Weighted inequalities involving Hardy operators

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Hardy inequality for non-negative functions

The (general) weighted Hardy inequality

$$\left(\int_a^b \left(\int_a^x f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}}. \qquad (1)$$

where a, b are real numbers, satisfying

$$-\infty \leq a < b \leq \infty,$$

w, v are weight4ed functions, i.e. measureble functions positive a.e. in (a, b)
p, q are real parameters, satisfying

$$0 < p,\, q \leq \infty.$$

The role of interval (a, b) is not important. It is easy to show that any general interval (a, b) can be reduced to the interval $(0, \infty)$.

The inequality (1) studied by many authors

1. $1 \le p \le q \le \infty$ (Muckenhoupt, Bredly, Kokilashvili, Tomaselli, Talenty, Artola, and more)
2. $1 \le q < p \le \infty$ (Mazja, Rosin)
3. $0 < q < 1 < p \le \infty$ (Sinnamon)
4. $0 < q < 1 = p$ (Sinnamon, Stepanov)

After result of Muckenhoupt(1972), where he show that (1) holds for $p = q$ if and only if

$$\sup_{x\in(0,\infty)} \left(\int_x^\infty w(y)dy\right)^{\frac{1}{p}} \left(\int_0^x v^{1-p'}(y)dy\right)^{\frac{1}{p'}} < \infty,$$

It is now named Muckenhoupt condition.

Nowadays exists many equivalent characterisations and many difference proofs.

A. Gogatishvili, A. Kufner, L.-E. Persson and A. Wedestig. An equivalence theorems for some scales of integral conditions related to Hardy's inequality with applications. Real Analysis and Exchange, 29(2003/04), no. 2, 867 - 880.

A. Gogatishvili, A. Kufner and L. E. Persson. Some new scales of characterization of Hardy?s inequality. Proc. Est. Acad. Sci. 59(2010), no. 1, 7 - 18.

Theorem 1

Let v, w be weights on $(0,\infty)$, $p \in [1,\infty)$ and $q \in (0,\infty)$. For $t \in (0,\infty)$, denote

$$V_p(t) = \begin{cases} \left(\int_0^t v^{1-p'}\right)^{\frac{1}{p'}} & \textit{if } p \in (1,\infty), \\ \operatorname*{ess\,sup}_{s\in(0,t)} \frac{1}{v(s)} & \textit{if } p = 1, \end{cases}$$

and

$$W(t) = \int_t^\infty w.$$

Then there exists a positive constant C such that (1) holds for every nonnegative measurable function f on $(0,\infty)$ if and only if $A < \infty$, where

$$A = \begin{cases} \sup_{t\in(0,\infty)} V_p(t)W(t)^{\frac{1}{q}} & \textit{if } p \le q, \\ \int_0^\infty W^{\frac{p}{p-q}}\, dV_p^{\frac{pq}{p-q}} & \textit{if } p > q, \end{cases}$$

in which the latter integral should be understood in the Lebesgue–Stieltjes sense with respect to the (monotone) function $V_p^{\frac{pq}{p-q}}$.

A.Gogatishvili, L.Pick. The two-weight Hardy inequality: a new elementary and universal proof arXiv:2109.15011

The general weighted Hardy inequality with kernel.

We say that $U\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ is a regular kernel if U is nonincreasing in the first variable, non-decreasing in the second variable and there exists a positive constant C such that

$$U(x,y) \le C(U(x,z) + U(z,x)) \quad \text{for every } x,z,y \in \mathbb{R},\ x \le z \le y. \tag{2}$$

We shall call C from (2) the constant of regularity of U.
Clearly, if U is a regular kernel and $0 < p < \infty$, then U^p is also a regular kernel.
definition of a regular kernel was introduced by Bloom and Kerman (1991).
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Let

$$\left(\int_0^\infty\left(\int_0^x U(x,t)f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}}\le C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}. \tag{3}$$

The inequality (3) was investigate by

- $1<p,q\le\infty$ Martin-Reyes and Sawyer(1989), Bloom and Kerman (1991), Oinarov (1991), Stepanov (1993)
- $0<q<1\le p<\infty$ Křepela (2017)

Theorem 2

Let U is regular kernel. Then the inequality (3) holds for all $f \geq 0$ if and only if

(I) $1 < p \leq q < \infty$

$$A_1 := \sup_{x\in(0,\infty)} \left(\int_x^b U(y,x)^q w(y)dy\right)^{\frac{1}{p}} V_p(a,x) < \infty,$$

$$A_2 := \sup_{x\in(0,\infty)} \left(\int_x^\infty w(y)dy\right)^{\frac{1}{p}} \left(\int_0^x U(x,y)^{p'} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} < \infty,$$

The best constant in (3) satisfies $C \approx A_1 + A_2$.

