \dots, x_N \rangle)$. A set $K \subset \mathbb{R}^N$ is *conic* if $\lambda K \subset K$ for all $\lambda > 0$.

Let $\mathscr{H}(K)$ denote the vector lattice of all positively homogeneous continuous functions $\varphi : K \to \mathbb{R}$. In what follows dt_k will stand for the kth coordinate function on $\mathbb{R}^N$, i. e. $dt_k : (t_1, \dots, t_N) \mapsto t_k$.

Definition 2.1. Let $[x_1, \dots, x_N] \subset K$ and $\varphi \in \mathscr{H}(K)$. Say that $\widehat{\varphi}(x_1, \dots, x_N)$ *exists* or *is well defined* in E and write $y = \widehat{\varphi}(x_1, \dots, x_N)$ whenever there is $y \in E$ such that $\omega(y) = \varphi(\omega(x_1), \dots, \omega(x_N))$ for all $\omega \in H(\langle x_1, \dots, x_N, y \rangle)$; cf. [21, Definition 3.1].

Theorem 2.2. *Let E be a uniformly complete vector lattice and $x_1, \dots, x_N \in E$. Assume that $K \subset \mathbb{R}^N$ is a conic set and $[x_1, \dots, x_N] \subset K$. Then $\widehat{\varphi}(x_1, \dots, x_N)$ exists for every $\varphi \in \mathscr{H}(K)$ and $\varphi \mapsto \widehat{\varphi}(x_1, \dots, x_N)$ is a unique lattice homomorphism from $\mathscr{H}(K)$ into E with $\widehat{dt_j}(x_1, \dots, x_N) = x_j$ for $j := 1, \dots, N$.*

$\lhd$ See [70, Theorem 3.5] and [69, Theorem 2.3]. $\rhd$

Now, we will turn to an extended version of the functional calculus [21, Theorem 4.12] on a unital f-algebra. Let D be a closed subset of $\mathbb{R}^N$. Denote by $\mathscr{B}(D)$ the f-algebra of continuous functions on D with polynomial growth; i.e., $\varphi \in \mathscr{B}(D)$ if and only if $\varphi \in C(D)$ and there are $n \in \mathbb{N}$ and $M \in \mathbb{R}_+$ satisfying $|\varphi(t)| \leqslant M(\mathbb{1} + w(t))^n$ $(t \in D)$, with $w(t) := |t_1| + \ldots + |t_N|$.

Lemma 2.3. *Let D be a closed subset of $\mathbb{R}^N$. To each $\varphi \in \mathscr{B}(D)$ there is $\bar{\varphi} \in \mathscr{B}(\mathbb{R}^N)$ such that $\bar{\varphi}|_D = \varphi$.*

$\lhd$ If $\varphi \in \mathscr{B}(D)$ and $|\varphi| \leqslant M(\mathbb{1} + w)^n$, then $M^{-1}\varphi(\mathbb{1} + w)^{-n} : D \to [-1, 1]$ is a continuous function, and by the Tietze–Urysohn theorem it admits some contunuous extension $\varphi' : \mathbb{R}^N \to [-1, 1]$. Clearly, $\varphi := M\varphi'(\mathbb{1} + w)^n$ belongs to $\mathscr{B}(\mathbb{R}^N)$ and $\bar{\varphi}|_D = \varphi$. $\rhd$

Definition 2.4. Consider an f-algebra E. Given an f-subalgebra $A \subset E$ and a finite tuple $\mathfrak{x} = (x_1, \dots, x_N) \in E^N$, denote by $\langle\!\langle\, \mathfrak{x} \,\rangle\!\rangle := \langle\!\langle x_1, \dots, x_N \rangle\!\rangle$ and $H_m(A)$ the f-subalgebra of E generated by $\{x_1, \dots, x_N\}$ and the set of all nonzero $\mathbb{R}$-valued multiplicative lattice homomorphisms on A, respectively. Denote by $[\,\mathfrak{x}\,]_m := [\,x_1, \dots, x_N\,]_m$ the closure of $\{(\omega(x_1), \dots, \omega(x_N)) : \omega \in H_m(\langle\!\langle\, \mathfrak{x} \,\rangle\!\rangle)\}$ in $\mathbb{R}^N$.

Definition 2.5. Let E be a uniformly complete f-algebra with unit element $\mathbb{1}$. Assume that $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$ satisfies the condition $[x_1, \dots, x_N]_m \subset D$ and take a continuous function $\varphi : D \to \mathbb{R}$. Say that the element $\widehat{\varphi}(x_1, \dots, x_N)$ *exists* or is *well-defined* in E provided that there is $y \in E$ satisfying $\omega(y) = \varphi(\omega(x_1), \dots, \omega(x_N))$ for all $\omega \in H_m(\langle\!\langle x_1, \dots, x_N, y, \mathbb{1} \rangle\!\rangle)$, cp. [21, Definition 4.2]. This is written down as $y = \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N)$.

According to [21, Corollary 2.5] $H_m(\langle\!\langle x_1, \dots, x_N, y, \mathbb{1} \rangle\!\rangle)$ separates the points of $\langle\!\langle x_1, \dots, x_N, y, \mathbb{1} \rangle\!\rangle$. Therefore, there is at most one $y \in E$ enjoying the above condition. Moreover, if H is a point separating subset of $H_m(\langle\!\langle x_1, \dots, x_N, y, \mathbb{1} \rangle\!\rangle)$ and $\omega(y) = \varphi(\omega(x_1), \dots, \omega(x_N))$ for all $\omega \in H$, then $y = \widehat{\varphi}(x_1, \dots, x_N)$; cp. [21, Lemma 4.3]. The following result should be compared with [21, Theorem 4.12].

Theorem 2.6. *Assume that E is a uniformly complete f-algebra with unit $\mathbb{1}$, $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$, and $D \subset \mathbb{R}^N$ is a subset of $\mathbb{R}^N$ containing $[\,\mathfrak{x}\,]_m$. Then $\widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \dots, x_N)$ exists for every $\varphi \in \mathscr{B}(D)$, and the mapping*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N) \quad \big(\varphi \in \mathscr{B}(D)\big)$$

is the unique multiplicative lattice homomorphism from $\mathscr{B}(D)$ to E such that $\widehat{\mathfrak{x}}(1_D) = \mathbb{1}$ and $\widehat{dt_j}(x_1, \dots, x_N) = x_j$ for all $j := 1, \dots, N$.

$\lhd$ Since $[\mathfrak{x}]_m$ is closed and $[\mathfrak{x}]_m \subset D$, we may assume without loss of generality that D is closed. If $D = \mathbb{R}^N$, then the claim is established in [21, Theorem 4.12]. Denote by h the corresponding multiplicative lattice homomorphism from $\mathscr{B}(\mathbb{R}^N)$ to E and let ϱ stands for the restriction operator $\varphi \mapsto \varphi|_D$ from $\mathscr{B}(\mathbb{R}^N)$ to $\mathscr{B}(D)$. It is easy that $\ker(\varrho) = \ker(h)$. Consequently, there is a linear operator $\widehat{\mathfrak{x}} : \mathscr{B}(D) \to E$, satisfying $\widehat{\mathfrak{x}} \circ \varrho = h$ [85, Theorem 2.3.8]. This $\widehat{\mathfrak{x}}$ is clearly a multiplicative lattice homomorphism. Also ϱ is surjective by Lemma 2.3, which implies the unicity of $\widehat{\mathfrak{x}}$. $\rhd$

3. Continuous Functional Calculus

A continuous functional calculus in a uniformly complete vector lattice can be constructed making use of Hörmander transform of convex functions, see [79].

Given a nonempty $D \subset \mathbb{R}^N$, let D_h stand for the conic hull of the set $\{1\} \times D$ in $\mathbb{R} \times \mathbb{R}^N$; i. e., $D_h := \operatorname{cone}(\{1\} \times D)$. Clearly, $D_h := \{(\lambda, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^N : \ \lambda > 0, \mathbf{t} \in \lambda D\}$. Take $\varphi : D \to \mathbb{R}$ and define $\varphi_h(\lambda, \mathbf{t}) := \lambda\varphi(\mathbf{t}/\lambda)$ $((\lambda, \mathbf{t}) \in D_h)$. Obviously, D_h is a conic subset of $\mathbb{R}^{N+1}$, while φ_h is a positively homogeneous function from D_h to $\mathbb{R}$. The function φ_h (as well as the set D_h) is often referred to as the *Hörmander transform* of φ (respectively, D). More details concerning the Hörmander transform as well as the following easy fact can be found in [86] and [110].

Proposition 3.1. *The Hörmander transform $\varphi \in C(D) \mapsto \varphi_h \in \mathscr{H}(D_h)$ is a lattice and linear isomorphism of the vector lattices $C(D)$ and $\mathscr{H}(D_h)$. In case D is a convex set, φ_h is a sublinear (superlinear) function if and only if φ is a convex* (concave) function.

Definition 3.2. Take $x_0, x_1, \dots, x_N \in E$. Write $(x_1, \dots, x_N) \prec x_0$ provided that $x_0 \neq 0$ and $\varepsilon(|x_1| + \dots + |x_N|) \leqslant x_0$ for some $\varepsilon > 0$. If $(x_1, \dots, x_N) \prec x_0$, then x_0 is a strong order unit in $\langle x_0, x_1, \dots, x_N \rangle$. Hence, $\omega(x_0) \neq 0$ for each nonzero $\omega \in H(x_0, x_1, \dots, x_N)$. By definition we put

$$[(x_1, \dots, x_N)/x_0] := \left\{ \left(\frac{\omega(x_1)}{\omega(x_0)}, \dots, \frac{\omega(x_N)}{\omega(x_0)} \right) \in \mathbb{R}^N : \right.$$
$$\left. 0 \neq \omega \in H(x_0, x_1, \dots, x_N) \right\}.$$

Clearly, $[(x_1, \dots, x_N)/x_0] \subset D$ if and only if $[x_0, x_1, \dots, x_N] \subset D_h$. It is also easy that $[(x_1, \dots, x_N)/x_0]$ is a compact subset of $\mathbb{R}^N$. Moreover, if L is a vector sublattice of E whith $x_0, x_1, \dots, x_N \in L$ and $H(L)$ separates the points of L, then $[(x_1, \dots, x_N)/x_0]$ is the inclusion least closed set that includes $\big\{(\omega(x_1)/\omega(x_0), \dots, \omega(x_N)/\omega(x_0)) \in \mathbb{R}^N : 0 \neq \omega \in H(L)\big\}$.

Definition 3.3. Let some tuple $x_0, x_1, \dots, x_N \in E$ be fixed. Consider $\varphi \in C(D)$, with $D \subset \mathbb{R}^N$, and suppose that $[(x_1, \dots, x_N)/x_0] \subset D$. Put by definition

$$y = x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0) \iff y = \widehat{\varphi_h}(x_0, x_1, \dots, x_N).$$

Given $y \in E$, we have $y = x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0)$ if and only if

$$\omega(y) = \omega(x_0)\varphi\big(\omega(x_1)/\omega(x_0), \dots, \omega(x_N)/\omega(x_0)\big)$$

for all $\omega \in H(\langle x_0, x_1, \dots, x_N, y \rangle)$, see Definition 2.1. It is straightforward that $x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0) = \widehat{\varphi}(x_1, \dots, x_N)$ whenever φ is positively homogeneous, and thus the Definitions 2.1 and 3.3 are in agrement.

Theorem 3.4. *Let E be a uniformly complete vector lattice, $x_0, x_1, \dots, x_N \in E$ and $\mathfrak{x} = (x_1, \dots, x_N)$. Assume that $(x_1, \dots, x_N) \prec x_0$ and $[(x_1, \dots, x_N)/x_0] \subset D \subset \mathbb{R}^N$. Then $x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0)$ exists for every $\varphi \in C(D)$ and the mapping*

$$(\mathfrak{x}/x_0)\hat{}: \varphi \mapsto x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0) \quad \big(\varphi \in C(D)\big)$$

is the unique lattice homomorphism from $C(D)$ to E satisfying $(\mathfrak{x}/x_0)\hat{}(1_D) = x_0$ and $(\mathfrak{x}/x_0)\hat{}(dt_j) = x_j$ for all $j := 1, \dots, N$. Moreover, $(\mathfrak{x}/x_0)\hat{}(C(D))$ coincides with the x_0-closure of the lattice $\langle x_0, x_1, \dots, x_N \rangle$.

Theorem 3.5. *Take a finite tuple $x_1, \dots, x_N$ of members of a uniformly complete f-algebra E with unit $\mathbb{1}$. If $D \subset \mathbb{R}^N$, $\varphi \in \mathscr{B}(D)$, and for some positive invertible $x_0 \in E$ we have $[(x_1, \dots, x_N)/x_0] \subset D$ and $(x_1, \dots, x_N) \prec x_0$ then $[x_1 x_0^{-1}, \dots, x_N x_0^{-1}]_m \subset D$ and*

$$x_0 \widehat{\varphi}(x_1 x_0^{-1}, \dots, x_N x_0^{-1}) = \widehat{\varphi}_h(x_0, x_1, \dots, x_N).$$

Remark 3.6. Theorem 3.5 says that the continuous functional calculus in uniformly complete vector lattices which is introduced in Definition 3.3 agrees with that in uniformly complete unital f-algebras which is given in Definition 2.5.

We need a slightly improved version of continuous functional calculus on uniformly complete f-algebras constructed in [21, Theorem 5.2].

Denote by $\mathscr{B}(\mathbb{R}^N_+)$ the f-algebra of continuous functions on $\mathbb{R}^N_+$ with polynomial growth; i.e., $\varphi \in \mathscr{B}(\mathbb{R}^N_+)$ if and only if $\varphi \in C(\mathbb{R}^N_+)$ and there are $n \in \mathbb{N}$ and $M \in \mathbb{R}_+$ satisfying $|\varphi(\mathbf{t})| \leqslant M(\mathbf{1} + w(\mathbf{t}))^n$ $(\mathbf{t} \in \mathbb{R}^N_+)$, where $\mathbf{t} := (t_1, \dots, t_N)$, $w(\mathbf{t}) := |t_1| + \ldots + |t_N|$ and $\mathbf{1}$ is the function identically equal to 1 on $\mathbb{R}^N_+$. Denote by $\mathscr{B}_0(\mathbb{R}^N_+)$ the set of all functions in $\mathscr{B}(\mathbb{R}^N_+)$ vanishing at zero. Let $\mathscr{A}(\mathbb{R}^N_+)$ stands for the set of all $\varphi \in \mathscr{B}(\mathbb{R}^N_+)$ such that $\lim_{\alpha \downarrow 0} \alpha^{-1} \varphi(\alpha \mathbf{t})$ exists uniformly on bounded subsets of $\mathbb{R}^N_+$. Evidently, $\mathscr{A}(\mathbb{R}^N_+) \subset \mathscr{B}_0(\mathbb{R}^N_+)$. Finally, let $\mathscr{H}(\mathbb{R}^N_+)$ denotes the set of all continuous positively homogeneous functions on $\mathbb{R}^N_+$.

Consider an f-algebra E. Denote by $H_m(E)$ the subset of $H(E)$ consisting of multiplicative functionals. We say that $\omega \in H(E)$ is *singular* if $\omega(xy) = 0$ for all $x, y \in E$. Let $H_s(E)$ denotes the set of singular members of $H(E)$. Given a finite tuple $\mathbf{x} = (x_1, \dots, x_N) \in E^N$, denote by $\langle\!\langle \mathbf{x} \rangle\!\rangle := \langle\!\langle x_1, \dots, x_N \rangle\!\rangle$ the f-subalgebra of E generated by $\{x_1, \dots, x_N\}$.

Definition 3.7. Let E be a uniformly complete f-algebra and $x_1, \dots, x_N \in E_+$. Take a continuous function $\varphi : \mathbb{R}^N_+ \to \mathbb{R}$. Say that the element $\widehat{\varphi}(x_1, \dots, x_N)$ *exists* or is *well-defined* in E provided that there is $y \in E$ satisfying

$$\begin{aligned} \omega(y) &= \varphi(\omega(x_1), \dots, \omega(x_N)) \quad \big(\omega \in H_m(\langle\!\langle x_1, \dots, x_N, y \rangle\!\rangle)\big), \\ \omega(y) &= \varphi_1(\omega(x_1), \dots, \omega(x_N)) \quad \big(\omega \in H_s(\langle\!\langle x_1, \dots, x_N, y \rangle\!\rangle)\big), \end{aligned} \tag{1}$$

cp. [21, Remark 5.3 (ii)]. This is written down as $y = \widehat{\varphi}(x_1, \dots, x_N)$.

Theorem 3.8. *Assume that E is a uniformly complete f-algebra and $x_1, \dots, x_N \in E_+$, and $\mathbf{x} := (x_1, \dots, x_N)$. Then $\widehat{\mathbf{x}}(\varphi) := \widehat{\varphi}(x_1, \dots, x_N)$ exists for every $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$, and the mapping $\widehat{\mathbf{x}} : \varphi \mapsto \widehat{\mathbf{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N)$ is the unique multiplicative lattice homomorphism from $\mathscr{A}(\mathbb{R}^N_+)$ to E such that $\widehat{dt}_j(x_1, \dots, x_N) = x_j$ for all $j := 1, \dots, N$. Moreover, $\widehat{\mathbf{x}}(\mathscr{A}(\mathbb{R}^N_+)) = \overline{\langle\!\langle x_1, \dots, x_N \rangle\!\rangle}$. (As in [21] $\overline{A}$ denotes the A-closure of A in E.)*

4. Envelope Representations

Definition 4.1. Given $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$, define the *up-conjugate* ϕ^* and the *down-conjugate* ϕ_* as follows:

$$\begin{aligned} \phi^*(\mathbf{t}) &:= \sup\{\langle \mathbf{s}, \mathbf{t} \rangle - \phi(\mathbf{s}) : \ \mathbf{s} \in \mathbb{R}^N\}, \\ \phi_*(\mathbf{t}) &:= \inf\{\langle \mathbf{s}, \mathbf{t} \rangle - \phi(\mathbf{s}) : \ \mathbf{s} \in \mathbb{R}^N\}. \end{aligned}$$

Also, put $\phi^{**} := (\phi^*)^*$ and $\phi_{**} := (\phi_*)_*$. Note that $\phi^{**} \leqslant \phi \leqslant \phi_{**}$.

