2-Local derivations on von Neumann algebras and AW^* - algebras

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Outline

Von Neumann algebras

Let H be a Hilbert space over the field $\mathbb{C}$ of complex numbers, and let $B(H)$ be the algebra of all bounded linear operators on H. Denote by $\mathbf{1}$ the identity operator on H, and let $P(H) = \{p \in B(H) : p = p^2 = p^*\}$ be the lattice of projections in $B(H)$. Consider a von Neumann algebra M on H, i.e. a *-subalgebra of $B(H)$ closed in the week operator topology and containing the operator $\mathbf{1}$. Denote by $\|\cdot\|_M$ the operator norm on M. The set $P(M) = P(H) \cap M$ is a complete orthomodular lattice with respect to the natural partial order on $M_h = \{x \in M : x = x^*\}$, generated by the cone M_+ of positive operators from M.

Von Neumann algebras

Every von Neumann algebra can be written uniquely as a sum of von Neumann algebras of types I, II_1 (finite), II_∞ (properly infinite, semifinite) and III (purely infinite).

Outline

Derivation

Given an algebra $\mathcal{A}$, a linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule).

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Each element $a \in \mathcal{A}$ implements a derivation D_a on $\mathcal{A}$ defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be inner derivations.

Local derivation

There exist various types of linear operators which are close to derivations. Recall that a linear map Δ of $\mathcal{A}$ is called a local derivation if for each $x \in \mathcal{A}$, there exists a derivation $D : \mathcal{A} \to \mathcal{A}$, depending on x, such that $\Delta(x) = D(x)$. This notion was introduced in 1990 independently by Kadison [a] and Larson and Sourour [b].

[a] R. V. Kadison, Local derivations, J. Algebra. 130 (1990), 494–509.

[b] D. R. Larson, A. R. Sourour, Local derivations and local automorphisms of $B(X)$, Proc. Sympos. Pure Math. 51. Providence, Rhode Island, 1990, Part 2, 187–194.

Local derivations

R.V.Kadison proved that each continuous local derivation of a von Neumann algebra M into a dual Banach M-bimodule is a derivation. This theorem gave way to studies on derivations on C^*-algebras, culminating with a result due to B.E. Johnson, which asserts that every local derivation of a C^*-algebra A into a Banach A-bimodule is automatically continuous, and hence is a derivation.[a] For details we refer to the papers.[b,c]

[a] B.E. Johnson, Local derivations on C^*-algebras are derivations, Transactions of the American Mathematical Society. 353 (2001), 313–325.

[b] Sh. A. Ayupov, K. K. Kudaybergenov, 2-local derivation on von Neumann algebras. Positivity, 19 (2015), 445-455.

[c] Sh. A. Ayupov, K. K. Kudaybergenov, A. M. Peralta, A survey on local and 2-local derivations on C^*-algebras and von Neumann algebras, Topics in Functional Analysis and Algebra, Contemporary Mathematics AMS, 672 (2016) 73-126.

2-local derivation

Recall that a map

$$\Delta : \mathcal{A} \to \mathcal{A}$$

(not linear in general) is called a 2-local derivation if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

2-local derivation

2-local derivation

Any derivation is a local and a 2-local derivation, but the converse is not true in general.

Example

Consider an algebra upper-triangular 2×2-matrix

$$\mathcal{A} = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} : a_{ij} \in \mathbb{C} \right\}.$$

Define operator Δ on $\mathcal{A}$ by

$$\Delta(A) = \begin{cases} 0, & \text{if } a_{11} \neq a_{22}, \\ \begin{pmatrix} 0 & 2a_{12} \\ 0 & 0 \end{pmatrix}, & \text{if } a_{11} = a_{22}. \end{cases}$$

Then Δ is a 2-local derivation, which is not a derivation[a].

[a] J. H. Zhang, H. X. Li, 2-Local derivations on digraph algebras, Acta Math. Sinica, Chinese series 49 (2006), 1401–1406.

Outline

The notion of 2-local derivations it was introduced in 1997 by P. Šemrl [a] and in this paper he described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. He proved that each 2-local derivation on this algebra is a (global) derivation. A similar description for the finite-dimensional case appeared later by S. O. Kim and J. S. Kim [b]

[a] Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc. 125 (1997) 2677–2680.