(II) $1 < q < p < \infty.$

$$A_3 := \left(\int_0^\infty \left(\int_x^\infty U(y,x)^q w(y)dy\right)^{\frac{p}{p-q}} (V_p(x))^{\frac{pp'q}{q'(p-q)}} v^{1-p'}(x)dx\right)^{\frac{p-q}{pq}} < \infty,$$

$$A_4 := \left(\int_0^\infty \left(\int_x^\infty w(y)dy\right)^{\frac{q}{p-q}} \left(\int_0^x U(x,y)^{p'} v^{1-p'}(y)dy\right)^{\frac{pq}{p'(p-q)}} w(x)dx\right)^{\frac{p-q}{pq}} < \infty,$$

The best constant in (3) satisfies $C \approx A_3 + A_4$.

Theorem 2

(III) $0 < q < 1 \le p < \infty$

$$A_4 < \infty,$$

$$A_5 := \left(\int_0^\infty \left(\int_x^b U(y,x)^q w(y) dy \right)^{\frac{q}{p-q}} \operatorname*{ess\,sup}_{a<t<x} U(x,t) \big(V_p(t)\big)^{\frac{pq}{p-q}} w(x) dx \right)^{\frac{p-q}{pq}} < \infty,$$

The best constant in (3) *satisfies* $C \approx A_4 + A_4$.

Problem 0.1

Fined simple proof of the general Hardy inequality with regular kernel.

Hardy inequality restricted on the cones of monotone functions

The boundedness of classical operators in Lorentz spaces is equivalent certain weighted Hardy inequality restricted on the cones of monotone functions.

Let $\mathbb{R}_+ := [0, \infty)$. Denote $\mathfrak{M}^+$ the set of all non-negative measurable functions on $\mathbb{R}_+$ and $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+$ ($\mathfrak{M}^\uparrow \subset \mathfrak{M}^+$) the subset of all non-increasing (non-decreasing) functions.

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C \left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}. \quad f \in \mathfrak{M}^\downarrow \tag{4}$$

The inequality (4) was caharacterized by Ariño, Muchenhoupt (1990) for $1 < p = q < \infty$ and $w = v$. In fact it is was also obtained by Boyd (1967). f Let $1 < p = q < \infty$ and $w = v$. Then the inequality (4) holds if and only if

$$B_p := \sup_{x \in (0,\infty)} x^p \left(\int_0^x w\right)^{-1} \int_x^\infty t^{-p} w < \infty.$$

- M. Carro, A. Gogatishvili, M. L. Gol'dman, H. Heinig, L. Pick, E. Sawyer, G. Sinnamon, J. Soria and V.D. Stepanov
- (Carro/Pick/Soria/Stepanov, 2001)

$$\left(\int_0^\infty \left(\int_0^x U(x,t)u(t)f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}. \quad f \in \mathfrak{M}^{\downarrow} \tag{5}$$

and dual version

$$\left(\int_0^\infty \left(\int_x^\infty U(t,x)u(t)f(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}. \quad f \in \mathfrak{M}^{\downarrow} \tag{6}$$

where U is regular kernel and u is weighed function.

1. $0 < q < \infty, 1 \leq p < \infty$.
2. $0 < p \leq q < \infty$.
3. $0 < q < p < 1$.

we will use following notation $V(t) = \int_0^t v(t)dt$.

The following results are from the following paper.