The Fenchel–Moreau duality theorem assets that $\phi^{**} = \phi$ if and only if ϕ is convex and lower semicontinuous, while $\phi_{**} = \phi$ if and only if ϕ is concave and upper semicontinuous); cp. [109]. Denote by $\mathscr{C}_\vee(\mathbb{R}^N, D)$ and $\mathscr{C}_\wedge(\mathbb{R}^N, D)$ the set of all lower semicontinuous convex fucntions $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and the set of upper semicontinuous concave functions $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ each of which is finite and continuous on $D \subset \mathbb{R}^N$. Let $\mathscr{B}_\vee(\mathbb{R}^N, D)$ and $\mathscr{B}_\wedge(\mathbb{R}^N, D)$ stand for the subsets of $\mathscr{C}_\vee(\mathbb{R}^N, D)$ and $\mathscr{C}_\wedge(\mathbb{R}^N, D)$ comprising the functions of polynomial growth on D.

Remark 4.2. If $\varphi \in \mathscr{C}_\vee(\mathbb{R}^N, D)$ and $\overline{\varphi}_h$ designates the closure of φ_h (see [109, § 7]), then $\overline{\varphi}_h \in \mathscr{H}_\vee(\mathbb{R}^N, D)$; cp. [86, Proposition 1.4]. Furthermore, $\widehat{\varphi}_h(x_1, \dots, x_N) = \widehat{\overline{\varphi}_h}(x_1, \dots, x_N)$ provided that $[x_1, \dots, x_N] \subset D$ since φ_h and $\overline{\varphi}_h$ coincide on D_h.

Theorem 4.3. *Let E be a uniformly complete vector lattice, while $x_0, x_1, \dots, x_N \in E$ and $\mathfrak{x} := (x_1, \dots, x_N) \prec x_0$. If $\varphi \in \mathscr{C}_\vee(\mathbb{R}^N; D)$, $\psi \in \mathscr{C}_\wedge(\mathbb{R}^N; D)$, and $[\,\mathfrak{x}/x_0\,] \subset D$ then*

$$x_0\widehat{\varphi}(\mathfrak{x}/x_0) = \sup_{\lambda \in \operatorname{dom}(\varphi^*)} \big\{ \langle \lambda, \mathfrak{x} \rangle - \varphi^*(\lambda)x_0 \big\},$$

$$x_0\widehat{\psi}(\mathfrak{x}/x_0) = \inf_{\lambda \in \operatorname{dom}(\varphi_*)} \big\{ \langle \lambda, \mathfrak{x} \rangle - \varphi_*(\lambda)x_0 \big\}.$$

Moreover, $x_0\widehat{\varphi}(\mathfrak{x}/x_0)$ $\big(x_0\widehat{\psi}(\mathfrak{x}/x_0)\big)$ is the uniform limit of an increasing (decreasing) sequence of finite suprema (infima) of linear combinations $-\varphi^*(\lambda)x_0 + \sum_{i=1}^N \lambda_i x_i$, with $\lambda = (\lambda_1, \dots \lambda_N) \in \operatorname{dom}(\varphi^*)$ $(\lambda \in \operatorname{dom}(\varphi_*))$.

Theorem 4.5. *Assume that E is a uniformely complete f-algebra with unit $\mathbb{1}$, while $x_1, \dots, x_N \in E$ and $\mathfrak{x} := (x_1, \dots, x_N)$. If $\varphi \in \mathscr{B}_\vee(\mathbb{R}^N; D)$, $\psi \in \mathscr{B}_\wedge(\mathbb{R}^N; D)$ and $[\,\mathfrak{x}\,]_m \subset D$, then*

$$\widehat{\mathfrak{x}}(\varphi) = \sup_{\lambda \in \operatorname{dom}(\varphi^*)} \big\{ \langle \lambda, \mathfrak{x} \rangle - \varphi^*(\lambda)\mathbb{1} \big\},$$

$$\widehat{\mathfrak{x}}(\psi) = \inf_{\lambda \in \operatorname{dom}(\varphi_*)} \big\{ \langle \lambda, \mathfrak{x} \rangle - \varphi_*(\lambda)\mathbb{1} \big\}.$$

Moreover, $\widehat{\varphi}(x_1, \dots, x_N)$ $\big(\widehat{\psi}(x_1, \dots, x_N)\big)$ is the order limit of an increasing (decreasing) sequence of finite suprema (infima) of linear combinations $\sum_{i=1}^N \lambda_i x_i - \varphi^*(\lambda)\mathbb{1}$ with $\lambda = (\lambda_1, \dots \lambda_N) \in \operatorname{dom}(\varphi^*)$ $((\lambda \in \operatorname{dom}(\varphi_*))$.

5. Convexity Inequalities

In this section we consider the abstractions of inequalities of Jensen, Hölder, and Minkowski types obtained in [69, Theorems 5.2, 5.5, 5.6].

Definition 5.1. Let E and F be vector lattices. An operator $f : E \to F \cup \{+\infty\}$ is said to be sublinear if $f(0) = 0$, $f(\lambda x) = \lambda f(x)$, and $f(x + y) \leqslant f(x) + f(y)$ for all $0 \leqslant \lambda \in \mathbb{R}$ and $x, y \in E$. An operator $g : E \to F \cup \{-\infty\}$ is *superlinear* provided that $-g$ is sublinear. Put $\operatorname{dom}(f) := \{x \in E : f(x) < +\infty\}$ and $\operatorname{dom}(g) := \{x \in E : g(x) > -\infty\}$. We say that f is increasing on $\operatorname{dom}(f)$ if $x \geqslant y$ implies $f(x) \geqslant f(y)$ for $x, y \in \operatorname{dom}(f)$. For more details concerning sublinear operators, see [78].

Given a convex cone $K \subset \mathbb{R}^N$, denote by $\mathscr{H}_\curlyvee(\mathbb{R}^N, K)$ $(\mathscr{H}_\curlywedge(\mathbb{R}^N, K))$ the set of all sublinear (superlinear) functions $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ $(\mathbb{R} \cup \{-\infty\})$ with the properties: a) ϕ is lower semicontinuous (upper semicontinuous), b) ϕ is continuous on $K \subset \operatorname{dom}(\phi)$, c) ϕ is increasing on $\operatorname{dom}(\phi)$ with respect to $\mathbb{R}^N_+$, d) $\mathbb{R}^N_+ - \operatorname{dom}(\phi) = \operatorname{dom}(\phi) - \mathbb{R}^N_+$.

Theorem 5.2. *Let E and F be relatively uniformly complete vector lattices, $f : E \to F \cup \{+\infty\}$ an increasing sublinear operator, and $g : E \to F \cup \{-\infty\}$ an increasing superlinear operator. Assume that $\varphi \in \mathscr{H}_{\curlyvee}(\mathbb{R}^N, K)$ and $\psi \in \mathscr{H}_{\curlywedge}(\mathbb{R}^N, K)$. If $x_1, \dots, x_N \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$ and $[x_1, \dots, x_N] \subset K$, $[f(x_1), \dots, f(x_N)] \subset K$ $[g(x_1), \dots, g(x_N)] \subset K$ then $\widehat{\varphi}(x_1, \dots, x_N) \in \operatorname{dom}(g)$, $\widehat{\psi}(x_1, \dots, x_N) \in \operatorname{dom}(f)$ and*

$$f\big(\widehat{\psi}(x_1, \dots, x_N)\big) \leqslant \widehat{\psi}\big(f(x_1), \dots, f(x_N)\big),$$
$$g\big(\widehat{\varphi}(x_1, \dots, x_N)\big) \geqslant \widehat{\varphi}\big(g(x_1), \dots, g(x_N)\big).$$

Remark 5.3. Let $\mathscr{H}_{\vee}(\mathbb{R}^N, K)$ and $\mathscr{H}_{\wedge}(\mathbb{R}^N, K)$ stand respectively for the set of lower semicontinuous convex functions $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and the set of upper semicontinuous concave functions $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ both finite and continuous on K. The inequalities in 5.2 remain valid if $\varphi \in \mathscr{H}_{\vee}(\mathbb{R}^N, K)$, $\psi \in \mathscr{H}_{\wedge}(\mathbb{R}^N, K)$, and $f, g : E \to F$ are positive linear operators. Moreover, in this case we may assume that E is a linear subspace of a uniformly complete vector lattice G provided that $\widehat{\varphi}(x_1, \dots, x_N)$, $\widehat{\psi}(x_1, \dots, x_N) \in E$ and F is Dedekind complete. Indeed, by The Kantorovich Extension Theorem [1, Theorem 1.32] f has a positive linear extension S to the order ideal G_0 generated by E in G, and hence it suffice to apply the above remark to S and G_0. For a fixed tuple $(x_1, \dots, x_N)$ we write $x_i \gg 0$ whenever $x_i \succ (x_1, \dots, x_N)$.

Corollary 5.4. *Let E and F be relatively uniformly complete vector lattices and let $f : E \to F \cup \{+\infty\}$ be an increasing sublinear mapping with $\operatorname{dom}(f) = E_+$. Then for $x_1, \dots, x_N \in E$ and $0 \leqslant \alpha_1, \dots, \alpha_N \in \mathbb{R}$, with $\alpha_1 + \ldots + \alpha_N = 1$ we have*

$$f\bigg(\prod_{i=1}^{N} |x_i|^{\alpha_i}\bigg) \leqslant \prod_{i=1}^{N} f(|x_i|)^{\alpha_i}.$$

The reverse inequality holds provided that $f : E \to F\{-\infty\}$ is superlinear, $\alpha_1 + \ldots + \alpha_N = 1$, $(-1)^k(1 - \alpha_1 - \ldots - \alpha_k)\,\alpha_1 \cdot \ldots \cdot \alpha_k \geqslant 0$ $(k := 1, \dots, N-1)$, and $|x_i| \gg 0$, $f(|x_i|) \gg 0$ for all i with $\alpha_i < 0$.

Corollary 5.5. *Let E and F be relatively uniformly complete vector lattices, $f : E \to F \cup \{+\infty\}$ be an increasing sublinear mapping with $\operatorname{dom}(f) = E_+$, and $x_1, \dots, x_N \in E$. If either and $0 < \alpha \leqslant 1$ or $\alpha < 0$, then*

$$f\Bigg(\bigg(\sum_{i=1}^{N} |x_i|^{\alpha}\bigg)^{1/\alpha}\Bigg) \leqslant \bigg(\sum_{i=1}^{N} f(|x_i|)^{\alpha}\bigg)^{1/\alpha}.$$

The reverse inequality holds if $f : E \to F \cup \{-\infty\}$ is increasing superlinear and $\alpha \geqslant 1$.

Remark 5.6. In the special case of vector lattices of measurable functions the first inequality in Theorem 5.2 was established by Haase [47, Proposition 1.1]. Similar results see in Bourbaki [16, Proposition I.1], Malygranda [97, Lemma 1], Mitrinović, Pečarić, Fink [99, p. 192]. Various classical and recent inequalities are related to Jensen's, Hölder's, and Minkowski's inequality (see [4, 49, 99, 100, 113]). Some of them can naturally be transferred into the environment of vector lattice by means of envelope representation (= quazilinearization) method, see [69]. For a Hölder type inequality involving positive linear operator between function lattices see Krengel [59, Lemma 7.4], Kreĭn, Petunin, and Semënov [58, pp. 61, 327], Maligranda [97, Remark 1], M. Haase [47, Remark 1.2 (5)]; see also Boulabiar [12] for Hölder type inequality involving positive operator between uniformly complete unital f-algebras.

6. Generalized Jessen Type Inequalities

In this section we present two results from [79]. Denote by $\mathscr{C}_\vee(\mathbb{R}^N, D)$ and $\mathscr{C}_\wedge(\mathbb{R}^N, D)$ the set of all lower semicontinuous convex fucntions $\varphi : \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ and the set of upper semicontinuous concave functions $\psi : \mathbb{R}^N \to \mathbb{R}\cup\{-\infty\}$ each of which is finite and continuous on $D \subset \mathbb{R}^N$. Let $\mathscr{B}_\vee(\mathbb{R}^N, D)$ and $\mathscr{B}_\wedge(\mathbb{R}^N, D)$ stand for the subsets of $\mathscr{C}_\vee(\mathbb{R}^N, D)$ and $\mathscr{C}_\wedge(\mathbb{R}^N, D)$ comprising the functions of polynomial growth on D.

Theorem 6.1. *Let E and F be uniformly complete vector lattices, and let $S : E \to F$ be a positive operator. Assume that $S(x_0) \neq 0$, $\varphi \in \mathscr{C}_\vee(\mathbb{R}^N, D)$ and $\psi \in \mathscr{C}_\wedge(\mathbb{R}^N, D)$ for some closed convex set $D \subset \mathbb{R}^N$. If a finite tuple $x_0, x_1, \dots, x_N \in E$ is such that $(x_1, \dots, x_N) \prec x_0$, $[(x_1, \dots, x_N)/x_0] \subset D$, and $[(S(x_1), \dots, S(x_N))/S(x_0)] \subset D$, then*

$$\begin{aligned} &S\big(x_0\widehat{\psi}(x_1/x_0, \dots, x_N/x_0)\big) \leqslant \\ &\qquad\qquad \leqslant S(x_0)\widehat{\psi}\big(S(x_1)/S(x_0), \dots, S(x_N)/S(x_0)\big), \\ &S\big(x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0)\big) \geqslant \\ &\qquad\qquad \geqslant S(x_0)\widehat{\varphi}\big(S(x_1)/S(x_0), \dots, S(x_N)/S(x_0)\big). \end{aligned}$$

Theorem 6.2. *Let E and F be uniformly complete vector lattices, $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$, and let $S : E \to F$ be a positive linear operator. Assume that $\varphi \in \mathscr{B}_\vee(\mathbb{R}^N, D)$ and $\psi \in \mathscr{B}_\wedge(\mathbb{R}^N, D)$ for some closed convex set $D \subset \mathbb{R}^N$. The following are valid:*

(1) *If E is an f-algebra with units $\mathbb{1}$, $[\mathfrak{x}]_m \subset D$, $(S(x_1), \dots, S(x_N)) \prec S(\mathbb{1})$, and $[(S(x_1), \dots, S(x_N))/S(\mathbb{1})] \subset D$, then*

$$\begin{aligned} S\big(\widehat{\psi}(x_1, \dots, x_N)\big) &\leqslant S(\mathbb{1})\widehat{\psi}\big(S(x_1)/S(\mathbb{1}), \dots, S(x_N)/S(\mathbb{1})\big), \\ S\big(\widehat{\varphi}(x_1, \dots, x_N)\big) &\geqslant S(\mathbb{1})\widehat{\varphi}\big(S(x_1)/S(\mathbb{1}), \dots, S(x_N)/S(\mathbb{1})\big); \end{aligned}$$

(2) *If F is an f-algebra with unit $\hat{\mathbb{1}}$, $[S(x_1), \dots, S(x_N)]_m \subset D$, and for some $x_0 \in E_+$ we have $S(x_0) = \hat{\mathbb{1}}$, $(x_1, \dots, x_N) \prec x_0$, and $[(x_1, \dots, x_N)/x_0] \subset D$, then*

$$\begin{aligned} S\big(x_0\widehat{\psi}(x_1/x_0, \dots, x_N/x_0)\big) &\leqslant \widehat{\psi}\big(S(x_1), \dots, S(x_N)\big), \\ S\big(x_0\widehat{\varphi}(x_1/x_0, \dots, x_N/x_0)\big) &\geqslant \widehat{\varphi}\big(S(x_1), \dots, S(x_N)\big); \end{aligned}$$

(3) *If both E and F are f-algebras with unit elements $\mathbb{1}$ and $\hat{\mathbb{1}}$, respectively, and in addition $S(\mathbb{1}) = \hat{\mathbb{1}}$, $[x_1, \dots, x_N]_m \subset D$, and $[S(x_1), \dots, S(x_N)]_m \subset D$, then*

$$\begin{aligned} S\big(\widehat{\psi}(x_1, \dots, x_N)\big) &\leqslant \widehat{\psi}\big(S(x_1), \dots, S(x_N)\big), \\ S\big(\widehat{\varphi}(x_1, \dots, x_N)\big) &\geqslant \widehat{\varphi}\big(S(x_1), \dots, S(x_N)\big). \end{aligned}$$

Remark 6.3. The generalizations of Jensen's inequality for convex functions φ are referred to as the *Jessen inequalities*; cp. [104, Theorems 2.4 and 2.6]. In the particular case when E is a vector space of real-valued functions and $F = \mathbb{R}$, for the inequalities in Theorem 6.2 (3) see [5, 102, 103, 104] where some other versions and generalizations of the Jessen inequality are also collected.