[b] S. O. Kim, J. S. Kim, Local automorphisms and derivations on M_n, Proc. Amer. Math. Soc. 132 (2004) 1389–1392.

2-local derivation on $B(H)$. Separabel case

The methods of the proofs above mentioned results essentially based the fact that an algebra $B(H)$ is single generated.
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$$A = \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 0 & \frac{1}{2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2^n} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

2-local derivation on $B(H)$. General case

In 2012 [a] we suggested a new technique and have generalized the above mentioned results for arbitrary Hilbert spaces. Namely we considered 2-local derivations on the algebra $B(H)$ of all bounded linear operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation.

[a] Sh. A. Ayupov, K. K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, J. Math. Anal. Appl. 395 (2012) 15–18.

2-local derivation on $B(H)$. General case

Our proof essentially use existence a faithful normal semi-finite trace on $B(H)$. Namely, the main ingredient of our paper is the following indentity

$$\operatorname{tr}(\Delta(x)y) = -\operatorname{tr}(x\Delta(y)) \tag{1}$$

for all $x \in B(H)$, and for finite-dimensional operator $y \in B(H)$, where tr is the canonical trace on $B(H)$.

A similar result for 2-local derivations on finite von Neumann algebras was obtained by Sh. A. Ayupov and etal. [a]. Finally, Sh. A. Ayupov and F. N. Arzikulov[b] extended all above results and give a short proof of this result for arbitrary semi-finite von Neumann algebras.

[a] Sh. A. Ayupov, K. K. Kudaybergenov, B. O. Nurjanov, A. K. Alauatdinov, Local and 2-local derivations on noncommutative Arens algebras, Mathematica Slovaca. 64 (2014) 1–10.

[b] Sh. A. Ayupov, F. N. Arzikulov, 2-local derivations on semi-finite von Neumann algebras, Glasgow Math. Jour. 56 (2014) 9–12.

Outline

Main result

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Theorem 3.1

Let M be an arbitrary von Neumann algebra. Then any 2-local derivation $\Delta : M \to M$ is a derivation.

Main result

For a self-adjoint subset $S \subseteq M$ denote by S' is the commutant of S, i.e.

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Let $g \in M$ be a self-adjoint element and let $\mathcal{W}^*(g) = \{g\}''$ be the abelian von Neumann algebra generated by the element g. Then there exists an element $a \in M$ such that

$$\Delta(x) = ax - xa$$

for all $x \in \mathcal{W}^*(g)$. In particular, Δ is additive on $\mathcal{W}^*(g)$.

Let $P(M)$ denote the lattice of all projections of the von Neumann algebra M. Recall that a map $m : P(M) \to \mathbb{C}$ is called a signed measure (or charge) if $m(e_1 + e_2) = m(e_1) + m(e_2)$ for arbitrary orthogonal projections e_1, e_2 in M. A signed measure m is said to be bounded if $\sup\{|m(e)| : e \in P(M)\}$ is finite.

We need the following version of the Gleason's Theorem[a].

[a] L.J. Bunce, J.D. Maitland Wright, The Mackey-Gleason problem, Bull. Amer. Math. Soc., 26 (1992) 288–293.

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[a] L.J. Bunce, J.D. Maitland Wright, The Mackey-Gleason problem, Bull. Amer. Math. Soc., 26 (1992) 288–293.

Gleason Theorem

Let $\mathcal{A}$ be a von Neumann algebra with no direct summand of Type I_2. Then each complex-valued finitely additive measure on $P(\mathcal{A})$ extends to a bounded linear functional on A.

Main result

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Lemma 3.2

Let M be an infinite von Neumann algebra. The restriction $\Delta|_{M_{sa}}$ of the 2-local derivation Δ onto the set M_{sa} of all self-adjoint of M is additive.

Lemma 3.3

There exists an element $a \in M$ such that $\Delta(x) = D_a(x) = ax - xa$ for all $x \in M_{sa}$.

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In order to prove this Lemma we consider the extension $\widetilde{\Delta}$ of $\Delta|_{M_{sa}}$ on M defined by:

$$\widetilde{\Delta}(x_1 + ix_2) = \Delta(x_1) + i\Delta(x_2),\ x_1, x_2 \in M_{sa}.$$

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$$\widetilde{\Delta}(x_1 + ix_2) = \Delta(x_1) + i\Delta(x_2),\ x_1, x_2 \in M_{sa}.$$

Lemma 3.4

If $\Delta|_{M_{sa}} \equiv 0$, then $\Delta \equiv 0$.