A. Gogatishvili and V. D. Stepanov. Reduction theorems for weighted integral inequalities on the cone of monotone functions. Russian Math. Surveys 68:4 597-664

Theorem 3

Let $0 < q \le \infty, 1 < p < \infty$ *and let* U *is regular kernel. Then the inequality* (5) *holds iff the following two inequalities are valid:*

$$\left(\int_0^\infty \left(\int_0^x U(x,t)u(t)\left(\int_t^\infty h\right)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p V^p v^{1-p}\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+, \tag{7}$$

and

$$\left(\int_0^\infty \left(\int_0^x U(x,t)u(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v\right)^{\frac{1}{p}}. \tag{8}$$

Theorem 4

Let $0 < q \le \infty, 1 < p < \infty$ *and let* U *is regular kernel. Then the inequality* (6) *holds iff the following two inequalities are valid:*

$$\left(\int_0^\infty \left(\int_x^\infty U(t,x)u(t)\left(\int_t^\infty h\right)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p V^p v^{1-p}\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+, \tag{9}$$

and

$$\left(\int_0^\infty \left(\int_x^\infty U(x,t)u(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v\right)^{\frac{1}{p}}. \tag{10}$$

Theorem 3 and Theorem 4 follows from general reduction theorem:
A mapping $\rho : \mathfrak{M}^+ \to [0,\infty)$ is called a monotone quasinorm if

(a) $\rho(\lambda f) = \lambda \rho(f)$ for all $\lambda \geq 0$ and $f \in \mathfrak{M}^+$;
(b) $\rho(f+g) \leq c(\rho(f) + \rho(g))$ for all $f, g \in \mathfrak{M}^+$, where c is a positive constant independent of f and g;
(c) $\rho(f) \leq c\rho(g)$ for almost every $x \in [0,\infty)$, if $f(x) \leq g(x)$ for almost every $x \in [0,\infty)$, where c is a positive constant independent of f and g.

Theorem 5

Let $1 < p < \infty$ and let ρ be any monotone quasinorm. Then the inequality

$$\rho(f) \leq C \left(\int_0^\infty f^p(x) v(x) dx \right)^{\frac{1}{p}}. \quad f \in \mathfrak{M}^\downarrow \tag{11}$$

holds if and only if the following two inequalities are valid:

$$\rho\left(\int_t^\infty h \right) \leq C \left(\int_0^\infty h^p V^p v^{1-p} \right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+, \tag{12}$$

and

$$\rho(\mathbf{1}) \leq C \left(\int_0^\infty v \right)^{\frac{1}{p}}. \tag{13}$$

In the case $0 < q < 1$, we know only discrete characterization of the inequality (9)
We have second reduction Theorem

Theorem 6

Let $1 < p < \infty$ and let ρ be any monotone quasinorm. Then the inequality (11) holds if and only if the following inequality is valid:

$$\rho\left(\frac{1}{V^2(t)}\int_0^t hV\right) \le C\left(\int_0^\infty h^p v^{1-p}\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+, \tag{14}$$

Theorem 7

Let $0 < q \le \infty, 1 < p < \infty$ *and let* U *is regular kernel. Then the inequality* (5) *holds iff the following two inequalities are valid:*

$$\left(\int_0^\infty \left(\int_0^x U(x,t)u(t)\left(\frac{1}{V^2(t)}\int_0^t hV\right)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p v^{1-p}\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+, \tag{15}$$

Theorem 8

Let $0 < q \le \infty, 1 < p < \infty$ *and let* U *is regular kernel. Then the inequality* (6) *holds iff the following two inequalities are valid:*

$$\left(\int_0^\infty \left(\int_x^\infty U(t,x)u(t)\left(\frac{1}{V^2(t)}\int_0^t hV\right)dt\right)^q w(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p v^{1-p}\right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+ \tag{16}$$

Theorem 9

Let $0 < p \leq 1$ $p \leq q < \infty$ *and let* U *is regular kernel. Then the inequality* (5) *holds iff the following two inequalities are valid:*

$$\sup_{y\in(0,\infty)} \left(\int_0^\infty \left(\int_0^{\min(x,y)} U(x,t)u(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} V(y)^{-\frac{1}{p}} < \infty. \qquad (17)$$

Theorem 10

Let $0 < p \leq 1$ $p \leq q < \infty$ *and let* U *is regular kernel. Then the inequality* (6) *holds iff the following two inequalities are valid:*

$$\sup_{y\in(0,\infty)} \left(\int_0^y \left(\int_x^y U(t,x)u(t)dt\right)^q w(x)dx\right)^{\frac{1}{q}} V(y)^{-\frac{1}{p}}, \qquad (18)$$