7. Beckenbach–Dresher Type Inequalities

In this section we consider a Beckenbach–Dresher type inequality in vector lattices. Let $(G,+)$ be a commutative semigroup, while E is a uniformly complete vector lattice, and $f_1,\dots,f_N : G \to E_+$. Assume that some set-valued map $\mathscr{F} : G \to \mathscr{P}(E_+)$ meets the following three conditions:

1) $(f_1(x),\dots,f_N(x)) \prec e$ for every $e \in \mathscr{F}(x)$,

2) $\mathscr{F}(x) + \mathscr{F}(y) \subset \mathscr{F}(x+y) - E_+$ for all $x,y \in G$, and

3) the infimum (supremum) of $\{e\widehat{\varphi}(\mathbf{f}(x)/e) : e \in \mathscr{F}(x)\}$ exists in E for each $x \in G$, where $\mathbf{f}(x) := (f_1(x),\dots,f_N(x)) \in E_+^N$.

Then, given a continuous function $\varphi : \mathbb{R}_+^N \to \mathbb{R}_+$, we have the operator $g : G \to E$ $(h : G \to E)$ well defined as

$$g(x) := \inf_{e\in\mathscr{F}(x)} \left(e\widehat{\varphi}\left(\frac{\mathbf{f}(x)}{e}\right)\right) \quad \left(h(x) := \sup_{e\in\mathscr{F}(x)} \left(e\widehat{\varphi}\left(\frac{\mathbf{f}(x)}{e}\right)\right)\right).$$

Theorem 7.1. *Suppose that the operators $g,h : G \to E$ are defined as above. Then the following assertions hold:*

(1) g is subadditive whenever $f_1,\dots,f_N$ are subadditive, and φ is an increasing convex function satisfying $\varphi(0)=0$;

(2) h is superadditive whenever $f_1,\dots,f_N$ are superadditive, and φ is an increasing concave function satisfying $\varphi(0)=0$.

For a single-valued map $\mathscr{F}(x) = \{f_0(x)\}$ $(x \in G)$ with $f_0 : G \to E_+$ we have the following particular case of Theorem 7.1, see [79].

Corollary 7.2. *Let $(G,+)$, E, and $f_1,\dots,f_N$ are the same as in Theorem 7.1. Suppose that an operator $f_0 : G \to E_+$ is such that $(f_1(x),\dots,f_N(x)) \prec f_0(x)$ for all $x \in G$. Then, given a continuous function $\varphi : \mathbb{R}_+^N \to \mathbb{R}_+$, we have the operator $h : G \to E$ well defined as*

$$h(x) := f_0(x)\widehat{\varphi}\big(f_1(x)/f_0(x),\dots,f_N(x)/f_0(x)\big).$$

Moreover,

(1) h is subadditive whenever f_0 is superadditive, $f_1,\dots,f_N$ are subadditive, and φ is an increasing convex function satisfying $\varphi(0)=0$;

(2) h is superadditive whenever $f_0,f_1,\dots,f_N$ are superadditive, and φ is an increasing concave function satisfying $\varphi(0)=0$.

Remark 7.3. The subadditivity of h (with convex φ, superadditive f_0, and subadditive $f_1,\dots,f_N$) means that for all $x,y \in G$ we have the *Peetre–Persson inequality*:

$$f_0(x+y)\widehat{\varphi}\left(\frac{\mathbf{f}(x+y)}{f_0(x+y)}\right) \leqslant f_0(x)\widehat{\varphi}\left(\frac{\mathbf{f}(x)}{f_0(x)}\right) + f_0(y)\widehat{\varphi}\left(\frac{\mathbf{f}(y)}{f_0(y)}\right).$$

The inequality holds in the opposite direction whenever φ is concave and $f_0,f_1,\dots,f_N$ are superadditive.

Remark 7.4. Theorem 7.1 in the particular case of $E = \mathbb{R}$ was obtained by Persson [107, Theorems 1 and 2], while Corollary 7.2 covers the “single-valued case” by Peetre and Persson [105]. A short history of the Beckenbach–Dresher inequality is presented in [117]. Some instances of the inequality are also addressed in [99, 103, 104].

8. Beckenbach–Dresher Type Inequalities in Uniformly Complete f-Algebras

In this section we preent the main results from [74]. Let $(G_1, +)$ and $(G_2, +)$ be commutative semigroups, while E is a uniformly complete f-algebra and $f_1, \dots, f_N : G_1 \to E_+$. Let $\mathscr{P}(M)$ stands for the power set of M. Assume that some set-valued map $\mathscr{F} : G_2 \to \mathscr{P}(\mathrm{Orth}(E)_+)$ meets the following three conditions:

(i) π^{-1} exists in $\mathrm{Orth}(E)$ for every $\pi \in \mathscr{F}(u)$ and $u \in G_2$,

(ii) $\mathscr{F}(u) + \mathscr{F}(v) \subset \mathscr{F}(u+v) - \mathrm{Orth}(E)_+$ for all $u, v \in G_2$, and

(iii) the infimum (the supremum) of $\{\pi\widehat{\varphi}(\pi^{-1}\mathbf{f}(u_1)) : \pi \in \mathscr{F}(u_2)\}$ exists in E for all $u_1 \in G_1$ and $u_2 \in G_2$, where $\mathbf{f}(u) := (f_1(u), \dots, f_N(u)) \in E_+^N$ and $\pi^{-1}\mathbf{f}(u) := (\pi^{-1}f_1(u), \dots, \pi^{-1}f_N(u)) \in E_+^N$.

Given a function $\varphi \in \mathscr{A}(\mathbb{R}_+^N)$ and a set-valued map $\mathscr{F} : G_2 \to \mathscr{P}(\mathrm{Orth}(E)_+)$ satisfying 3 (i–iii), we have the operator $g : G_1 \times G_2 \to E$ $(h : G_1 \times G_2 \to E)$ well defined as

$$\begin{gathered} g(u_1, u_2) := \inf_{\pi \in \mathscr{F}(u_2)} \big\{\pi\widehat{\varphi}\big(\pi^{-1}\mathbf{f}(u_1)\big)\big\} \\ \Big(h(u_1, u_2) := \sup_{\pi \in \mathscr{F}(u_1)} \big\{\pi\widehat{\varphi}\big(\pi^{-1}\mathbf{f}(u_2)\big)\big\}\Big). \end{gathered} \tag{2}$$

Theorem 8.1. *Suppose that the operators $g, h : G_1 \times G_2 \to E$ are defined as in (2). Then:*

(1) g is subadditive whenever $f_1, \dots, f_N$ are subadditive and $\varphi \in \mathscr{A}(\mathbb{R}_+^{\mathbb{N}})$ is increasing and convex;

(2) h is superadditive whenever $f_1, \dots, f_N$ are superadditive, and $\varphi \in \mathscr{A}(\mathbb{R}_+^{\mathbb{N}})$ is increasing and concave.

Remark 8.2. If the hypotheses of 3 (i–iii) are fulfilled for some fixed $u_1, v_1 \in G_1$ and $u_2, v_2 \in G_2$ then the inequalities hold:

$$\begin{aligned} g(u_1 + v_1, u_2 + v_2) &\leqslant g(u_1, u_2) + g(v_1, v_2), \\ h(u_1 + v_1, u_2 + v_2) &\geqslant h(u_1, u_1) + h(v_1, v_2). \end{aligned}$$

Remark 8.3. An f-algebra E can be identified with $\mathrm{Orth}(E)$ if and only if E has a unit element. Thus, Theorem A remains valid if E is a uniformly complete unitary f-algebra and the set-valued map $\mathscr{F} : G \to \mathscr{P}(E_+)$ satisfies the condition 3 (i–iii) with $\mathrm{Orth}(E)$ replaced by E. Moreover, we can take $\varphi \in \mathscr{B}_0(\mathbb{R}_+^{\mathbb{N}})$ in subadditive case and $\varphi \in \mathscr{B}(\mathbb{R}_+^{\mathbb{N}})$ with $\varphi(0) \geqslant 0$ in superadditive case, see [21, Theorem 4.12].

Corollary 8.4. *Let $G := G_1 = G_2$, $g_0(u) := g(u, u)$, $h_0(u) := h(u, u)$ for all $u \in G$, and $\varphi \in \mathscr{A}(\mathbb{R}_+^{\mathbb{N}})$. Then $g_0 : G \to E$ is subadditive whenever $f_1, \dots, f_N$ are subadditive and φ is increasing and convex, while $h_0 : G \to E$ is superadditive whenever $f_1, \dots, f_N$ are superadditive, and φ is increasing and concave.*

Remark 8.5. Corollary 8.4 in the particular case of $E = \mathbb{R}$ was obtained by Persson [107, Theorems 1 and 2]. Of course, Corollary 8.4 is equivalent to Theorem A. Indeed, under the hypotheses of Theorem A put $G := G_1 \times G_2$, $u = (u_1, u_2) \in G_1 \times G_2$, $\widetilde{f_i}(u) := f_i(u_1)$ $(i := 1, \dots, N)$, $\widetilde{\mathscr{F}}(u) := \mathscr{F}(u_2)$. If $f_1, \dots, f_N$ are subadditive then $\widetilde{f}_1, \dots, \widetilde{f}_N$ are also subadditive and $g_0 : G \to E$ is subadditive by Corollary 8.4. Since $g_0(u) := \inf_{\pi \in \widetilde{\mathscr{F}}(u)} \big\{\pi\widehat{\varphi}\big(\pi^{-1}\widetilde{\mathbf{f}}(u)\big)\big\} = g(u_1, u_2)$, we get the subadditivity of g. Similar argument work for the superadditive case.

For a single-valued map $\mathscr{F}(x) = \{f_0(x)\}$ $(x \in G)$ with $f_0 : G \to \mathrm{Orth}(E)_+$ we have the following particular case of the above Theorem, see [79]. Denote $\frac{x}{\pi} := \pi^{-1}x$.

Corollary 8.6. *Suppose that $f_1, \dots, f_N$ are subadditive, $f_0 : G \to \mathrm{Orth}(E)_+$ is superadditive, and $f_0(u)$ is invertible in $\mathrm{Orth}(E)$ for every $u \in G$. Then, given an increasing continuous convex function $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$ and $u_1, v_1 \in G_1$, $u_2, v_2 \in G_2$, the inequality holds:*

$$f_0(u_2+v_2)\widehat{\varphi}\left(\frac{\mathbf{f}(u_1+v_1)}{f_0(u_2+v_2)}\right) \leqslant f_0(u_2)\widehat{\varphi}\left(\frac{\mathbf{f}(u_1)}{f_0(u_2)}\right) + f_0(v_2)\widehat{\varphi}\left(\frac{\mathbf{f}(v_1)}{f_0(v_2)}\right). \tag{3}$$

The reverse inequality holds in (3) *whenever $f_0, f_1, \dots, f_N$ are superadditive, and φ is an increasing concave function.*

Putting $G := G_1 = G_2$ $u := u_1 = u_2$, and $v := v_1 = v_2$, we get, see [105] and [107].

Corollary 8.7. *Under the hypotheses of Corollary* 2 *the Peetre–Persson inequality holds:*

$$f_0(u+v)\widehat{\varphi}\left(\frac{\mathbf{f}(u+v)}{f_0(u+v)}\right) \leqslant f_0(u)\widehat{\varphi}\left(\frac{\mathbf{f}(u)}{f_0(u)}\right) + f_0(v)\widehat{\varphi}\left(\frac{\mathbf{f}(v)}{f_0(v)}\right). \tag{4}$$

If $f_0, f_1, \dots, f_N$ are superadditive and φ is increasing and concave, then (4) *is reversed.*

Remark 8.8. Corollary 8.7 covers the "single-valued case" by Peetre and Persson [105]. A short history of the Beckenbach–Dresher inequality is presented in [117]. Some instances of the inequality are also addressed in [99, 104].

9. Jensen Type Inequalities for Positive Multilinear Operators

Maligranda [97, Theorem 1] proved that, given a positive bilinear operator T from $E \times F$ to $L^0(\Omega, \Sigma, \mu)$ with ideal spaces E and F on measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ respectively, the inequality

$$T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) \leqslant C\varphi(T(|x_0|, |y_0|), T(|x_1|, |y_1|))$$

holds for any $x_0, x_1 \in E$ and $y_0, y_1 \in F$, provided that the parameters φ, φ_0, φ_1 are nonnegative concave positively homogeneous continuous functions on $\mathbb{R}^2_+$ and $\varphi_0(1, s)\varphi_1(1, t) \leqslant C\varphi(1, st)$ for some $C > 0$ and all $s, t > 0$. Then it was used to prove an interpolation theorem for positive bilinear operators on Calderón–Lozanovskiĭ spaces [97], see also [101].

The aim of this section is to present a generalization of the mentioned Maligranda result for positive multilinear operators between uniformly complete vector lattices with a broader class of parameter functions.

Definition 9.1. Recall that the *Hadamard product* $\mathbf{s} \circ \mathbf{t}$ of vectors $\mathbf{s} := (s_1, \dots, s_m) \in \mathbb{R}^m$ and $\mathbf{t} := (t_1, \dots, t_m) \in \mathbb{R}^m$ is defined as $\mathbf{s} \circ \mathbf{t} := (s_1 t_1, \dots, s_m t_m)$. A tuple $\bar{K} := (K_0, K_1, \dots, K_m)$ of conic sets $K_0, K_1, \dots, K_m$ in $\mathbb{R}^N$ is said to be *multiplicative* if $\mathbf{s}_1 \circ \ldots \circ \mathbf{s}_m \in K_0$ for all $\mathbf{s}_j \in K_j$ $(j := 1, \dots, m)$. A tuple $(\varphi_0, \varphi_1, \dots, \varphi_m)$ is called *submultiplicative* (*supermultiplicative*) on $\bar{K}$, if $K_i \subset \mathrm{dom}(\varphi_i)$ for all $i := 0, 1, \dots, m$ and

$$\varphi_0(\mathbf{s}_1 \circ \ldots \circ \mathbf{s}_m) \leqslant \varphi_1(\mathbf{s}_1) \cdot \ldots \cdot \varphi_m(\mathbf{s}_m)$$
$$\big(\varphi_0(\mathbf{s}_1 \circ \ldots \circ \mathbf{s}_m) \geqslant \varphi_1(\mathbf{s}_1) \cdot \ldots \cdot \varphi_m(\mathbf{s}_m)\big)$$

for all $\mathbf{s}_j \in K_j$ $(j := 1, \dots, m)$.

Definition 9.2. Let $E_1,\dots,E_m$ and G be vector lattices. A multilinear (m-linear) operator $S: E_1\times\ldots\times E_m\to G$ is called *positive,* if $S(x_1,\dots,x_m)\geqslant 0$ for all $0\leqslant x_i\in E_i$ $(i=1,\dots,m)$.

Theorem 9.3. *Let $E_1,\dots,E_m$ and G be uniformly complete vector lattices, $x_{1,j},\dots,x_{N,j}\in E_j$ $(j=1,\dots,m)$. Suppose that $\bar K=(K_0,K_1,\dots,K_m)$ is a multiplicative tuple of conic sets in $\mathbb{R}^N$ and consider two tuples $(\varphi_0,\varphi_1,\dots,\varphi_m)$ and $(\psi_0,\psi_1,\dots,\psi_m)$ with $\varphi_j,\psi_j\in\mathscr{H}(K_j)$ $(j=1,\dots,m)$, $\varphi_0\in\mathscr{H}_\wedge(\mathbb{R}^N,K_0)$ and $\psi_0\in\mathscr{H}_\vee(\mathbb{R}^N,K_0)$ which are supermultiplicative and submultiplicative on $\bar K$, respectively. Let $[x_{1,j},\dots,x_{N,j}]\subset K_j$, for all $j=1,\dots,m$. Then for every positive m-linear operator $B: E_1\times\ldots\times E_m\to G$ with $[B(x_{1,1},\dots,x_{1,m}),\dots,B(x_{N,1},\dots,x_{N,m})]\subset K_0$ we have*

$$
\begin{aligned}
&B\big(\widehat\varphi_1(x_{1,1},\dots,x_{N,1}),\dots,\widehat\varphi_m(x_{1,m},\dots,x_{N,m})\big)\leqslant\\
&\qquad\leqslant\widehat\varphi_0\big(B(x_{1,1},\dots,x_{1,m}),\dots,B(x_{N,1},\dots,x_{N,m})\big),\\
&B\big(\widehat\psi_1(x_{1,1},\dots,x_{N,1}),\dots,\widehat\psi_m(x_{1,m},\dots,x_{N,m})\big)\geqslant\\
&\qquad\geqslant\widehat\psi_0\big(B(x_{1,1},\dots,x_{N,1}),\dots,B(x_{N,1},\dots,x_{N,m})\big).
\end{aligned}
$$

Remark 9.4. Two different proofs of Maligranda's inequality are presented in [97]. The first stemming from Astashkin [2] starts with the simple case of step functions and then employs a density argument. The second one uses the specific lower envelope representation of Calderón–Lozanovskiĭ concave functions. Our approach involves different tools: it rely upon extended homogeneous functional calculus [69, 70] and Fremlin's tensor product of Archimedean vector lattices [39]. Details can be found in [76].

10. Inequalities for Bilinear Operators: A Transfer Principle

In this section we present a *transfer principle* from [73] which enables us to transform inequalities with semi-inner products to inequalities containing positive semidefinite symmetric bilinear operators with values in a vector lattice. Concerning bilinear operators on vector lattices see [17, 20, 67, 72].

Definition 10.1. Let E, F, and G be vector lattices and X be a real vector space. A bilinear operator $B: X\times X\to G$ is said to be *symmetric* if $B(x,y)=B(y,x)$ for all $x,y\in X$ and *positive semidefinite* if $B(x,x)\geqslant 0$ for every $x\in X$. A *semi-inner product* on X is a positive semidefinite symmetric form $(\cdot,\cdot): X\times X\to\mathbb{R}$. A bilinear operator $B: E\times E\to G$ is called *orthosymmetric* if $|x|\wedge|y|=0$ implies $B(x,y)=0$ for all $x,y\in E$ or, equivalently, $B(|x|,|x|)=B(x,x)$ $(x\in E)$, see [17, 20, 67].