For details of the proof we refer to
Sh.A.Ayupov, K. K.Kudaybergenov, "2-Local derivations on von Neumann algebras"POSITIVITY, 19, 2015, No.3, 445-455.

Outline

AW^*- algebras

The notion of AW^*-algebras was introduced by Kaplansky as an abstract generalization of von Neumann algebras. Namely, AW^*-algebra is a C^*-algebra such that the left annihilator of any subset is a principal left ideal generated by a projection. He showed that much of the "non spatial theory"of von Neumann algebras can be extended to AW^*-algebras.

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Every von Neumann algebra is a AW^*-algebra but the converse is not true as was shown by Dixmier with an abelian example [J. Dixmier, Sur certain espaces consideres par M.H.Stone, Summa Brasil Math. 2 (1951), 151-182].

AW^*- algebras

Kaplansky proved that an AW^*-algebra of type I is a von Neumann algebra if and only if its center is a von Neumann algebra and conjectured that this is true for general AW^*-algebras. But in 1970 Takenouchi and Dyer independently showed this to be false by providing examples of type III AW^*-algebras which are not von Neumann algebras [O. Takenouchi, A non-W*, AW*-factor, Lect. Notes Math., vol. 650 (1978), 135-139], [J.Dyer, Concerning AW*-algebras, Notices Amer. Math. Soc., 17 (1970), 788].

AW^*- algebras

In the present section we extend our above results concerning 2-local derivations to the case of arbitrary AW^*-algebras. First we consider 2-local derivations on matrix algebras over unital semi-prime Banach algebras. Namely, we prove that if $\mathcal{A}$ is a unital semi-prime Banach algebra with the inner derivation property then any 2-local derivation on the algebra $M_{2^n}(\mathcal{A})$, $n \geq 2$, is a derivation. We apply this result to AW^*-algebras and prove that any 2-local derivation on an arbitrary AW^*-algebra is a derivation.

Outline

If $\Delta : \mathcal{A} \to \mathcal{A}$ is a 2-local derivation, then from the definition it easily follows that Δ is homogenous. At the same time,

$$\Delta(x^2) = \Delta(x)x + x\Delta(x) \tag{2}$$

for each $x \in \mathcal{A}$.

In the paper of M. Bresar
[M. Bresar, Jordan derivations on semi-prime rings. Proc. Amer. Math. Soc., 104 (1988), 1003-1006]
it is proved that any Jordan derivation (i.e. a linear map satisfying the above equation) on a semi-prime algebra is a derivation. Therefore, in the case semi-prime algebras in order to prove that a 2-local derivation $\Delta : \mathcal{A} \to \mathcal{A}$ is a derivation, it is sufficient to prove that $\Delta : \mathcal{A} \to \mathcal{A}$ is additive.

We say that an algebra $\mathcal{A}$ has the inner derivation property if every derivation on $\mathcal{A}$ is inner. Recall that an algebra $\mathcal{A}$ is said to be semi-prime if $a\mathcal{A}a = 0$ implies that $a = 0$.
Let $M_n(\mathcal{A})$ be the algebra of $n \times n$-matrices over $\mathcal{A}$ and assume that $n \geq 2$.

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Let $M_n(\mathcal{A})$ be the algebra of $n \times n$-matrices over $\mathcal{A}$ and assume that $n \geq 2$.

Lemma 4.1

Let $\mathcal{A}$ be a unital Banach algebra with the inner derivation property. Then the algebra $M_n(\mathcal{A})$ also has the inner derivation property.

The following theorem is the main result of this section.

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Theorem 4.2

Let $\mathcal{A}$ be a unital semi-prime Banach algebra with the inner derivation property and let $M_{2^n}(\mathcal{A})$ be the algebra of $2^n \times 2^n$-matrices over $\mathcal{A}$. Then any 2-local derivation Δ on $M_{2^n}(\mathcal{A})$ is a derivation.

2-Local derivations on matrix algebras

The proof of Theorem 4.2. is rather technical and consists of two steps. For details we refer to our paper [Sh.A.Ayupov, K.K.Kudaybergenov, 2-Local derivations on matrix algebras over semi-prime rings and on AW*-algebras. IOP Publishing. Journal of Physics: Conference Series 697 (2016), 1-11]

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In the first step we shall show additivity of Δ on the the subalgebra of diagonal matrices from $M_{2^n}(\mathcal{A})$.