$q < p \leq 1$

$$V(t) := \int_0^t v, \quad U_u(x,y) := \int_0^y U(x,z)u(z)dz.$$

Theorem 11

Let $0 < q < p \leq 1, 1/r := 1/q - 1/p$. *Let* $U(x,y)$ *be a regular kernel. The inequality* (5) *is equivalent with one of the following inequalities:*

$$\left(\int_0^\infty \left(\int_0^x U(x,y)\left(\int_y^\infty h\right)^{\frac{1}{p}} u(y)dy\right)^q w(x)dx\right)^{\frac{p}{q}} \leq C_1^p \int_0^\infty hV, \ f \in \mathfrak{M}^+, \tag{19}$$

$$\left(\int_0^\infty \left(\int_0^x U_u^p(x,y)h(y)u(y)dy + U_u(x,x)\int_x^\infty h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \tag{20}$$

$$\leq C_2^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{21}$$

$$\left(\int_0^\infty \left(\sup_{y \geq x} U_u^p(y,x)\int_y^\infty h\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leq C_3^p \int_0^\infty hV, \ h \in \mathfrak{M}^+, \tag{22}$$

The proof is base on the following estimates: If $0 < p < 1$

$$\left(\sup_{0<y<x} U_u^p(x,y) \int_y^\infty h \right)^{\frac{1}{p}} \le \int_0^x U(x,y)u(y) \left(\int_y^\infty h \right)^{\frac{1}{p}} dy \lesssim \left(\int_0^x U_u^p(y,x)h(y)dy \right)^{\frac{1}{p}}$$

Denote

$$\widetilde{U}_u(y,x) := \int_x^y U(z,x)u(z)dz.$$

Theorem 12

Let $0 < q < p \leq 1, 1/r := 1/q - 1/p$. *TheThe inequality* (6) *is equivalent with one of the following inequalities:*

$$\left(\int_0^\infty \left(\int_x^\infty U(y,x)\left(\int_y^\infty h\right)^{\frac{1}{p}} u(y)dy\right)^q w(x)dx\right)^{\frac{p}{q}} \leq C_1^p \int_0^\infty hV \ \ h \in \mathfrak{M}+, \quad (23)$$

$$\left(\int_0^\infty \left(\int_x^\infty \widetilde{U}_u^p(y,x)h(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leq C_2^p \int_0^\infty hV, \ \ h \in \mathfrak{M}^+, \quad (24)$$

$$\left(\int_0^\infty \left(\sup_{y\geq x} \widetilde{U}_u^p(y,x) \int_y^\infty h\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leq C_3^p \int_0^\infty hV, \ \ h \in \mathfrak{M}^+, \quad (25)$$

Moreover, $C_1 \approx C_2 \approx C_3$.

The proof is base on the following estimates: If $0 < p < 1$

$$\left(\sup_{y\geq x}\widetilde{U}_u^p(y,x)\int_y^\infty h\right)^{\frac{1}{p}} \leq \int_x^\infty U(y,x)u(y)\left(\int_y^\infty h\right)^{\frac{1}{p}}dy \lesssim \left(\int_x^\infty \widetilde{U}_u^p(y,x)h(y)dy\right)^{\frac{1}{p}}$$

Observation this fact in the following paper

A. Gogatishvili, L. Pick and T. Ünver. Weighted inequalities for discrete iterated kernel operators, Math. Nachr. 295, (2022), no. 11, 2171 – 2196

we have following theorem:

Theorem 13

Let $1 \le p < \infty$ and $0 < q < \infty$ and U be a regular kernel. Then the following three statements are equivalent.