Denote $I:=\{1,\dots,n\}$, $J:=\{1,\dots,m\}$, and $N=mn$. Fix a bijection σ from $I\times J$ onto $\{1,\dots,N\}$. Consider positively homogeneous continuous mappings $\Phi,\Phi':\mathbb{R}^N\to\mathbb{R}^k$, $\Psi,\Psi':\mathbb{R}^N\to\mathbb{R}^l$ and denote ($\mathbf{t}\in\mathbb{R}^N$ and $\mathbf{u}\in E^N$):

$$
\begin{gathered}
\Phi(\mathbf{t})=(\varphi_1(\mathbf{t}),\dots,\varphi_k(\mathbf{t})),\quad \Phi'(\mathbf{t})=(\varphi_1'(\mathbf{t}),\dots,\varphi_k'(\mathbf{t})),\\
\Phi(\mathbf{t})\Phi'(\mathbf{t}):=(\varphi_1(\mathbf{t})\varphi_1'(\mathbf{t}),\dots,\varphi_k(\mathbf{t})\varphi_k'(\mathbf{t})),\\
\widehat\Phi(\mathbf{u})\circ\widehat\Phi'(\mathbf{u}):=(\widehat\varphi_1(\mathbf{u})\circ\widehat\varphi_1'(\mathbf{u})\dots,\widehat\varphi_k(u)\circ\widehat\varphi_k'(u)).
\end{gathered}
$$

Similar meaning have the symbols $\Psi(\mathbf{t})\Psi'(\mathbf{t})$ and $\widehat{\Psi}(\mathbf{u}) \circ \widehat{\Psi}'(\mathbf{u})$. Clearly, φ_i, φ_i' $(1 \leqslant i \leqslant k)$ and ψ_j, ψ_j' $(1 \leqslant j \leqslant l)$ are positively homogeneous continuous real valued functions on $\mathbb{R}^N$.

Theorem 10.2 (Transfer Principle)**.** *Let E and F be uniformly complete vector lattices, X be a real vector space, and $x_i, y_j \in X$ for all $i \in I$ and $j \in J$. Assume that $\varphi \in \mathscr{H}_\wedge(\mathbb{R}^k)$ and $\psi \in \mathscr{H}_\vee(\mathbb{R}^k)$. If for any semi-inner product $(\cdot,\cdot)$ on X the inequality*

$$\psi\big(\Psi(\mathbf{t})\Psi'(\mathbf{t})\big) \leqslant \varphi\big(\Phi(\mathbf{t})\Phi'(\mathbf{t})\big)$$

holds with $\mathbf{t} = (t_1, \dots, t_N) \in \mathbb{R}^N$, $t_{\sigma(i,j)} := (x_i, y_j)$ $\big((i,j) \in I \times J\big)$, then for any positive semidefinite symmetric bilinear operator $\langle\cdot,\cdot\rangle$ from $X \times X$ to E and any positive orthosymmetric bilinear operator $\circ : E \times E \to F$ the inequality

$$\widehat{\psi}\big(\widehat{\Psi}(\mathbf{u}) \circ \widehat{\Psi}'(\mathbf{u})\big) \leqslant \widehat{\varphi}\big(\widehat{\Phi}(\mathbf{u}) \circ \widehat{\Phi}'(\mathbf{u})\big)$$

holds with $\mathbf{u} = (u_1, \dots, u_N) \in E^N$, $u_{\sigma(i,j)} := \langle x_i, y_j\rangle$ $\big((i,j) \in I \times J\big)$. If the equality holds in semi-inner product case and $\circ$ is a lattice bimorphism, then equality holds also in the case of symmetric positive semidefinite bilinear operators.

Remark 10.3. The inequalities in Theorem 10.2 may contain in both parts arbitrary finite but equal number of factors with respect to $\cdot$ and $\circ$, see [73]. One can also state and prove Theorem 10.2 with positively homogeneous functions defined on some conic sets of finite-dimensional space using the extended homogeneous functional calculus. But in this case the necessary compatibility conditions become too awkward.

Remark 10.4. To produce a new inequality for bilinear operators by means of the above Transfer Principle one have only to analyze the structure of a given inequality for semi-inner product identifying in it the functions φ, ψ, φ_i, φ_i', ψ_i, and ψ_i' and rewriting it, if there is a need, in an appropriate form.

Corollary 10.5. *Let X be a real vector space E be a vector lattice, and $\langle\cdot,\cdot\rangle$ be a positively semidefinite symmetric bilinear operator from $X \times X$ to E. Let F be another vector lattice and $\circ : E \times E \to F$ be a positive orthosymmetric bilinear operator. Then the following general form of the classical Cauchy–Bunyakowski inequality holds:*

$$\langle x, y\rangle \circ \langle x, y\rangle \leqslant \langle x, x\rangle \circ \langle y, y\rangle \quad (x, y \in X).$$

Remark 10.6. Corollary 10.5 was announced in [65] and proved in [20, Theorem 3.8], see also [17, Theorem 32]. Previously, Huijsmans and de Pagter [52] proved the inequality with $\circ$ replaced by the multiplication of a semiprime f-algebra $E = F$. The semiprimeness assumption was removed by Bernau and Huijsmans [8] and the result was established for any almost f-algebra E by Buskes and van Rooij [22].

Remark 10.7. Corollary 10.5 is the simplest particular case of Theorem 10.3, since only coordinate functions are used: $\varphi_1(t) = dt_1$, $\varphi_1'(t) = dt_2$, $\psi_1(t) = dt_3$, $\psi_1'(t) = dt_4$ $(t = (t_1, t_2, t_3, t_4))$ with $t_1 := (x, x)$, $t_2 := (y, y)$, and $t_3 = t_4 := (x, y)$. Nevertheless, the proof of the Transfer Principle uses exactly the same tools as needed for the proof of this very special case (cp. [20, Theorem 3.8]). Several illustrative examples demonstrating the strength of the Transfer Principle are considered in [73]. A great deal of inequalities admitting generalization by means of the Transfer Principle is spread everywhere in the literature, see [33, 34, 35, 99, 114].

Part II. Multilinear Operators and Polynomials in Vector Lattices

In this part we collect three sorts of results: 1) Radon–Nikodým type theorems and the Maharam extension of orthosymmetric multilinear operators, 2) the results on representation and extension of orthogonally additive polynomials on vector lattices, and 3) Strassen type result on the existence of a multilinear operator with given marginals.

1. Introduction

Polynomials in infinitely many variables or polynomials defined on infinite-dimensional spaces have been explored since the late of 1800's. The study of polynomials on vector lattices is of more recent date.

It was shown in [29] that the space of s-homogeneous polynomials on an infinite dimensional Banach space with an unconditional basis does not have an unconditional basis. Later it was discovered in [42] that homogeneous polynomials with unconditionally convergent monomial expansions coincide with the homogeneous polynomials that are regular with respect to the Banach lattice structure of the domain. A polynomial is regular if it is representable as the difference of two positive polynomials, while the positivity means that the generating symmetric multilinear operator is positive. From this starting point an increasing attention is attracted to order properties of polynomials.

The study of bilinear operators within the framework of the theory of vector lattice originated about sixty years ago. Some historical remarks, main trends, and results are reflected in survey papers [17] and [68]. In [112] the class of orthogonally additive homogeneous polynomials on Banach lattice was introduced. An inseparable companion of orthogonal additivity turns out to be orthosymmetry introduced in [22]. The aim of this part is to provide new information about the order structure of orthogonally additive homogeneous polynomials and orthosymmetric multilinear operators. One of the main tools is the following: each bounded orthogonally additive homogeneous polynomial acting from an Archimedean vector lattice into a separated convex bornological space can be represented as the composite of an exponentiation-like mapping and a bounded linear operator, provided that the bornological space is complete or the vector lattice is uniformly complete.

The unexplained terms of use below can be found in [1, 64, 78, 98]. All vector lattices in this paper are assumed real and Archimedean.

2. Orthosymmetry

Definition 2.1. A multilinear operator $\varphi : E^s \to F$ is called *positive* if $\varphi(x_1, \dots, x_s) \geqslant 0$ for all $0 \leqslant x_1, \dots, x_s \in E$ and *orthoregular* if it can be written as the difference of two positive orthosymmetric operators. We say that φ is *orthosymmetric*, if $\varphi(x_1, \dots, x_s) = 0$ provided that $|x_i| \wedge |x_j| = 0$ for some pair of indices $i \neq j$ and *symmetric*, if $\varphi(x_1, \dots, x_s) = \varphi(x_{\sigma(1)}, \dots, x_{\sigma(s)})$ for all $x_1, \dots, x_s \in E$ and every permutation σ of $\{1, \dots, s\}$.

Definition 2.2. An s-linear operator $\varphi : E_1 \times \dots \times E_s \to F$ is called a *lattice s-morfism* if for every $k = 1, \dots, s$ and for all $x_i \in E_i^+$ $(i = 1, \dots, k-1, k+1, \dots, s)$

the map $x_k \mapsto \varphi(x_1, \dots, x_k, \dots, x_s)$ $(x_k \in E_k)$ is a lattice homomorphism (x_0 and x_{s+1} should be omitted).

Definition 2.3. Let $2 \leqslant s \in \mathbb{N}$ and E be an Archimedean vector lattice. The pair $(E^{s\odot}, \odot_s)$ is called an *s-power* of E if

(1) $E^{s\odot}$ is a vector lattice;

(2) $\odot_s : E \times \cdots \times E \to E^{s\odot}$ is a symmetric lattice s-morphism;

(3) for any Archimedean vector lattice F and every symmetric lattice s-morphism $\varphi : E \times \cdots \times E \to F$ there exists a unique lattice homomorphism $S : E^{s\odot} \to F$ such that $S \circ \odot_s = \varphi$.

Remark 2.4. Definition 2.3 was introduced by K. Boulabiar and G. Buskes [14]. They also established that the s-power exists for any Archimedean vector lattice. The case $s = 2$ was examined earlier by G. Buskes and A. van Rooij [23].

Theorem 2.5. *Let $s \in 2, 3, \dots$ and E be a vector lattice. Then E has a unique (up to a lattice isomorphism) s-power $(E^{s\odot}, \odot_s)$.*

Definition 2.6. A *bornology* on a set X is an increasing (relative to $\subset$) filter $\mathfrak{B}$, the elements of which form covering of X. In this case the sets of $\mathfrak{B}$ are called *bounded.* A *base* of a bonology $\mathfrak{B}$ on X is any basis of the filter $\mathfrak{B}$. A map acting between the sets with bornology is called *bounded* if the image of any bounded set is bounded.

Definition 2.7. The bornology on a vector space E over $\mathbb{K}$ is called *vector bornology* if the maps $E \times E \ni (x, y) \mapsto x + y \in E$ and $\mathbb{K} \times E \ni (\lambda, x) \mapsto \lambda x \in E$ are bounded. A *bornological vector space* is a pair $(E, \mathfrak{B})$, consisting of a vector space E and a vector bornology $\mathfrak{B}$ on E. A bornological vector space $(E, \mathfrak{B})$, whose bornology $\mathfrak{B}$ is stable under the formation of convex hulls will be called *a convex bornological space.*

Let $L_b(E, F)$ stands for the set of bounded linear operators, acting from E to F, and $L_b(^sE, F)$ stands for the set of bounded orthosymmetric s-linear operators, acting from E^s to F. Further details concerning bornological spaces can be found in [11].

Theorem 2.8. *Let E be a uniformly complete vector lattice, F a convex bornological space or a locally convex space, and $\varphi : E \times \cdots \times E \to F$ a bounded orthosymmetric s-linear map. Then the map $T_\varphi : E^{s\odot} \to F$ defined by*

$$T_\varphi(x) = \varphi(x, |x|, \dots, |x|) \quad (x \in E)$$

is a unique bounded linear map with $\varphi = T_\varphi \circ \odot_s$. The correspondence $\varphi \mapsto T_\varphi$ is an isomorphism of $L_b(^sE, F)$ onto $L_b(E, F)$.

Remark 2.9. This theorem extends the result which was established in [14, Theorem 5.1] by K. Boulabier and G. Buskes for positive orthosymmetric multilinear operators, provided that E and F are uniformly complete vector lattices. It is easy to see that the given arguments work also for any order bounded orthosymmetric operator.

Theorem 2.10. *Let E be a vector lattice and F be a convex bornological space or locally convex space. Then any bounded orthosymmetric multilinear map from E^s to F is symmetric.*

Remark 2.11. This theorem was established in [16, Theorem 2] for the case when E and F are Archimedean vector lattices and the multilinear map is positive.

The case of positive and bounded bilinear operators in vector lattices is considered in [22, Colorrary 2] and [20, Theorem 3.4], respectively. In [9, Theorem 14] it is shown that when E and F are vector lattices the condition of order boundedness of bilinear operator can be weaken to continuity with respect to uniform convergence.

3. Orthogonally Additive Polynomials

The class of orthogonally additive polynomials was introduce by Sundaresan [112] and received much attention [10, 25, 53, 106]. It turned out that, very often, a polynomial is orthogonally additive if and only if its generating multilinear operator is orthosymmetric. To state the result we need some definition.

Definition 3.1. Let E and F be vector spaces and s an integer $\geqslant 1$. The map $P : E \to F$ is called a *homogeneous polynomial of degree* s (or an *s-homogeneous polynomial*), if there exists an s-linear map $\varphi : E^s \to F$, such that $P(x) = \varphi(x, \dots, x)$ for all $x \in E$. Thus, a homogeneous polynomial P of degree s admits a representation $P = \varphi \circ D_s$, where $D_s : E \to E^s$ is a diagonal imbedding $E \ni x \mapsto (x, \dots, x) \in E^s$. Let us agree that a homogeneous polynomial of degree $s = 0$ is a constant map $e \mapsto f \in F$ $(e \in E)$.

Definition 3.2. For a homogeneous polynomial $P : E \to F$ of degree s there exists a unique symmetric multilinear map $\varphi : E^s \to F$, called *a generating* map, such that $P(x) = \varphi(x, \dots, x)$. The generating map $\varphi : E^s \to F$ can be represented as

$$\varphi(x_1, \dots, x_s) = \frac{1}{s!}\Delta_{x_s}\Delta_{x_{s-1}} \dots \Delta_{x_1}P,$$

where Δ_h is the *difference* operator defined as $\Delta_h f(x) = f(x+h)-f(x)$, see [32, 84]. Moreover, $\Delta_{h_1,\dots,h_k}P(x) = 0$ for all $x \in E$, if $k > s$ (see [84, Lemma 15.9.2]).

Definition 3.3. A homogeneous polynomial is said to be *positive* if the generating symmetric multilinear operator is positive and *regular* if it is representable as the difference of two positive polynomials.

Definition 3.4. A map $P : E \to F$ is a *polynomial* (not necessary homogeneous) *of degree* $\leqslant s$, if there exists an integer s and k-homogeneous polynomial P_k $(k = 0, 1, \dots, s)$ such that $P = P_0 + P_1 + \dots + P_s$. Here $P_0 = \mathrm{const}$ and P_1 is a linear operator.

Definition 3.5. Let E be a vector lattice. A polynomial P defined on E is said to be *orthogonally additive* if $P_0(x + y) = P_0(x) + P_0(y)$ for any disjoint $x, y \in E$, where $P_0(x) := P(x) - P(0)$. Clearly, a homogeneous polynomial P is orthogonally additive if $P(x + y) = P(x) + P(y)$ for disjoint $x, y \in E$. Recall that two elements $x, y \in E$ are said to be *disjoint* if $|x| \wedge |y| = 0$.

Definition 3.6. A bornological space $(E, \mathfrak{B})$ is called *separated* if $\{0\}$ is the only bounded vector subspace of E.

Proposition 3.7. *Let E be a vector lattice and F a separated bornological vector space. A polynomial $P : E \to F$, $P = P_0+P_1+\dots+P_s$ is orthogonally additive if and only if the homogeneous polynomials P_k $(k = 1, \dots, s)$ are orthogonally additive.*

Theorem 3.8. *Let E be a vector lattice and F be a convex bornological space. A bounded s-homogeneous polynomial $P : E \to F$ is orthogonally additive if and only if its generating s-linear map $\varphi : E^s \to$ is orthosymmetric.*

4. Radon–Nikodým Type Theorems for Multilinear Operators and Polynomials

Definition 4.1. Let E and G be vector lattices and let B be a positive multilinear operator from E^s into G. Say that B is *order interval preserving* or possesses the *Maharam property* if, for every $x_1, \dots, x_s \in E_+$ and $0 \leqslant g \leqslant B(x_1, \dots, x_s) \in G_+$, there exist $0 \leqslant u_1 \leqslant x_1, \dots, 0 \leqslant u_s \leqslant x_s$ such that $g = B(u_1, \dots, u_s)$ or, in short, $B\,(\,[0, x_1] \times \dots \times [0, x_s]\,) = [0, B(x_1, \dots, x_s)]$ for all $x_1, \dots, x_s \in E_+$. A positive order continuous multilinear operator with the Maharam property is called a *multilinear Maharam operator*.

Remark 4.2. In [15] the notion of almost right (or left) interval preserving bilinear operator was considered and a bilinear version of Arendt's theorem on duality between lattice homomorphisms and interval preserving operators was proved [15, Theorem 14], cf. [1, Theorem 7.4].

Definition 4.3. Let A be another positive multilinear operator from E^s into G. Then B is said to be *absolutely continuous* with respect to A whenever $B(x_1, \dots, x_s) \in A(x_1, \dots, x_s)^{\perp\perp}$ for all $0 \leqslant x_1, \dots, x_s \in E$. Evidently, any $B \in A^{\perp\perp}$ is absolutely continuous with respect to A.