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In the first step we shall show additivity of Δ on the the subalgebra of diagonal matrices from $M_{2^n}(\mathcal{A})$.

In the second step of our proof we show that if a 2-local derivation Δ on a matrix algebra equals to zero on all diagonal matrices and on the linear span of matrix units, then it is identically zero on the whole algebra.

The condition on the algebra $\mathcal{A}$ to be a Banach algebra was applied in the proof only for the invertibility of elements of the forms $\mathbf{1} + x$, where $x \in \mathcal{A}$, $\|x\| < 1$. In this connection the following question naturally arises.

The condition on the algebra $\mathcal{A}$ to be a Banach algebra was applied in the proof only for the invertibility of elements of the forms $\mathbf{1} + x$, where $x \in \mathcal{A}$, $\|x\| < 1$. In this connection the following question naturally arises.

Problem 4.3.

Does Theorem 4.2. hold for arbitrary (not necessarily normed) algebra $\mathcal{A}$ with the inner derivation property?

Outline

In this section we apply Theorem 4.2. to the description of 2-local derivations on AW^*-algebras.

An application to AW^*-algebras

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Theorem 5.1

Let $\mathcal{A}$ be an arbitrary AW^*-algebra. Then any 2-local derivation Δ on $\mathcal{A}$ is a derivation.

An application to AW^*-algebras

Proof.Let us first note that any AW^*-algebra is semi-prime. It is also known that AW^*-algebra has the inner derivation property [D. Olesen, Derivations AW^*-algebras are inner, Pacific J. Math., 53, 555-561 (1974)].

An application to AW^*-algebras

Let z be a central projection in $\mathcal{A}$. Since $D(z) = 0$ for an arbitrary derivation D, it is clear that $\Delta(z) = 0$ for any 2-local derivation Δ on $\mathcal{A}$. Take $x \in \mathcal{A}$ and let D be a derivation on $\mathcal{A}$ such that $\Delta(zx) = D(zx), \Delta(x) = D(x)$. Then we have $\Delta(zx) = D(zx) = D(z)x + zD(x) = z\Delta(x)$. This means that every 2-local derivation Δ maps $z\mathcal{A}$ into $z\mathcal{A}$ for each central projection $z \in \mathcal{A}$. So, we may consider the restriction of Δ onto $e\mathcal{A}$. Since an arbitrary AW^*-algebra can be decomposed along a central projection into the direct sum of an abelian AW^*-algebra, and AW^*-algebras of type $\mathrm{I}_n, n \geq 2$, type I_∞,, type II and type III , we may consider these cases separately.

An application to AW^*-algebras

Let $\mathcal{A}$ be an abelian AW^*-algebra. It is well-known that any derivation on a such algebra is identically zero. Therefore any 2-local derivation on an abelian AW^*-algebra is also identically zero.

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If $\mathcal{A}$ is an AW^*-algebra of type I_n, $n \geq 2$, with the center $Z(\mathcal{A})$, then it is isomorphic to the algebra $M_n(Z(\mathcal{A}))$. By Theorem 4.2. there exists a derivation D on $\mathcal{A} \equiv M_n(Z(\mathcal{A}))$ such that $\Delta \equiv D$. So, Δ is a derivation.

Let the AW^*-algebra $\mathcal{A}$ have one of the types I_∞, II or III. Then the halving Lemmata for type I_∞, type II and type III AW^*-algebras from
[I. Kaplansky, Projections in Banach algebras, Ann. Math. 53, 235-249 (1951)],
imply that the unit of the algebra $\mathcal{A}$ can be represented as a sum of mutually equivalent orthogonal projections e_1, e_2, e_3, e_4 from $\mathcal{A}$. Then the map $x \mapsto \sum\limits_{i,j=1}^{4} e_i x e_j$ defines an isomorphism between the algebra $\mathcal{A}$ and the matrix algebra $M_4(\mathcal{B})$, where $\mathcal{B} = e_{1,1}\mathcal{A}e_{1,1}$. Therefore Theorem 4.2. implies that any 2-local derivation on $\mathcal{A}$ is a derivation. The proof is complete.

THANKS FOR YOUR ATTENTION!