(i) *There exists a constant C_1 such that*

$$\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{0<y\le x} U(y,x)\int_0^y f\right)^q w(x)dx\right)^{\frac{1}{q}} \le C_1 \left(\int_0^\infty f^p v\right)^{\frac{1}{p}} \quad f \in \mathcal{M}_+ \tag{26}$$

(ii) *There exists a constant C_2 such that*

$$\left(\int_0^\infty \left(\int_0^x U(y,x)^p f(y)dy\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_2 \int_0^\infty f(x)V_p(0,x)^{-p}dx \quad f \in \mathcal{M}_+ \tag{27}$$

(iii) *There exists a constant C_3 such that*

$$\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{0<y\le x} U(y,x)^p \int_0^y f\right)^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \le C_3 \int_0^\infty f(x)V_p(0,x)^{-p}dx \quad f \in \mathcal{M}_+ \tag{28}$$

Moreover, if C_1, C_2 and C_3 are the best constants in (26), (27) and (28), respectively, then $C_1 \approx C_2^{\frac{1}{p}} \approx C_3^{\frac{1}{p}}$.

The proof is based on the so call discretizing techniques.

Note that the kernel U_u and $\widetilde{U}_u$ are not regular kernels. The Theorem 13 for more general kernel then regular kernel which cover the U_u and $\widetilde{U}_u$ kernel unknown.

Definition 14

Let $\lambda > 0$. We say that a non-negative function h is λ-*quasiconcave* if h is equivalent to a non-decreasing function on $(0, \infty)$ and $\frac{h(t)}{t^\lambda}$ is equivalent to a non-increasing function on $(0, \infty)$. We denote by Ω_λ the family of λ-*quasiconcave* functions. We say that h is *quasiconcave* when $\lambda = 1$ and we write that $h \in \Omega$.

A. Gogatishvili and J. S. Neves, Weighted norm inequalities for positive operators restricted on the cone of λ-quasiconcave functions, accepted in Proc. Roy. Soc. Edinburgh Sect. A 150, no. 1, 17-39, 2020

A. Gogatishvili and J. S. Neves, Weighted norm inequalities for positive operators restricted on the cone of λ-quasiconcave functions?corrigendum. Proc. Roy. Soc. Edinburgh Sect. A 152,no. 2, 542-543. 2022.

Theorem 15

Let $\lambda > 0$ *and* $1 \le p < \infty$. *Let* ρ *be any monotone quasinorm. Then the inequality*

$$\rho(f) \le C_1 \left(\int_0^\infty (f(t))^p v(t)\,dt \right)^{\frac{1}{p}}, \; f \in \Omega_\lambda, \tag{29}$$

holds if, and only if, the following three inequalities are valid:

$$\rho\left(\int_0^x h + x^\lambda \int_x^\infty t^{-\lambda} h \right) \le C_2 \left(\int_0^\infty h^p(x) \frac{x^{\lambda p(1-p)} \left(\int_0^x t^{\lambda p} v\right)^{1-p} \left(\int_x^\infty v\right)^{1-p}}{\left(\int_0^x t^{\lambda p} v + x^{\lambda p} \int_x^\infty v\right)^{1-2p}}\,dx \right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+; \tag{30}$$

$$\rho\left(T(\mathbf{1})\right) \le C_3 \left(\int_0^\infty v \right)^{\frac{1}{p}}; \tag{31}$$

$$\rho\left(T\left(x^\lambda\right)\right) \le C_4 \left(\int_0^\infty x^{\lambda p} v(x)\,dx \right)^{\frac{1}{p}}. \tag{32}$$

Definition 16

A pair (X_0, X_1) of Banach spaces X_0 and X_1 is called a compatible couple if there is some Hausdorff topological vector space, say $\mathcal{X}$, in which each of X_0 and X_1 is continuously embedded.

Note That (L^1, L^∞) is a compatible couple because both L^1 and L^∞ are continuously embedded in the Housdorf space $\mathcal{M}_0$ of measurable functions that are finite a.e.

Applications

K-interpolation method or real method. K-interpolation method or real method. Interpolation constructions are obtained with the help of the K-functional. As usual we denote by $K(t, f, X_0, X_1)$ the K-functional of the couple X_0, X_1, i.e.,

$$K(t, f, X_0, X_1) = \inf_{f=f_0+f_1} \|x_0\|_{f_0} + t\|f_1\|_{X_1}, \quad t > 0,$$

where the infimum is taken over all representations of x as a sum of $f_0 \in X_0$ and $f_1 \in X_1$.