Theorem 4.4. *Let E and G be Dedekind complete vector lattices, $B : E^s \to G$ be a positive orthosymmetric s-linear operator and $B = \Phi_B \odot_s$ for a uniquely defined positive linear operator $\Phi_B : E^{s\odot} \to G$. The following conditions are equivalent:*

(1) *B is order interval preserving.*

(2) *For at least one $i := 1, \dots, s$ the map $x \mapsto B(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_s)$ is order interval preserving for all $0 \leqslant x_k \in E$, $k \neq i$ (x_0 and x_{s+1} are omitted).*

(3) *For any $0 \leqslant x \in E$ and $0 \leqslant g \leqslant B(x, \dots, x)$ there exists $u \in E$, $0 \leqslant u \leqslant x$ such that $g = B(u, \dots, u)$.*

(4) *Φ_B is order interval preserving.*

Theorem 4.5. *Let E, G, B, and Φ_B be as in 4.4. Then B is a Maharam operator if and only if Φ_B is a Maharam operator.*

Making use of Theorems 4.4 and 4.5 we are able to transfer some results from linear Maharam operators to orthosymmetric multilinear Maharam operators.

Theorem 4.6 (Multilinear Radon–Nikodým Theorem)**.** *Let E and G be Dedekind complete vector lattices and let B and A be orthosymmetric order continuous positive multilinear operators from E^s to G with A possessing the Maharam property. Then the following assertions are equivalent:*

(1) $B \in \{A\}^{\perp\perp}$.

(2) *B is absolutely continuous with respect to A.*

(3) *There exists an orthomorphism $0 \leqslant \rho \in \mathrm{Orth}^\infty(E)$ such that*

$$B(x_1, \dots, x_s) = A(x_1, \dots, x_{i-1}, \rho x_i, x_{i+1}, \dots, x_s) \quad (x_1, \dots, x_s \in \mathscr{D}(\rho)).$$

(4) *There exists an increasing sequence of positive orthomorphisms (ρ_n), $\rho_n \in \mathrm{Orth}(E)$, such that for any $i := 1, \dots, s$ the representation holds:*

$$B(x_1, \dots, x_s) = \sup_n A(x_1, \dots, x_{i-1}, \rho x_i, x_{i+1}, \dots, x_s) \quad (x_1, \dots, x_s \in E_+).$$

Theorem 4.7 (Polynomial Radon–Nikodým Theorem)**.** *Let E and G be Dedekind complete vector lattices and let P and Q be order continuous positive* *orthogonally additive polynomials from E to G with Q possessing the Maharam property. Then the following assertions are equivalent:*

(1) $P \in \{Q\}^{\perp\perp}$.

(2) *P is absolutely continuous with respect to Q.*

(3) *There exists an orthomorphism $0 \leqslant \rho \in \mathrm{Orth}^{\infty}(E)$ such that*

$$P(x) = Q(\rho x) \quad (x \in \mathscr{D}(\rho)).$$

(4) *There exists an increasing sequence of positive orthomorphisms (ρ_n), $\rho_n \in \mathrm{Orth}(E)$, such that the representation holds:*

$$P(x) = \sup_n Q(\rho x) \quad (x \in E_+).$$

Remark 4.8. The class of linear Maharam operators was first studied by D. Maharam in [95] (see also the survey paper [96]). W. A. J. Luxemburg and A. R. Schep [94] extended a portion of Maharam's theory to the case of positive operators in Dedekind complete vector lattices. The terms "Maharam property" and "Maharam operator" were introduced in [94] and [62], respectively (more details see in [64]). The Maharam property transplanted to the entourage of convex operators is presented in [78]. Every linear Maharam operator is an interpretation of some order continuous linear functional in an appropriate Boolean-valued model, see [64]. This Boolean-valued status of the concept of Maharam operator was established in [62]. Bilinear versions of 4.3–4.6 were proved in [72]. Theorem 4.7 was obtained by Z. A. Kusraeva.

5. Maharam Extension of Orthosymmetric Multilinear Operators

The Maharam extension and its functional representation (well known in the linear case, see [64, Sections 4.5 and 6.3] and [93]) can be developed for orthosymmetric positive multilinear operators. It was done in [116] by B. B. Tasoev. Denote by $\mathbb{P}(E)$ the Boolean algebra of band projections in E.

Definition 5.1. A positive multilinear operator $B : E^s \to F$ is called *strictly positive* if $B(|x|, \dots, |x|) = 0$ implies $x = 0$ for every $x \in E$.

Theorem 5.2. *Let E and F be vector lattices with F Dedekind complete. Let B be a strictly positive orthosymmetric multilinear operator from E^s into F. Then there exist a Dedekind complete vector lattice $\overline{E}$, a lattice isomorphism j from E into $\overline{E}$, and a multilinear Maharam operator $\overline{B} : \overline{E}^s \to F$ such that the following hold:*

(1) $B(x_1, \dots, x_s) = \overline{B}(jx_1, \dots, jx_s) \ \ (x_1, \dots, x_s \in E)$.

(2) *The order ideal in $\overline{E}$ generated by $j(E)$ coincides with $\overline{E}$.*

(3) *There is an f-algebra isomorphism $h : \mathrm{Orth}(F) \to \mathrm{Orth}(\overline{E})$ such that*

$$\pi B(x_1, \dots, x_s) = \overline{B}(jx_1, \dots, jx_{i-1}, h(\pi)jx_i, jx_{i+1}, \dots, jx_s)$$
$$(i := 1, \dots, s;\ x_1, \dots, x_s \in E;\ \pi \in \mathrm{Orth}(F)_+).$$

(4) *E is dense in $\overline{E}$ in the sense that for all $z \in \overline{E}$ and $0 < \varepsilon \in \mathbb{R}$ there is $z_\varepsilon \in \overline{E}$, a partition $(\pi_\xi) \subset \mathbb{P}(F)$ of the projection $[\overline{B}(z, z)] \in \mathbb{P}(F)$, and a family $(x_\xi) \subset E$*

such that

$$\overline{B}(z_\varepsilon, |z_\varepsilon|, \dots, |z_\varepsilon|) = o\text{-}\sum \pi_\xi \overline{B}(jx_\xi, |jx_\xi|, \dots, |jx_\xi|),$$
$$\big|\overline{B}(z_\varepsilon, |z_\varepsilon| \dots, |z_\varepsilon|) - \overline{B}(z, |z|, \dots, |z|)\big| \leqslant \varepsilon \overline{B}(|z|, \dots, |z|).$$

Theorem 5.3. *Let E, $\overline{E}$, F, B, $\overline{B}$, and j be as in Theorem 5.1. For any operator $D \in \{B\}^{\perp\perp}$ there exists a unique operator $\overline{D} \in \{\overline{B}\}^{\perp\perp}$ such that $D(x_1, \dots, x_s) = \overline{D}(jx_1, \dots, jx_s)$ for all $x_1, \dots, x_s \in E$. The correspondence $D \mapsto \overline{D}$ is a lattice isomorphism of Dedekind complete vector lattices $\{B\}^{\perp\perp}$ and $\{\overline{B}\}^{\perp\perp}$.*

Let A be a nonempty set, $\mathscr{A}$ a σ-algebra of its subsets, and $\mathscr{N}$ a σ-ideal in $\mathscr{A}$. Let $M(A, \mathscr{A}, \mathscr{N})$ be the space of cosets of measurable functions on A. We will suppose that the measurable space $(\mathscr{A}, \mathscr{N})$ is of *countable type;* i. e., an arbitrary family $(A_\alpha) \subset \mathscr{A} \backslash \mathscr{N}$ with $A_\alpha \cap A_\beta \in \mathscr{N}$ $(\alpha \neq \beta)$ is at most countable. In this event $M(A, \mathscr{A}, \mathscr{N})$ is an order complete vector lattice. Let F be an order-dense ideal in $M(A, \mathscr{A}, \mathscr{N})$. A sequence $(A_n) \subset \mathscr{A}$ of pairwise disjoint sets is called a *partition of a measurable set* $A_0 \in \mathscr{A}$ if $\chi_{A_0} = \sup \chi_{A_n}$ in F, where χ_C stands for the characteristic function of C.

Let P be a σ-compact topological space. Denote by $\mathscr{A} \times \mathscr{B}$ the σ-algebra generated by the rectangles $C \times B$ where $B \subset P$ is an arbitrary Baire set and $C \in \mathscr{A}$.

Assume that $\varphi : \mathscr{A} \times \mathscr{B} \to F$ is a positive countably additive measure, $L^0(A \times P, \varphi)$ is the space of (φ-equivalence classes of) almost everywhere finite functions measurable with respect to σ-algebra $\mathscr{A} \times \mathscr{B}$, and $L^1(A \times P, \varphi)$ is the order dense ideal of φ-integrable functions.

Let μ be a regular Borel measure on P, and let $L^1(P, \mu)$ be the vector lattice of cosets of real μ-measurable functions on P. Assume that E is an order dense ideal in $L^1(P, \mu)$ containing the identically one function 1_P. Given $e \in E$, the coset of the function $(s, t) \mapsto e(t)$ $\big((s, t) \in A \times P\big)$ is identified with e. Denote $E^{(s)} := \{e^s : e \in E\}$.

Theorem 5.4. *For any order continuous strictly positive orthosymmetric multilinear operator $B : E^{(s)} \to F$ there exists a unique countably additive ample measure $\varphi : \mathscr{A} \times \mathscr{B} \to F$ such that*

$$B(x_1, \dots, x_s) = \int\limits_{A \times B} x_1(t), \dots, x_s(t)\, d\varphi(s, t) \quad (x_1, \dots, x_s \in E).$$

Moreover, for any order bounded orthosymmetric multilinear operator $D \in \{B\}^{\perp\perp}$ there exists a unique (up to φ-equivalence) φ-measurable function $\mathscr{K}_D$ such that

$$D(x_1, \dots, x_s) = \int\limits_{A \times B} \mathscr{K}_D(s, t) x_1(t), \dots, x_s(t)\ d\varphi(s, t) \quad (x_1, \dots, x_s \in E).$$

The correspondence $D \mapsto \mathscr{K}_D$ is a lattice isomorphism from $\{B\}^{\perp\perp} \subset BL_o^{\sim}(E; F)$ onto an order dense ideal $L_\varphi := \{g \in L^0(A \times P, \varphi) :\ g \cdot \jmath(E) \subset L^1(A \times P, \varphi)\}$ in $L^0(A \times P, \varphi)$.

6. Representation of Orthogonally Additive Polynomials

In this section we present the representation results for orthogonally additive polynomials obtained in [80].

Definition 6.1. *A disk* is an absolutely convex set. Let E be a vector space. A disk $A \subset E$ is called *a comletant disk* if the space $E_A = \bigcup_{k=1}^{\infty} nA$ semi-normed by the gauge of A is a Banach space. A convex bornological space is called *a complete convex bornological space* if its bornology has a base consisting of comletant disks.

We denote the space of bounded linear operators from E to F by $L_b(E, F)$ and the space of E-valued s-homogeneous orthogonally additive bounded polynomials by $\mathscr{P}_o({}^sE, F)$.

Theorem 6.2. *Let E be a uniformly complete vector lattice and F be a convex bornological space. Then for any orthogonally additive order bounded s-homogeneous polynomial $P : E \to F$ there exists a unique bounded linear operator $S : E^{s\odot} \to F$ such that $P = S \circ \odot_s \circ D_s$, i. e.*

$$P(x) = S(x^{s\odot}) = S(\underbrace{x \odot \cdots \odot x}_{s \text{ times}}) \quad (x \in E). \tag{5}$$

Moreover, we have $\mathscr{P}_o({}^sE, F) \simeq L_b(E^{s\odot}, F)$.

Theorem 6.3. *Let E be a vector lattice and F be a complete convex bornological space or quasicomplete locally convex space. Then for any orthogonally additive order bounded s-homogeneous polynomial $P : E \to F$ there exists a unique bounded linear operator S such, that the representation* (5) *holds and $\mathscr{P}_0({}^sE, F) \simeq L_b(E^{s\odot}, F)$.*

Theorem 6.4. *Let E be a uniformly complete vector lattice and F be a convex bornological space. Then for any bounded orthogonally additive polynomial P : $E \to F$ of degree $s \geqslant 1$ there exists a unique collection of bounded linear operators $S_k : E^{k\odot} \to F$ $(k := 1, \dots, s)$ and a constant $S_0 \in F$ such, that*

$$P(x) = S_0 + \sum_{k=1}^{s} S_k(x^{k\odot}) \quad (x \in E). \tag{6}$$

Theorem 6.5. *Let E be a vector lattice and F a complete bornological space or quasicomplete locally convex spaces. Then for any order bounded orthogonally additive polynomial $P : E \to F$ of degree $s \geqslant 1$ there exists a unique collection of order bounded linear operators $S_k : E^{k\odot} \to F$ $(k := 1, \dots, s)$ and a constant $S_0 \in F$ such, that* (6) *holds.*

Corollary 6.6. *Let E and F be vector lattices and at least one of them is uniformly complete. Then for any order bounded orthogonally additive polynomial $P : E \to F$ of degree $s \geqslant 1$ there exists a unique collection of order bounded linear operators $S_k : E^{k\odot} \to F$ $(k := 1, \dots, s)$ and a constant $S_0 \in F$ such that* (6) *holds.*

Remark 6.7. Since for a Banach function lattice E we have $E^{s\odot} = E^{(s)} = \{f^s : f \in E\}$ and $f^{s\odot} = f^s$ for every $f \in E$, Theorem 6.2 extends the representation result [10, Theorem 2.3] by Benyamini Y., Lassalle S., Llavona J. G.

7. Kernel Representation of Orthogonally Additive Polynomials

In this section we present some kernel representation result for homogeneous polynomials obtained in [82].

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be σ-finite measure spaces and $(\Omega, \Sigma, \mu) := (\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2, \mu_1 \otimes \mu_2)$ be their product. Consider ideal function spaces $E \subset L^0(\Omega_1, \Sigma_1, \mu_1)$ and $F \subset L^0(\Omega_2, \Sigma_2, \mu_2)$.

Definition 7.1. Let E be a vector lattice. A homogeneous polynomial P on E is said to be *orthogonally additive* if $P(x+y) = P(x) + P(y)$ for any disjoint $x, y \in E$.

Definition 7.2. A k-homogeneous polynomial $P : E \to F$ is said to admit *a kernel representation,* if there exists a μ-measurable function of two variables $K : \Omega_2 \times \Omega_1 \to \mathbb{R}$, such that for every $x \in E$ for μ_1-almost all $s \in \Omega_2$ the function $t \mapsto K(s,t)x^k(t)$ is μ_1-integrable on Ω_1 and

$$(Px)(s) = \int\limits_{\Omega_1} K(s,t)\, x^k(t)\, d\mu_1(t) \quad (x \in E).$$

Theorem 7.3. *Let $P : E \to F$ be an orthogonally additive k-homogeneous polynomial. Then the following are equivalent:*

(1) *P admits a kernel representation.*

(2) *If $0 \leqslant x_n \leqslant x \in E$ $(n \in \mathbb{N})$ and $x_n \to 0$ in a measure μ_1, then $Px_n \to 0$ μ_2-almost everywhere.*

(3) *P satisfies the following conditions:* (a) *if $\mu_1(B_n) \to 0$ $(B_n \in \Sigma_1)$ and $\chi_{B_n} \leqslant x \in E$ $(n \in \mathbb{N})$, then $P(\chi_{B_n}) \to 0$ μ_2-almost everywhere;* (b) *if $0 \leqslant x_n \leqslant x \in E$ $(n \in \mathbb{N})$ and $x_n \to 0$ μ_1-almost everywhere, then $Px_n \to 0$ μ_2-almost everywhere.*

Theorem 7.4. *Let F be an order continuous Köthe function space. Then every orthogonally additive k-homogeneous polynomial $P : L^k(\Omega_1, \Sigma_1, \mu_1) \to F$ admits a kernel representation.*

Definition 7.5. Let E be a Banach lattice. A k-homogeneous polynomial $P : E \to F$ is said to be *majorizing* if there exists $f \in F_+$ such that $|P(x)| \leqslant f\|x\|^k$ for all $x \in E$.

Definition 7.6. For $1 < p \in \mathbb{R}$ and a Köthe function space E define $(E^{(p)}, \|\cdot\|_p)$ by $E^{(p)} := \{u \in L^0(\Omega_1, \Sigma_1, \mu_1) : |u|^p \in E\}$ and $\|u\|_p := (\||u|^p\|_E)^{1/p}$.

Theorem 7.7. *Let E be an order continuous Köthe function space. Then every orthogonally additive k-homogeneous majorizing polynomial $P : E^{(k)} \to L^0(\Omega_2, \Sigma_2, \mu_2)$ admits a kernel representation.*

Remark 7.8. The proof of Theorem 6.3 can be reduced to Buhvalov's criterion of a kernel representability of linear operators [55, Ch. XI, § 1, Theorem 1] by the use of representation results from § 6. (Bukhvalov's characterization of kernel operators was obtained in [18], see also [19].) Theorems 7.4 and 7.7 are deduced in a similar way from [55, Ch. XI, § 1, Theorem 6] and [55, Ch. XI, § 1, Theorem 5], respectively.