$$(X_0, X_1)^K_{w,p} := \left\{ f \in X_0 + X_1 : \quad \|x\|^K_{w,p} := \left(\int_0^\infty K(t, x, X_0, X_1)^p w(t) dt \right)^{\frac{1}{p}} < \infty \right\}$$

$$\varphi_i(x) := \left(\int_0^\infty \min(t, x)^{p_i} w_i(t) dt \right)^{\frac{1}{p_i}} \quad i = 0, 1$$

let $\frac{\varphi_0(x)}{\varphi_1(x)}$ is increasing and define the function σ by equality $x = \frac{\varphi_0(\sigma(x))}{\varphi_1(\sigma(x))}$.

$$K(t, f, (X_0, X_1)_{w_0,p_0}, (X_0, X_1)_{w_1,p_1}) \approx \left(\int_0^{\sigma(x)} K(t, f, X_0, X_1)^{p_0} w_0(t) dt \right)^{\frac{1}{p_0}}$$

$$\left(\int^\infty \right)^{\frac{1}{p_1}}$$

(33) is satisfies if and only if

$$\left(\int_0^{\sigma(x)} K(t,f,X_0,X_1)^{p_0} w_0(t)dt\right)^{\frac{1}{p_0}} \lesssim t\left(\int_0^{\infty} K(t,f,X_0,X_1)^{p_1} w_1(t)dt\right)^{\frac{1}{p_1}} \quad (34)$$

$$t\left(\int_{\sigma(x)}^{\infty} K(t,f,X_0,X_1)^{p_1} w_1(t)dt\right)^{\frac{1}{p_1}} \lesssim \left(\int_0^{\infty} K(t,f,X_0,X_1)^{p_0} w_0(t)dt\right)^{\frac{1}{p_0}} \quad (35)$$

Let $p_0 \le p_1$ and $\frac{1}{r} = \frac{1}{p_0} - \frac{1}{p_1}$
(34) holds if and only if

$$\left(\int_0^{x} \frac{\varphi_0(t)^{r-p_0}}{\varphi_1(t)^r} w_0(t)dt\right)^{\frac{1}{r}} \lesssim \frac{\varphi_0(x)}{\varphi_1(x)} \quad (36)$$

and (35) holds if and only if

$$\frac{\varphi_1(x)}{\varphi_0(x)} \quad \text{is quasi-decreasing.}$$

(36) $\iff$ there is exists small $\varepsilon > 0$ such that $\frac{\varphi_0(x)^{1-\varepsilon}}{\varphi_1(x)}$ is quasi-increasing.

Proposition 17

The formula (33) hold if and only if there is exists small $\varepsilon > 0$ such that $\frac{\varphi_0(x)^{1-\varepsilon}}{\varphi_1(x)}$ is quasi-increasing.

The result of what is known form follows sufficient condition that (33) is held if exists small $\varepsilon > 0$ such that $\frac{\varphi_0(x)}{\varphi_1(x)x^{\varepsilon}}$ is quasi-increasing.
Let

$$w_0(t) = \frac{1}{t}\left(\ln\left(\frac{e}{t}\right)\right)^{\alpha_0}\chi_{(0,1)}(t) \quad w_1(t) = \frac{1}{t}\left(\ln\left(\frac{e}{t}\right)\right)^{\alpha_1}\chi_{(0,1)}(t)$$

$$\varphi_0(t) \approx t\left(\ln\left(\frac{e}{t}\right)\right)^{\alpha_0+\frac{1}{p_0}}\chi_{(0,1)}(t) + \chi_{(1,infty)}(t)$$

$$\varphi_1(t) \approx t\left(\ln\left(\frac{e}{t}\right)\right)^{\alpha_1+\frac{1}{p_1}}\chi_{(0,1)}(t) + \chi_{(1,infty)}(t)$$

$\frac{\varphi_0(x)}{\varphi_1(x)x^{\varepsilon}}$ is quasi-decreasing for every $\varepsilon > 0$
if $\alpha_0 + \frac{1}{p_0} < \alpha_1 + \frac{1}{p_1}$ and $0 < \varepsilon < \alpha_1 + \frac{1}{p_1} - \alpha_0 + \frac{1}{p_0}$, than $\frac{\varphi_0(x)^{1-\varepsilon}}{\varphi_1(x)}$ is quasi-increasing.
Let

$$w_0(t) = \frac{1}{t}\left(\ln\left(\frac{e}{t}\right)\right)^{\alpha_0}\chi_{(0,1)}(t)$$

$$w_1(t) = \frac{1}{t}\left(\ln\left(\frac{e}{t}\right)\right)^{\alpha_0+\frac{1}{p_0}-\frac{1}{p_1}}\left(\ln\left(e\ln\left(\frac{e}{t}\right)\right)\right)^{\beta}\chi_{(0,1)}(t)$$