8. Extension of Orthogonally Additive Polynomials

In [91, Theorem 14] Loan established that a positive homogeneous polynomial defined on a majorizing sublattice of a vector lattice with values in a Dedekind complete vector lattice admits an extension to the ambient lattice wich is again a positive homogeneous polynomial. If the polynomial under consideration is orthogonally additive, then its positive extension can be also chosen orthogonally additive. But a stronger statement, the existence of a “simultaneous extension”, can be proved using the representation results from § 6. Moreover, a Buck–Phelps type characterization of extreme extensions of orthogonally additive homogeneous polynomials can be also deduced. These results obtained in [80] are presented below.

Let $\mathscr{P}^r_{\mathrm{oa}}({}^sE, F)$ stands for a set of all regular s-homogeneous orthogonally additive polynomials from E to F (see Definitions 3.3 and 3.5).

Definition 8.1. A sublattice G of a vector lattice E is called *majorizing*, if for every $x \in E$ there exists $g \in G$ such that $x \leqslant g$.

Denote by $\mathscr{R}_p$ the restriction map $P \mapsto P|_G$ from $\mathscr{P}^r_{oa}({}^sE, F)$ onto $\mathscr{P}^r_{oa}({}^sG, F)$.

Theorem 8.2. *Let G be a majorizing sublattice of a vector lattice E and F be a Dedekind complete vector lattice. Then there exists an order continuous lattice homomorphism $\widehat{\mathscr{E}} : \mathscr{P}^r_{\mathrm{oa}}({}^sG, F) \to \mathscr{P}^r_{\mathrm{oa}}({}^sE, F)$ (a "simultaneous extension" operator) such that $\mathscr{R}_p \circ \widehat{\mathscr{E}} = I$, where I is the identity operator in $\mathscr{P}^r_{\mathrm{oa}}({}^sG, F)$.*

Let $P : G \to F$ be a positive orthogonally additive s-homogeneous polynomial. Denote by $\mathscr{E}_+(P)$ a set of all positive orthogonally additive s-homogeneous extensions of P to E. Then $\mathscr{E}_+(P)$ is a convex set. If G is majorizing then $\mathscr{E}_+(P) \neq \varnothing$.

Definition 8.3. An extreme point of $\mathscr{E}_+(P)$ is called an *extreme extension* of P.

Theorem 8.4. *Let E and G be uniformly complete vector lattices, F be a Dedekind complete vector lattice, and G be a sublattice of E. Assume that the set $\mathscr{E}_+(P)$ is nonempty for a positive orthogonally additive s-homogeneous polynomial $P : E \to F$. A polynomial $\widehat{P} \in \mathscr{E}_+(P)$ is an extreme point of $\mathscr{E}_+(P)$ if and only if*

$$\inf \left\{ \widehat{P}\left(\left|(x^s + u^s)^{\frac{1}{s}}\right|\right) : \ u \in G \right\} = 0 \quad (x \in E).$$

Remark 8.5. Theorem 8.2 is a combination of Corollary 6.6 and the existence of "simultaneous extension" operator for linear regular operators, see [64, Theorem 3.4.11]. Theorem 8.4 is proved by reducing to the case of linear positive operator, i. e. to the Lipecki–Plachke–Tomsen Theorem [1, Theorem 2.7] making use of Corollary 6.6. Similar results for positive orthosymmetric bilinear operators see in [20] and [83], respectively.

9. Bilinear Operators with Given Marginals

In his celebrated paper V. Strassen gave a necessary and sufficient condition for the existence of a positive measure with given marginals [115, Theorem 7]. This result was generalized by many authors in different directions, see [50, 56, 57] and the references therein. In [75] an attempt was made to extend Strassen's theorem to multilinear operator with values in a Dedekind complete vector lattice (Kantorovich space). The results are presented below. Consider a vector spaces X, Y and a Dedekind complete vector lattice E. Let $BL(X, Y; G)$ and $\mathbb{P}(E)$ stand for the space of all bilinear operators from $X \times Y$ to G and the Boolean algebra of band projections in E, respectively.

Definition 9.1. A set $\mathscr{U}$ of bilinear operators from $X \times Y$ into E is called *weakly* (*order*) *bounded* if the set $\{B(x, y) : \ B \in \mathscr{U}\}$ is order bounded in E for all $x \in X$ and $y \in Y$, and weakly o-closed if it is close with respect to point-wise order convergence.

Definition 9.2. A partition of unity in $\mathbb{P}(E)$ is a pair-wise disjoint family of band projections $(\pi_\xi)_{\xi \in \Xi}$ with $\sum_{\xi \in \Xi} \pi_\xi = I_E$. If $(B_\xi)_{\xi \in \Xi}$ is a family of operators in $BL(X, Y; E)$ and the operator $B \in BL(X, Y; E)$ is such that $B(x, y) =$

o-$\sum_{\xi\in\Xi}\pi_\xi B_\xi(x,y)$ for all $x\in X$ and $y\in Y$, then B is called the *mixing* of $(B_\xi)_{\xi\in\Xi}$ by a partition of unity $(\pi_\xi)_{\xi\in\Xi}$. Denote by $\mathrm{mix}(\mathscr{U})$ the set of all mixings with $(B_\xi)\subset\mathscr{U}$.

Definition 9.3. The set $\mathscr{U}\subset BL(X,Y;E)$ is called *cyclic* if $\mathscr{U}=\mathrm{mix}(\mathscr{U})$, i. e. $\mathscr{U}$ contains all mixings of its members.

Remark 9.4. A key role in Strassen's proof plays Alaoglu's theorem. Thus, looking for the Strassen type theorems for multilinear operators it is desirable to have some operator versions of Alaoglu's theorem. The approach presented in [75] relay upon the intrinsic characterization of subdifferentials (see Theorem 9.5 below). This fact was established in [79] making use of Boolean valued analysis approach. A standard proof was found in [63]. Details can be found in [78].

Theorem 9.5. *A weakly order bounded set of linear operators is a subdifferential if and only if it is convex, cyclic, and weakly o-closed.*

Making use of Theorem 9.5 and the "linearization" via algebraic tensor product or Fremlin's tensor product we arrive at following two results.

Theorem 9.6. *Let X and Y be vector spaces, while G and H be Dedekind complete vector lattices. Fix $e\in X$, $f\in Y$, $S\in L(X,H)$, and $T\in L(Y,H)$. Assume that $\mathscr{D}$ is a convex, cyclic, weakly closed and weakly bounded subset in $BL(X,Y;G)$ and $Q:G\to H$ is a sublinear Maharam operator. The following are equivalent:*

(1) *There exist $B\in\mathscr{D}$ and $R\in\partial Q$ such that $S=R\circ B(\cdot,f)$ and $T=R\circ B(e,\cdot)$.*

(2) *For any $x\in X$ and $y\in Y$ the inequality holds*

$$Sx+Ty\leqslant\sup_{D\in\mathscr{D}}\Big\{Q\big(D(x,f)+D(e,y)\big)\Big\}.$$

Theorem 9.7. *Let E, F, G H be vector lattices with G and H Dedekind complete. Fix $e\in E^+$, $f\in F^+$, $S\in L^+(E,H)$, and $T\in L^+(F,H)$. Assume that $\mathscr{D}$ is a convex, cyclic, weakly closed and weakly bounded subset in $BL^+(E,F;G)$ and $Q:G\to H$ is a sublinear Maharam operator. Then the following are equivalent:*

(1) *There exist $B\in\mathscr{D}$ and $R\in\partial Q$ such that $S\leqslant R\circ B(\cdot,f)$ and $T\leqslant R\circ B(e,\cdot)$.*

(2) *For every $x\in E$ and $y\in F$ the inequality holds*

$$Sx+Ty\leqslant\sup_{D\in\mathscr{D}}\Big\{Q\big(D(x^+,f)+D(e,y^+)\big)\Big\}.$$

Part III. Continuous and Measurable Bundles of Banach Lattices

The following problems concerning Banach spaces of continuous and measurable sections were studied: homogeneous functional calculus and analytic representations of some classes of dominated operators in Banach lattices of sections; Banach–Stone type theorem for Banach spaces of continuous sections; multiplicative representation and desintegration in Banach section space.

1. Introduction

The theory of Banach bundles stemming from J. von Neumann proved to have a vast area of applications in analysis. In particular, continuous and measurable Banach bundles are often used for representing various functional-analytical objects, see [38, 41, 43, 44, 51, 64, 111]. It was shown in [61] that a Banach–Kantorovich space over a Dedekind complete vector lattice is linearly isometric to the space of almost global sections of a suitable continuous Banach bundle over an extremally diconnected compactum. However, uniqueness of the representing bundle was not established. It was A. E. Gutman [43] who found a class of uniqueness for this representation result, the class of ample (or complete) continuous Banach bundles.

The next problem was to specify this representation result if the norming lattice is an ideal function space. In this case the representing object is measurable Banach bundle and the uniqueness class is formed of liftable Banach bundles, i.e. measurable Banach bundles admitting lifting; this result is also due to A. E. Gutman [44]. It turned out that ample continuous Banach bundles and liftable measurable Banach bundles have other advantages and interesting applications [40, 45, 46, 64].

The aim of this part is to specify the Gutman's theory of liftable Banach bundles for bundles of Banach lattices and to find some new applications.

All unexplained terms can be found in [43, 44, 45, 64], and [98]. All vector lattices in this paper are real and Archimedean.

2. Measurable Bundles of Banach Lattices

In this section, we give a brief exposition of measurable bundle of Banach lattices and corresponding vector lattices of measurable sections.

Definition 2.1. Let Ω be a nonempty set. A *bundle of Banach lattices over* Ω is a mapping $\mathscr{X}$ defined on Ω and associating a Banach lattice $\mathscr{X}_\omega := \mathscr{X}(\omega) := \big(\mathscr{X}(\omega), \|\cdot\|_{\mathscr{X}(\omega)}\big) := \big(\mathscr{X}_\omega, \|\cdot\|_\omega\big)$ with every point $\omega \in \Omega$. The value $\mathscr{X}_\omega$ of a bundle $\mathscr{X}$ is called its *stalk* over ω. A mapping s defined on a nonempty set $\mathrm{dom}(s) \subset \Omega$ is called a *section* over $\mathrm{dom}(s)$ if $s(\omega) \in \mathscr{X}_\omega$ for every $\omega \in \mathrm{dom}(s)$. A section over Ω is called *global*. Let $S(\Omega, \mathscr{X})$ stands for the set of all global sections of $\mathscr{X}$ endowed with the structure of a vector lattice by letting $u \leqslant v \Leftrightarrow (\forall \omega \in \Omega)\ u(\omega) \leqslant v(\omega)$ and $(\alpha u + \beta v)(\omega) = \alpha u(\omega) + \beta v(\omega)$ $(\omega \in \Omega)$, where $\alpha, \beta \in \mathbb{R}$ and $u, v \in S(\Omega, \mathscr{X})$. For each section $s \in S(\Omega, \mathscr{X})$ we define its point-wise norm by $|\!|\!|s|\!|\!| : \omega \mapsto \|s(\omega)\|_{\mathscr{X}(\omega)}$ $(\omega \in \Omega)$. A set of sections S is called *stalkwise dense* in $\mathscr{X}$ if the set $\{s(\omega) : s \in S\}$ is dense in $\mathscr{X}(\omega)$ for every $\omega \in \Omega$.

Definition 2.2. Now consider a nonzero measure space (Ω, Σ, μ) with the direct sum property. Let $\mathscr{X}$ be a bundle of Banach lattices over Ω. A set of sections

$\mathscr{C} \subset S(\Omega, \mathscr{X})$ is called a *measurability structure* on $\mathscr{X}$, if it satisfies the following conditions:

(a) $\mathscr{C}$ is a vector lattice, i. e. $\lambda_1 c_1 + \lambda_2 c_2 \in \mathscr{C}$, $|c| \in \mathscr{C}$ $(\lambda_1, \lambda_2 \in \mathbb{R}; c_1, c_2 \in \mathscr{C})$;

(b) $|||c||| : \Omega \to \mathbb{R}$ is measurable for every $c \in \mathscr{C}$;

(c) the set $\mathscr{C}$ is stalkwise dense in $\mathscr{X}$.

If $\mathscr{C}$ is a measurability structure in $\mathscr{X}$ then we call the pair $(\mathscr{X}, \mathscr{C})$ a *measurable bundle of Banach lattices* over (Ω, Σ, μ). We shall usually write simply $\mathscr{X}$ instead of $(\mathscr{X}, \mathscr{C})$.

Definition 2.3. Let $(\mathscr{X}, \mathscr{C})$ be a measurable bundle of Banach lattices over Ω. Denote by $S_\sim(\Omega, \mathscr{X})$ the set of all sections of $\mathscr{X}$ defined almost everywhere on Ω. We say that $s \in S_\sim(\Omega, \mathscr{X})$ is a *step-section*, if $s = \sum_{k=1}^n \chi_{A_k} c_k$ for some $n \in \mathbb{N}$, $A_1, \dots, A_n \in \Sigma$, $c_1, \dots, c_n \in \mathscr{C}$. A section $u \in S_\sim(\Omega, \mathscr{X})$ is called *measurable* if, for every $K \in \Sigma$ with $\mu(K) < +\infty$, there is a sequence $(s_n)_{n\in\mathbb{N}}$ of step-sections such that $s_n(\omega) \to u(\omega)$ for almost all $\omega \in K$. The set of all measurable sections of $\mathscr{X}$ is denoted by $\mathscr{L}^0(\Omega, \Sigma, \mu, \mathscr{X})$ or $\mathscr{L}^0(\mu, \mathscr{X})$ for brevity.

Suppose that $\mathscr{X}$ is a measurable Banach bundle over (Ω, Σ, μ). Consider the equivalence relation $\sim$ in the set $\mathscr{L}^0(\mu, \mathscr{X})$: $u \sim v$ means that $u(\omega) = v(\omega)$ for almost all $\omega \in \Omega$. The coset containing $v \in \mathscr{L}^0(\mu, \mathscr{X})$ is denoted by $\tilde{v}$ or $v^\sim$. The quotient set $L^0(\mu, \mathscr{X}) := L^0(\Omega, \Sigma, \mu, \mathscr{X}) := \mathscr{L}^0(\mu, \mathscr{X})/{\sim}$ is a vector lattice under the operations and ordering defined as $(s\tilde{u} + t\tilde{v}) = (su + tv)^\sim$ $(s, t \in \mathbb{R})$ and $\tilde{u} \leqslant \tilde{v}$ provided that $u(\omega) \leqslant v(\omega)$ for almost all $\omega \in \Omega$. It is clear that the vector lattice $L^0(\mu, \mathscr{X})$ is uniformly complete. For every $\tilde{v} \in \mathscr{L}^0(\mu, \mathscr{X})/{\sim}$ we may define its (vector) *norm* $\lvert v\rvert := \lvert\tilde{v}\rvert := |||v|||^\sim \in L^0(\mu)$.

Let E be an order dense ideal in $L^0(\Omega, \Sigma, \mu)$ and $E(\mathscr{X})$ stands for the set of all $u \in L^0(\mu, \mathscr{X})$ with $\lvert u\rvert \in E$. Endow $E(\mathscr{X})$ with the operations, ordering, and E-valued norm induced from $L^0(\Omega, \mathscr{X})$. If E is a normed lattice then $E(\mathscr{X})$ is also a normed lattice under the “mixed” norm $|||u||| := \|\lvert u\rvert\|_E$ $\big(u \in E(\mathscr{X})\big)$.

Theorem 2.4. *If a measure space (Ω, Σ, μ) possesses the direct sum property and if $\mathscr{X}$ is a measurable bundle of Banach lattices over Ω, then the pair $\big(L^0(\Omega, \mathscr{X}), \lvert\cdot\rvert\big)$ is a Banach–Kantorovich lattice over $L^0(\Omega, \Sigma, \mu)$. If E is an ideal of $L^0(\Omega)$ then $\big(E(\mathscr{X}), \lvert\cdot\rvert\big)$ is a Banach–Kantorovich lattice over E. If, in addition, E is a Banach lattice, then $\big(E(\mathscr{X}), |||\cdot|||\big)$ is a Banach lattice.*

Remark 2.5. Recall that a *Banach–Kantorovich space* over a Dedekind complete vector lattice E is a vector spaces X with a decomposable norm $\lvert\cdot\rvert : X \to E$ which is norm complete with respect to order convergence in E. Decomposability means that for all $e_1, e_2 \in E_+$ and $x \in X$, with $\lvert x\rvert = e_1 + e_2$ there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\lvert x_k\rvert = e_k$ $(k := 1, 2)$. If a Banach–Kantorovich space is in addition a vector lattice and the norm is monotone $(|x| \leqslant |y| \Rightarrow \lvert x\rvert \leqslant \lvert y\rvert)$ then it is called a *Banach–Kantorovich lattice*. Any Banach–Kantorovich lattice is a lattice ordered module over the f-algebra $\mathrm{Orth}(\lvert X\rvert)$. A detailed presentation see in [64].

3. Liftable Bundles of Banach Lattices

Let $\mathscr{X}$ be a measurable bundle of Banach lattices over Ω. Since the measure space (Ω, Σ, μ) possesses the direct sum property, we can consider a fixed lifting $\rho : L^\infty(\Omega) \to \mathscr{L}^\infty(\Omega)$, see [54, 87].

Definition 3.1. A mapping $\rho_{\mathscr{X}} : L^{\infty}(\mu, \mathscr{X}) \to \mathscr{L}^{\infty}(\mu, \mathscr{X})$ is called a *lifting* of $L^{\infty}(\mu, \mathscr{X})$ *associated with* ρ if, for all $u, v \in L^{\infty}(\mu, \mathscr{X})$ and $e \in L^{\infty}(\mu)$, the following hold:

(1) $\rho_{\mathscr{X}}(u) \in u$ and $\operatorname{dom}(\rho_{\mathscr{X}}(u)) = \Omega$;

(2) $|\rho_{\mathscr{X}}(u)| = \rho(|u|)$;

(3) $\rho_{\mathscr{X}}(u + v) = \rho_{\mathscr{X}}(u) + \rho_{\mathscr{X}}(v)$;

(4) $|\rho_{\mathscr{X}}(u)| = \rho_{\mathscr{X}}(|u|)$;

(5) $\rho_{\mathscr{X}}(eu) = \rho(e)\rho_{\mathscr{X}}(u)$;

(6) $\{\rho_{\mathscr{X}}(u) : u \in L^{\infty}(\Omega, \mathscr{X})\}$ is stalkwise dense in $\mathscr{X}$.