I. Ahmed, A. Fiorenza and A. Gogatishvili. Holmstedt's formula for the K-functional: the limit case $\theta_0 = \theta_1$. Math. Nachr. 296 (2023), no. 12, 5484 — 5492.

I. Ahmed, A. Fiorenza and A. Gogatishvili. A generalized version of Holmstedt's formula for the K-functional, Colloquium Mathematicum 176 (2024), no. 1, 77 – 85.

- $(\mathcal{R}, \mu)$ is a totally σ-finite measure space with a non-atomic measure μ,
- $\mathcal{M}$ is the set of all μ-measurable functions on $\mathcal{R}$ whose values lie in $[-\infty, \infty]$,
- $\mathcal{M}^+$ is the class of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$.

The non-increasing rearrangement of $f \in \mathcal{M}(\mathcal{R}, \mu)$ is the function f^* defined by

$$f^*(t) = \inf\{\lambda : \mu(\{x \in \mathcal{R} : |f(x)| > \lambda\}) \le t\}, \quad t \in [0, \infty).$$

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The *maximal non-increasing rearrangement* of f, f_u^{**} defined by ,

$$f_u^{**}(t) = \frac{1}{\int_0^t u(s)ds} \int_0^t f^*(s)u(s)ds, \quad t \in (0, \infty)$$

If $u = 1$ we will write f^{**}

$0 < p < \infty$, Lorentz spaces $\Lambda^p(v)$ (Lorentz 1951)

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p v(t) dt \right)^{\frac{1}{p}} < \infty.$$

$0 < p < \infty$, Lorentz spaces $\Lambda^p(v)$ (Lorentz 1951)

$$\|f\|_{\Lambda^p(v)} := \left(\int_0^\infty (f^*(t))^p v(t) dt \right)^{\frac{1}{p}} < \infty.$$

$$\|f\|_{\Gamma_u^p(v)} := \left(\int_0^\infty (f_u^{**}(t))^p v(t) dt \right)^{\frac{1}{p}} < \infty.$$

$0 < p < \infty$, the space $\Gamma_u^p(v)$ (Gogatishvili Pick 2003)
if $u = 1$ we write $\Gamma^p(v)$ (Sawyer 1990)

$0 < p < \infty$, $\mathfrak{M} = \{f \in \mathcal{M}^+ : f^*(\infty) = 0\}$,

$$\|f\|_{S^p(v)} = \left(\int_0^\infty \left[f^{**}(t) - f^*(t) \right]^p v(t)\, dt \right)^{\frac{1}{p}} < \infty$$

$v(t) = t^{\frac{1}{q} - \frac{1}{p}}$ (Bennett, De Vore, Sharpley 1981)
$p = q = \infty$ Weak-L^∞

(Carro/Gogatishvili/Martin/Pick 2005)

Remark 18

If $X \subset Y$ and the identity operator is continuous from X to Y, i.e., $\exists c : \| \mathrm{I}(z) \|_Y \leq c \|z\|_X$ for all $z \in X$, we say that X is embedded into Y and write $X \hookrightarrow Y$.

$0 < p, q \le \infty$

$$\Lambda^p(v) \hookrightarrow \Lambda^q(w)$$
$$\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$$
$$\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$$
$$\Gamma^p_u(v) \hookrightarrow \Gamma^q_u(w)$$
$$\Lambda^p(v) \hookrightarrow S^q(w)$$