Definition 3.2. We say that $\mathscr{X}$ is a *liftable bundle of Banach lattices* provided that there exists a lifting of $L^{\infty}(\Omega)$ and a lifting of $L^{\infty}(\Omega, \mathscr{X})$ associated with it.

Remark 3.3. This definition (with 2.6 (4) excluded) for general measurable Banach bundles was introduced by A. E. Gutman [44]. He also demonstrated that 2.6 (6) cannot be derived from other properties of lifting. I. G. Ganiev adapted this notion to bundles of Banach lattices by adding property 2.6 (4).

Definition 3.4. Measurable bundles of Banach lattices $\mathscr{X}$ and $\mathscr{Y}$ over (Ω, Σ, μ) are said to be *isometrically isomorphic* if there exists a family of isometric lattice isomorphisms $h_{\omega} : \mathscr{X}(\omega) \to \mathscr{Y}(\omega)$ $(\omega \in \Omega)$ such that $\tilde{h}(\mathscr{L}^{0}(\mu, \mathscr{X})) = \mathscr{L}^{0}(\mu, \mathscr{Y})$, where the section $v : \omega \mapsto h_{\omega}(u(\omega))$ is measurable for every $u \in \mathscr{L}^{0}(\mu, \mathscr{X})$ and $\tilde{h}$ is defined by $\tilde{h}(u) := v$.

Theorem 3.5. *Every Banach–Kantorovich lattice X over an order-dense ideal $E \subset L^{0}(\Omega)$ is linearly isometric to $E(\mathscr{X})$ for some liftable bundle of Banach lattices $\mathscr{X}$ over Ω. Moreover, such a bundle $\mathscr{X}$ is unique to within an isometric isomorphism.*

Theorem 3.6. *Let $\mathscr{X}$ be a liftable bundle of Banach lattices over Ω. Then there exists* (*a unique to within an isometric isomorphism*) *liftable bundle of Banach lattices $\mathscr{X}'$* (*called a dual bundle*) *such that the following hold:*

(1) *at each point $\omega \in \Omega$, the stalk $\mathscr{X}'(\omega)$ is a Banach sublattice of $\mathscr{X}(\omega)'$;*

(2) *if $v \in \mathscr{L}^{0}(\mu, \mathscr{X})$ and $v' \in \mathscr{L}^{0}(\mu, \mathscr{X}')$, then $\langle v, v' \rangle \in \mathscr{L}^{0}(\mu)$ where $\langle v, v' \rangle : \omega \mapsto \langle v(\omega), v'(\omega) \rangle_{\omega}$;*

(3) *for all $v \in \mathscr{L}^{\infty}(\mu, \mathscr{X})$ and $v' \in \mathscr{L}^{\infty}(\mu, \mathscr{X}')$, we have $\rho(\langle v, v' \rangle^{\sim}) = \langle \rho_{\mathscr{X}}(\widetilde{v}), \rho_{\mathscr{X}'}(\widetilde{v}') \rangle$, where $\rho_{\mathscr{X}}$ and $\rho_{\mathscr{X}'}$ are respective liftings of $\mathscr{X}$ and $\mathscr{X}'$ associated with ρ;*

(4) *if a bounded mapping $v' : \omega \mapsto v'(\omega)$ is such that, for every $v \in \mathscr{L}^{\infty}(\mu, \mathscr{X})$ the function $\langle v, v' \rangle$ is measurable and $\rho(\langle v, v' \rangle) = \langle \rho_{\mathscr{X}}(v), v' \rangle$, then $v' \in \mathscr{L}^{\infty}(\mu, \mathscr{X}')$;*

(5) *if $\mathbf{u}' \in L^{\infty}(\mu, \mathscr{X}')$, then for every $\omega \in \Omega$ we have*

$$\rho\big(|\mathbf{u}'|\big)(\omega) = \sup\{\rho\big(\langle \mathbf{u}, \mathbf{u}' \rangle\big)(\omega) : \ \mathbf{u} \in L^{\infty}(\mu, \mathscr{X}), \ |\mathbf{u}| \leqslant \mathbb{1}\}.$$

Remark 3.7. The idea of a measurability structure has been proposed by N. Dinculeanu [31, Notes and remarks, 8] as early as in 1966, but has not been much studied in subsequent twenty five years. In a series of papers A. E. Gutman undertook a systematic study of this conception and built a nice theory of measurable Banach bundles, see [43, 44, 45, 46]. A different approach to defining measurability of sections see in [111].

4. Lattices of Measurable Vector Functions

The vector lattices $E(X)$ and $E_w(X')$ are Banach–Kantorovich spaces, see [64, 2.3.7 and 2.3.10]. By Theorem 3.5 there exist liftable bundles of Banach lattices $\mathscr{X}$ and $\mathscr{Y}$ such that $E(X) \simeq E(\mathscr{X})$ and $E_w(X') \simeq E(\mathscr{Y})$.

Definition 4.1. We say that $(\mathscr{X}, \mathscr{Y})$ is a *representing pair of measurable Banach bundles* for $(E(X), E_w(X'))$.

It turns out that one can take $\mathscr{Y} = \mathscr{X}'$, while $\mathscr{X}$ is in a sense a "liftable hull" of a trivial bundle $\Omega \times \{X\}$.

Theorem 4.2. *Let X be a Banach lattice and (Ω, Σ, μ) a measure space with the direct sum property. There exists a liftable measurable bundle of Banach lattices $\mathscr{X} := (\mathscr{X}(\omega))_{\omega\in\Omega}$ over Ω, unique to within a ρ-isometry, and such that if $\mathscr{X}' := (\mathscr{X}'(\omega))_{\omega\in\Omega}$ is the dual measurable Banach bundle, then*

(1) *X is a Banach sublattice of each stalk $\mathscr{X}(\omega)$ and $\mathscr{X}'(\omega)$ is a Banach sublattice of $\mathscr{X}(\omega)'$ for all $\omega \in \Omega$;*

(2) *the respective liftings $\rho_{\mathscr{X}}$ and $\rho_{\mathscr{X}'}$ of $\mathscr{X}$ and $\mathscr{X}'$ are module preserving, are associated with ρ, and $\rho_{\mathscr{X}}(\widetilde{c}) = c$ for all constant functions $c : \Omega \to X$;*

(3) *for every section $u \in \mathscr{L}^0(\Omega, \mathscr{X})$ the function $\overline{u}$ coinciding with u on $u^{-1}(X)$ and vanishing on $\Omega \setminus u^{-1}(X)$ is contained in $\mathscr{L}^0(\mu, X)$;*

(4) *the mapping sending the coset of $u \in \mathscr{L}^0(\mu, \mathscr{X})$ to the coset of $\overline{u} \in \mathscr{L}^0(\mu, X)$ is a lattice isomorphism and an isometry of $L^0(\mu, \mathscr{X})$ onto $L^0(\mu, X)$;*

(5) *for every section $v \in \mathscr{L}^0(\mu, \mathscr{X}')$ the function $v_X : \omega \mapsto v(\omega)|_X$ from Ω to X' is contained in $\mathscr{L}^0_w(\mu, X')$;*

(6) *the mapping sending the coset of $v \in \mathscr{L}^0(\mu, \mathscr{X}')$ to the coset of $v_X \in \mathscr{L}^0_w(\mu, X')$ is a lattice isomorphism and an isometry of $L^0(\mu, \mathscr{X}')$ onto $L^0_w(\mu, X')$.*

Assume that K and φ are the same as in Definition 2.1 of Part I.

Proposition 4.3. *Let $u_1, \dots, u_N \in \mathscr{L}^0(\Omega, \Sigma, \mu, X)$, and $[\tilde{u}_1, \dots, \tilde{u}_N] \subset K$. Then there exists a measurable set $\Omega_0 \subset \Omega$ such that $\mu(\Omega \setminus \Omega_0) = 0$, $[u_1(\omega), \dots, u_N(\omega)] \subset K$ for all $\omega \in \Omega_0$, and $\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N)$ is the equivalence class of the measurable vector-function $\omega \in \Omega_0 \mapsto \widehat{\varphi}(u_1(\omega), \dots, u_N(\omega)) \in X$. Moreover,*

$$|\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N)| \leqslant \|\varphi\|(|\tilde{u}_1| \vee \dots \vee |\tilde{u}_N|).$$

Remark 4.4. According to 4.2 (6) the spaces $L^0(\mu, \mathscr{X}')$ and $L^0_w(\mu, X')$ are isometric and lattice isomorphic, while the corresponding equivalence classes in $L^0(\mu, \mathscr{X}')$ are essentially larger than in $L^0_w(\mu, X')$. So, it may happen that there is a measurable section with some nice properties but no equivalent vector function enjoy them. Therefore, one can expect that Theorem 4.2 leads to a better understanding of the structure of lattices of measurable vector-valued functions.

Remark 4.5. At the same time the Banach lattice $E(\mathscr{X})$ deserves an independent study. Let $\mathscr{P}$ be a property of a Banach lattice and $F \in (\mathscr{P})$ means that the Banach lattice F possesses the property $\mathscr{P}$. Then the following question naturally arises: is is true that

$$E(\mathscr{X}) \in (\mathscr{P}) \ \Leftrightarrow \ E \in (\mathscr{P}) \text{ and } \mathscr{X}_\omega \in (\mathscr{P}) \text{ for almost all } \omega \in \Omega.$$

Much is known about this question for the Banach lattice of measurable vector-valued functions [27, 30, 87, 88]. But in the case of Banach lattice of measurable sections we have a collection of challenging problems.

5. Representation of Dominated Operators

In this section we present two representation theorems for $\widehat{\varphi}(T_1,\dots,T_N)$ with dominate operators $T_1,\dots,T_N$ obtained in [71]. First, we recall two types of *dominated operators*, see [64, 4.1.3 (3, 4)].

Definition 5.1. Let X be a Banach space and E an ideal space. An operator $S : X \to E$ is dominated if the image of the unit ball in X is order bounded in E. The element $\lvert S\rvert$ defined as

$$\lvert S\rvert = \sup\{|Sx| :\ x \in X,\ \|x\| \leqslant 1\}$$

is called the *abstract norm* of S. The linear space of all dominated operators $M(X,E)$ is denoted also by $L_A(X,F)$ and is called the space of operators with abstract norm. If X is a Banach lattice then $M(X,E)$ is a Dedekind complete vector lattice. Actually, the exact dominant is presented by the mapping $t \mapsto t\lvert S\rvert$ $(t \in \mathbb{R})$.

Definition 5.2. An operator $S : E \to Y$ is dominated if there exists a positive functional e^* on E such that $\|Te\| \leqslant \langle |e|, e^*\rangle$ $(e \in E)$. The exact dominant is calculated as follows:

$$\lvert T\rvert e = \sup\left\{\sum_{k=1}^{n} |Te_k\| :\ e_1,\dots,e_n \in E_+,\ \sum_{k=1}^{n} e_k = e,\ n \in \mathbb{N}\right\} \quad (e \in E_+).$$

Put $\widehat{\varphi}(u_1(\omega),\dots,u_N(\omega)) = 0$ whenever $u_1,\dots,u_N \in \mathscr{L}^0(\mu, X')$ but $\widehat{\varphi}(u_1(\omega),\dots,u_N(\omega))$ cannot be correctly defined in X', i.e. $[u_1,\dots,u_N]$ is not contained in K.

Theorem 5.3. *Let X be a Banach lattice, E an ideal space on (Ω,Σ,μ), and $(\mathscr{X},\mathscr{X}')$ a representing pair of measurable Banach bundles for $(E(X), E_w(X'))$. Consider $\varphi \in \mathscr{H}(\mathbb{R}^N, K)$ and $S_1,\dots,S_N \in M(X,E)$ with $[S_1,\dots,S_N] \subset K$ and put $e := \lvert S_1\rvert + \dots + \lvert S_N\rvert$, $S := \widehat{\varphi}(S_1,\dots,S_N)$. Then there exist global measurable sections $v_1,\dots,v_N \in \mathscr{L}^0(\Omega,\mathscr{X}')$ such that*

(1) $\tilde v_1,\dots,\tilde v_N \in E(\mathscr{X}')$;

(2) $[v_1(\omega),\dots,v_N(\omega)] \subset K$ *for all* $\omega \in \Omega$;

(3) *the map* $\omega \mapsto \widehat{\varphi}(v_1(\omega),\dots,v_N(\omega))$ $(\omega \in \Omega)$ *is a global measurable section of* $\mathscr{X}'$ *and for all* $x \in X$ *and* $\omega \in \Omega$ *we have*

$$\rho_e(Sx)(\omega) = \langle x, \widehat{\varphi}(v_1(\omega),\dots,v_N(\omega))\rangle;$$

(4) $\rho_e(\lvert S\rvert)(\omega) = \|\widehat{\varphi}(v_1(\omega),\dots,v_N(\omega))\|_{\mathscr{X}'(\omega)}$ $(\omega \in \Omega)$.

For any Banach space X the mapping which sends a dominated operator $S \in M_F(E,X')$ to the restriction $h(S) := S'|_X$ of its adjoint $S' : X'' \to F$ to X is an isomorphism of $M(E,X')$ onto $M(X,F)$; moreover, $\lvert S\rvert = \lvert h(S)\rvert$ for all S

Theorem 5.4. *Let X be a Banach lattice, E an ideal space over (Ω,Σ,μ) with point separating Köthe dual E', identified with $E^* := \{e^* \in L^0(\mu) : (\forall e \in E)\, ee^* \in L^1(\mu)\}$, and $(\mathscr{X},\mathscr{X}')$ a representing pair of measurable Banach bundles for $(E(X), E_w(X'))$. Let the dominated operators $S_1,\dots,S_N \in M_F(E,X')$ with $[S_1,\dots,S_N] \subset K$ are given, and $S := \widehat{\varphi}(S_1,\dots,S_N)$. Then there exist global measurable sections $v_1,\dots,v_N \in \mathscr{L}^0(\Omega,\mathscr{X}')$ such that*

(1) $\tilde v_1,\dots,\tilde v_N \in F(\mathscr{X}')$;

(2) $[v_1(\omega), \dots, v_N(\omega)] \subset K$ *for all* $\omega \in \Omega$;

(3) *for every* $\varphi \in \mathscr{H}(C, K)$, *the function* $\omega \mapsto \widehat{\varphi}(v_1(\omega), \dots, v_N(\omega))$ $(\omega \in \Omega)$ *is a measurable section of* $\mathscr{X}'$ *and the representation holds*

$$\langle x, S(e) \rangle = \int\limits_{\Omega} e(\omega) \langle x, \widehat{\varphi}(v_1(\omega), \dots, v_N(\omega)) \rangle \, d\mu(\omega) \quad (e \in E,\ x \in X);$$

(4) *the function* $\omega \mapsto \|\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\|$ $(\omega \in \Omega)$ *is measurable and*

$$|S|(e) = \int\limits_{\Omega} e(\omega) \|\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\| \, d\mu(\omega) \quad (e \in E).$$

6. Continuous Bundles of Banach Lattices

In this section we give a brief exposition of continuous bundles of Banach lattices and corresponding vector lattices of continuous sections. Denote by Q an extremally disconnected compact Hausdorff space.

Definition 6.1. Let $\mathscr{X}$ be a bundle of Banach lattices over Q. A set of global sections $\mathscr{C} \subset S(Q, \mathscr{X})$ is called a *continuity structure* on $\mathscr{X}$ if it satisfies the following conditions:

(a) $\mathscr{C}$ is a vector lattice, i. e. $\alpha c_1 + \beta c_2 \in \mathscr{C}$, $|c| \in \mathscr{C}$ $(\alpha, \beta \in \mathbb{R};\ c_1, c_2 \in \mathscr{C})$;

(b) the pointwise norm $|||c||| : Q \to \mathbb{R}$ is continuous for every $c \in \mathscr{C}$;

(c) $\mathscr{C}$ is stalkwise dense in $\mathscr{X}$.

Definition 6.2. The pair $(\mathscr{X}, \mathscr{C})$ is called a *continuous bundle of Banach lattices over* Q, whenever $\mathscr{C}$ is continuity structure on $\mathscr{X}$.

Definition 6.3. Let $(\mathscr{X}, \mathscr{C})$ be a continuous bundle of Banach lattices over Q. We say that section $u \in S(D, \mathscr{X})$ over $D \subset Q$ is $\mathscr{C}$-*continuous* at a point $q \in D$ if the function $|||u - c|||$ is continuous at q for every $c \in \mathscr{C}$. A section $u \in S(D, \mathscr{X})$ is $\mathscr{C}$-continuous if it is $\mathscr{C}$-*continuous* at every point $q \in D$.

We shall write simply $\mathscr{X}$ instead of $(\mathscr{X}, \mathscr{C})$ and continuous instead of $\mathscr{C}$-continuous. Denote by $C_\infty(Q, \mathscr{X})$ the space of its (extended) almost global continuous sections, see [64, § 2.4].

Theorem 6.4. *Let* $\mathscr{X}$ *be a continuous bundle of Banach lattices over extremal compact* Q. *Then* $C_\infty(Q, \mathscr{X})$ *is Banach–Kantorovich lattice over* $C_\infty(Q)$. *If* E *is an order-dense ideal in* $C_\infty(Q)$ *then* $E(\mathscr{X}, |\cdot|)$, *is Banach–Kantorovich lattice over* E. *If in addition* E *is a Banach lattice, then* $E(\mathscr{X})$, $|||\cdot|||$ *is also a Banach lattice.*

Theorem 6.5. *Every Banach–Kantorovich lattice* $\mathscr{X}$ *over an order-dense ideal* $E \subset C_\infty(Q)$ *is lattice isometric to* $E(\mathscr{X})$ *for some ample continuous bundle* $\mathscr{X}$ *of Banach lattices over* Q. *Moreover, such a bundle* $\mathscr{X}$ *is unique to within a lattice isometry.*

Theorem 6.6. *Let* X *be a Banach lattice. Then there exist an ample bundle of Banach lattices* $\mathscr{X}$ *such that:*

(a) *at every point* $q \in Q$ *the lattice* X *is a dense sublattice of the stalk* $\mathscr{X}(q)$;

(b) *for each section* $u \in \mathscr{X}$ *there exists a comeager supset* $Q_0 \subset Q$ *such that* $u(q) \in X$ *for all* $q \in Q_0$;

(c) *for every ideal* $E \subset C_\infty(Q)$ *spaces* $E(X)$ *and* $E(\mathscr{X})$ *are isometrically lattice isomorphic.*

Let (Ω, Σ, μ) a be measure space with the direct sum property and Q the Stone space of the Boolean algebra $B(\Omega) := \sum/\mu^{-1}(0)$. Denote by $\tau : \Omega \to Q$ the canonical immersion of Ω into Q corresponding to the lifting τ of $L^\infty(\Omega)$ [64, 1.2.7 (3)]. Let $\mathscr{Y}$ be an ample continuous bundle of Banach lattices over Q and $\mathscr{X} = \mathscr{Y} \circ \tau$.

If $\mathscr{C}$ is a continuous structure in $\mathscr{Y}$, then the set $\mathscr{C} \circ \tau$ is a measurability structure in $\mathscr{X}$, since $|c| \circ \tau = |c \circ \tau|$, $|||c \circ \tau||| = |||c||| \circ \tau$ and $|||c||| \circ \tau$ is a measurable function. The bundle $\mathscr{Y} \circ \tau$ is always regarded as a measurable Banach bundle with respect to the measurability structure $\mathscr{C} \circ \tau$.

Theorem 6.7. *Let (Ω, Σ, μ) be measure space with the direct sum property. The mapping $v \mapsto (v \circ \tau)^\sim$ is a lattice isometry of Banach–Kantorovich lattices $C_\infty(Q, \mathscr{Y})$ and $L^0(\Omega, \mathscr{X})$, associated with isomorphism $(e \mapsto (e \circ \tau)^\sim) : C_\infty(Q) \to L^0(\Omega)$. The image of $C(Q, \mathscr{Y})$ under this lattice isometry is $L^\infty(\Omega, \mathscr{X})$.*

Theorem 6.7 describes a method of constructing a liftable Banach bundle given an ample continuous bundle of Banach lattices over the corresponding Stone space. The following result shows that every liftable measurable bundle of Banach lattices can be obtained in such a way.

Theorem 6.8. *Let $\mathscr{X}$ be a ρ-invariant measurable bundle of Banach lattices over Ω that has a lifting associated with ρ. Then there exists an ample continuous bundle of Banach lattices $\hat{\mathscr{X}}$ over Q (unique to within a lattice isometry) such that $\mathscr{X} = \hat{\mathscr{X}} \circ \tau$ and $\rho(u) = \hat{u} \circ \tau$ for all $u \in L^\infty(\Omega, \mathscr{X})$.*

Remark 6.9. The theory of ample continuous Banach bundles was developed by A. E. Gutman [43]–[46], see also [64]. The corresponding theory of measurable bundles of Banach lattices was developed in [40] and [71]. This section presents the main results of a forthcoming paper by S. N. Tabuev and starts the study of ample continuous bundles of Banach lattices.

7. Banach–Stone Type Problem

One of the important classical theorems of functional analysis states that the categories of compact Hausdorff spaces and Banach spaces of continuous functions defined on these compacts are isomorphic. In particular, this fact is the starting point of the thriving area of mathematics, noncommutative geometry. If we replace the spaces of continuous functions by the spaces of continuous vector-valued functions then it may happen that for some Banach spaces X and Y the Banach spaces $C(Q, X)$ and $C(P, Y)$ are isomorphic for non-homeomorphic compacts Q and P. So a natural question arises: what extra conditions must be satisfied by the isomorphism $T : C(Q, X) \to C(P, Y)$ and the spaces X and Y in order to guarantee the existence of a homeomorphism $\varphi : P \to Q$. This problem with various modifications is known as Banach–Stone problem. There is a rich literature devoted to different aspects of this problem [6, 24, 28, 36, 37]. At the same time the theory of continuous Banach bundles allow us to consider spaces of vector-functions as special case of bundles with constant stalk. So, it is worth to consider the extended versions of Banach–Stone type theorems for section spaces associated with continuous bundles of Banach lattices. Below we present a result from [108] by M. A. Pliev and S. N. Tabuev.

Let Q and P be compact Hausdorff spaces, $\mathscr{X}$ and $\mathscr{Y}$ be continuous bundles of Banach lattices over Q and P, respectively. Then the spaces $C(Q, \mathscr{X})$ and $C(P, \mathscr{Y})$

of global continuous section are also Banach lattices with the point-wise ordering. The norm of a section f is given by $|||f||| = \||\|f(\cdot)\|_{\mathscr{X}(\cdot)}\|_{C(Q)}$. Take arbitrary $t \in Q$ and $s \in Q$ and introduce the sets

$$M_t = \{f \in C(Q, \mathscr{X}) : \ f(t) = 0\}; \quad N_s = \{g \in C(K, \mathscr{Y}) : \ g(s) = 0\}.$$

The sets M_t and N_s are closed order ideals in $C(Q, \mathscr{X})$ and $C(P, \mathscr{Y})$, respectively.

Definition 7.1. Let T be a lattice isomorphism from $C(Q, \mathscr{X})$ onto $C(P, \mathscr{X})$. We say that T *satisfy the property* $\mathscr{P}$ if, for every $f \in C(Q, \mathscr{X})$, we have

$$(\forall s \in P)\, (Tf)(s) \neq 0 \Leftrightarrow \ (\forall t \in Q)\, f(t) \neq 0.$$

Observe an important property of lattice isomorphisms with the property $\mathscr{P}$.

Proposition 7.2. *Let an isomorphism $T : C(Q, \mathscr{X}) \to C(P, \mathscr{Y})$ satisfy the property $\mathscr{P}$. Then for every $t \in Q$ there exists a unique $s \in K$ such that $T(M_t) = N_s$.*

Theorem 7.3. *Let $T : C(Q, \mathscr{X}) \to C(P, \mathscr{Y})$ be a lattice isomorphism with the property $\mathscr{P}$. Then Q and P are homeomorphic and the representation holds:*

$$(Tf)(s) = H(s)f(\psi(s)) \quad (\forall\, f \in C(Q, \mathscr{X}); \ s \in P);$$

where $\psi : P \to Q$ is a homeomorphism and H is a homomorphism of continuous Banach bundles $\mathscr{X}$ and $\mathscr{Y}$, with $s \mapsto H(s) \in \mathscr{L}(\mathscr{X}_{\psi(s)}, \mathscr{Y}_s)$.

Remark 7.4. If Q is extremally disconnected then $\|T\| = \sup_{s\in K} \|H(s)\|$. In the special case of constant bundles $\mathscr{X}$ and $\mathscr{Y}$ we obtain the result in [28].

8. Multiplicative Representation

Definition 8.1. Let X be a Banach space. A *Markushevich basis* (*M-basis* for short) of X is a family $(x_i, x'_i)_{i\in I}$, where $x_i \in X$ and $x'_i \in X'$, such that:

(1) $x_i(x'_j) = \delta_{ij}$ (the Kronecker delta symbol) for every $i, j \in I$;

(2) $X = \operatorname{span}\{x_i : \ i \in I\}$;

(3) $\{x'_i : \ i \in I\}$ separates the points of X (i. e. for each $x \in X\backslash\{0\}$ there is $i \in I$ such that $x'_i(x) \neq 0$).

It is well known that every separable Banach space has an M-basis, see [48]. Moreover, every weakly compactly generated Banach space has an M-basis, see [48, Corollary 5.2]. For a detailed presentation we refer to [48].

Definition 8.2. A liftable measurable Banach bundle $\mathscr{X}$ over (Ω, Σ, μ) is said to have a *generalized M-basis* (or *GM-basis* for short) if there is a family $(\varphi_i, \varphi'_i)_{i\in I}$ of measurable section, where $\varphi_i \in L_\infty(\mu, \mathscr{X})$ and $\varphi'_i \in L_\infty(\mu, \mathscr{X}')$, such that

1) $\langle \varphi'_i, \varphi_j \rangle = \delta_{ij} 1_\Omega$;

2) for every order ideal E of $L_0(\mu)$ there exists an order dense subspace E_0 of E such that $\|f(\cdot) - \sum_{k=1}^n h_k(\cdot)\varphi_k(\cdot)\|_{\mathscr{X}(\cdot)} \to 0$ a. e. as $n \to \infty$ for every step-section $f \in E(\mathscr{X})$; $h_k \in E_0$ for every $1 \leqslant k \leqslant n$; φ_k are the elements of GM-basis;

3) for every measurable section $f \in L_0(\mathscr{X})$, $f \neq 0$, there exist φ'_{i_0} and a measurable set $A \in \Sigma$, $\mu(A) > 0$, so that $g(\omega) := \langle \varphi'_{i_0}, f \rangle(\omega) > 0$ for every $\omega \in A$.

Example 8.3. Let X be a Banach space which have an M-basis $(x_i, x'_i)_{i\in I}$ and (Ω, Σ, μ) a finite measure space. Consider a constant Banach bundle $\Omega \times X$ and its liftable hull X_Ω, see [71, Definition 3.2 and Theorem 3.3]. Then the measurable

Banach bundle X_Ω have a GM-basis $(\varphi_i, \varphi_i')_{i\in I}$ such that $\varphi_i = x_i 1_\Omega$; $\varphi_i' = x_i' 1_\Omega$ for every $i \in I$.

Definition 8.4. Let (Ω, Σ, μ) be a finite measure space. A liftable Banach bundle $\mathscr{X}$ over Ω is said to have the *S-property* if for every measurable sections $f, g \in L_0(\mu, \mathscr{X})$ there exists a measurable subbundle $\mathscr{Y}_{f,g}$ of $\mathscr{X}$ such that $f(t), g(t) \in \mathscr{Y}_{f,g}(t)$ for almost every all $t \in \Omega$ and $\mathscr{Y}_{f,g}$ has a GM-basis.

Example 8.5. Let X be a Banach space and (Ω, Σ, μ) a finite measure space. Consider a constant Banach bundle $\Omega \times X$ and its liftable hull X_Ω. We can prove that the measurable Banach bundle X_Ω have the S-property. Given two measurable section $f, g \in L_0(\mu, X_\Omega)$, there exists a measurable set $A \in \Sigma$, $\mu(\Omega \backslash A) = 0$, such that $f(t), g(t) \in X$ for every $t \in A$. Then $f_1 := f|_A : A \to X$ and $g_1 := g|_A : A \to X$ are Bochner μ-measurable vector-functions and the Banach space $Y := \overline{\operatorname{span}(f_1(A) \cup g_1(A))}$ is separable and therefore has an M-basis. So, we can take a measurable subbundle $\mathscr{Y}_{f,g}$ of X_Ω defined by $\mathscr{Y}_{f,g}(t) := X_\Omega(t) \cap Y$.

Theorem 8.6. *Let $\mathscr{X}$ be a liftable Banach bundle and with the S-property. Suppose the set of step-sections $\mathrm{St}(\mathscr{X})$ is norm dense in $E(\mathscr{X})$. Let $T : E(\mathscr{X}) \to E(\mathscr{X})$ be a linear continuous operator. The following statements are equivalent:*

(1) T is a multiplication operator, i. e. there is $g_0 \in L_\infty(\mu)$ such that $T(f) = g_0 f$ for all $f \in E(\mathscr{X})$.

(2) The equality $T(g\langle f, \varphi'\rangle\varphi) = g(\langle T(f), \varphi'\rangle\varphi)$ holds for every $g \in L_\infty(\mu)$, $f \in E(\mathscr{X})$, $\varphi \in L_\infty(\mu, \mathscr{X})$, and $\varphi' \in L_\infty(\mu, \mathscr{X}')$.

Remark 8.7. Theorem 8.6 is the main result of a forthcoming paper by M. A. Pliev.

9. Desintegration

In this section we present a Strassen type desintegration result by E. K. Basaeva and M. A. Pliev [5]. For desintegration in Dedekind complete vector lattices, see [78].

Definition 9.1. We say that a measurable Banach bundle $\mathscr{X}$ is *separable*, if there exists a countable stalkwise dense set of measurable sections in $\mathscr{L}^0(\mu, \mathscr{X})$.

Let E be an ideal space over $L_0(\Omega)$. Define

$$E(\mathscr{X}) := \{v \in L_0(\Omega, \Sigma, \mu, \mathscr{X}) : |v| \in E\}.$$

It is known that $E(\mathscr{X})$ is *bo*-complete lattice normed space and $L_0(\Omega, \mathscr{X})$ is its universal extension, see [64, Theorems 2.5.3 and 2.5.4(1)].

Proposition 9.2. *Let (Ω, Σ, μ) be a measure space with the direct sum property, $\mathscr{X}$ be a liftable Banach bundle over (Ω, Σ, μ) and F be a Banach space. Then there exists a unique liftable Banach bundle $\mathscr{Z}$ over Ω, such that the stalk $\mathscr{Z}(\omega)$ is a subspace of $\mathscr{L}(\mathscr{X}(\omega), F)$ for every $\omega \in \Omega$.*

Let $\mathscr{X}$ be a liftable Banach bundle over (Ω, Σ, μ), $E \subset L_0(\Omega, \Sigma, \mu)$ an ideal space and E' a Köthe dual of E, i. e.

$$E' = \left\{ e' \in L_0(\Omega, \Sigma, \mu) : (\forall e \in E) \int\limits_\Omega |e'e|\, d\mu(\omega) < \infty \right\}.$$

Hypotheses 9.3. Consider a family $(K_\omega)_{\omega\in\Omega}$ of continuous sublinear operators $K_\omega : \mathscr{X}_\omega \to E$ such that for every $u \in E(\mathscr{X})$ the vector-function $\omega \mapsto K_\omega(u(\omega)) :=$

$K(\omega, u(\omega))$ is Bochner measurable and the function $\omega \mapsto \|K_\omega\|$ is dominated by some measurable function from E'. Then we can define a sublinear operator $K : E(\mathscr{X}) \to F$ by

$$K(u) = \int_\Omega K(\omega, u(\omega))d\mu(\omega) \quad (u \in E(\mathscr{X})),$$

where the integral is considered as Bochner.

An explicit description of the subdifferential ∂K of K is an important problem in convex analysis, see [78]. The first result of this kind for functionals on a separable Banach space was given in a celebrated paper by Strassen [115, Theorem 1]. Generalisation of this result for the case of functionals defined on a space of Banach sections is due to Kusraev [66].

Definition 9.4. Denote by $\int_\Omega \partial K_\omega d\mu(\omega)$ the set of all linear operators from $E(\mathscr{X})$ to F representable as

$$u \mapsto \int_\Omega \langle u(\omega), u'(\omega)\rangle d\mu(\omega),$$

where $u'(\cdot)$ is a measurable section in $\mathscr{Z}$ with $u'(\omega) \in \partial K_\omega$ for all $\omega \in \Omega$ and $\|u'(\cdot)\| \in E'$. Let $E(\mathscr{X})^\star$ stands for the set of all linear operators $T : E(\mathscr{X}) \to L_1(\Omega, \Sigma, \mu, F)$ for which there exists an element $e' \in E'$ such that

$$|Tu| \leqslant e'\|u(\cdot)\| \quad (u \in E(\mathscr{X})).$$

If $\mathscr{X}$ is a separable liftable Banach bundle then there is a convenient description for the space $E(\mathscr{X})^\star$.

Proposition 9.5. *If $\mathscr{X}$ is a separable liftable Banach bundle, then the space of measurable sections $E'(\mathscr{Z})$ and the space of operators $E(\mathscr{X})^\star$ are linearly isometric. The isometry is defined by assigning to measurable section $v \in E'(\mathscr{Z})$ the operator $T : E(\mathscr{X}) \to L_1(\Omega, \Sigma, \mu, F)$ given by $T_v : u \mapsto \langle u, v\rangle$.*

Given a family $(K_\omega)_{\omega\in\Omega}$, we define an operator $R : E(\mathscr{X}) \to L_1(\Omega, \Sigma, \mu, F)$ by

$$Ru := \pi(K(\omega, u(\omega))) \quad (u \in E(\mathscr{X})),$$

where $\pi(g)$ is the equivalence class of measurable vector-function g. It is obvious that R is a sublinear operator and $\partial K = \int_\Omega \partial R \, d\mu(\omega)$.

Theorem 9.6. *Let (Ω, Σ, μ) be a measure space with the direct sum property and $\mathscr{X}$ be a separable liftable Banach bundle. Let the family of operators $(K_\omega)_{\omega\in\Omega}$ is the same as in* 9.3. *Then the representation holds:*

$$\partial K = \int_\Omega \partial K_\omega \, d\mu(\omega).$$

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