$0 < p, q \le \infty$

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$$\Gamma_u^p(v) \hookrightarrow \Lambda^q(w)$$
$$\Gamma_u^p(v) \hookrightarrow \Gamma_u^q(w)$$
$$\Lambda^p(v) \hookrightarrow S^q(w)$$

$$\begin{aligned} S^p(v) \hookrightarrow S^q(w) \quad &\Longleftrightarrow \quad \Lambda^p(\tilde{v}) \hookrightarrow \Lambda^q(\tilde{w}) \\ S^p(v) \hookrightarrow \Gamma^q(w) \quad &\Longleftrightarrow \quad \Lambda^p(\tilde{v}) \hookrightarrow \Gamma^q(\tilde{w}) \\ \Gamma^p(v) \hookrightarrow S^q(w) \quad &\Longleftrightarrow \quad \Gamma^p(\tilde{v}) \hookrightarrow \Lambda^q(\tilde{w}) \\ S^p(v) \hookrightarrow \Lambda^q(w) \quad &\Longleftrightarrow \quad \Lambda^p(\tilde{v}) \hookrightarrow S^q(\tilde{w}) \end{aligned}$$

with

$$\tilde{v}(t) = v(1/t)t^{p-2}, \quad \tilde{w}(t) = w(1/t)t^{q-2}$$

$0 < p, q \le \infty$

$$\Lambda^p(v) \hookrightarrow \Lambda^q(w)$$
$$\Lambda^p(v) \hookrightarrow \Gamma^q_u(w)$$
$$\Gamma^p_u(v) \hookrightarrow \Lambda^q(w)$$
$$\Gamma^p_u(v) \hookrightarrow \Gamma^q_u(w)$$
$$\Lambda^p(v) \hookrightarrow S^q(w)$$

$$\begin{aligned}
S^p(v) \hookrightarrow S^q(w) &\iff \Lambda^p(\tilde{v}) \hookrightarrow \Lambda^q(\tilde{w})\\
S^p(v) \hookrightarrow \Gamma^q(w) &\iff \Lambda^p(\tilde{v}) \hookrightarrow \Gamma^q(\tilde{w})\\
\Gamma^p(v) \hookrightarrow S^q(w) &\iff \Gamma^p(\tilde{v}) \hookrightarrow \Lambda^q(\tilde{w})\\
S^p(v) \hookrightarrow \Lambda^q(w) &\iff \Lambda^p(\tilde{v}) \hookrightarrow S^q(\tilde{w})
\end{aligned}$$

with

$$\tilde{v}(t) = v(1/t)t^{p-2}, \quad \tilde{w}(t) = w(1/t)t^{q-2}$$

Remark 19

M. Carro, A. Garcia del Amo, A. Gogatishvili, M.L. Gol'dman, H. Heinig, S. Lai, L. Maligranda, J. Martin, C. Neugebauer, R. Oinarov, L. Pick, E. Sawyer, G. Sinnamon, J. Soria and V.D. Stepanov, and many more.

- M. L. Gol'dman, H. P. Heinig, and V. D. Stepanov. On the principle of duality in Lorentz spaces. Canad. J. Math., 48(5): 959 - 979, 1996.
- Gol'dman, M.L., Functions spaces and their applications, Patrice Lumumba Univ., (1991), 35–67.
- Gol'dman, M.L., On integral inequalities on a cone of functions with monotonicity properties, Soviet Math. Dokl. (2)44(1992), 581–587.
- Gol'dman, M.L., Function spaces, differential operators and nonlinear analysis, Teubner Texte, Math. 133(1993), 274–279.

$$\Lambda^p(v) \hookrightarrow \Lambda^q(w) \quad \text{(Sawyer 1990, Stepanov 1993)}$$

It is easy to see that

$$\Lambda^p(v) \hookrightarrow \Lambda^q(w) \iff \Lambda^{\frac{p}{q}}(v) \hookrightarrow \Lambda^1(w)$$

Sawyer (1990)

$$\Lambda^p(v) \hookrightarrow \Lambda^1(w), \quad 1 < p < \infty$$

Stepanov (1993) $0 < p < 1$

Now it is know as Sawyer's duality theorem

$$\Lambda^p(v) \hookrightarrow \Lambda^q(w)$$

$$\Updownarrow$$

$$\left(\int_0^\infty (f^*(t))^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f^*(t))^p v(t)dt\right)^{\frac{1}{p}}, \quad f \in \mathcal{M},$$

$$\Updownarrow$$

$$\left(\int_0^\infty \left(\int_t^\infty h\right)^{\frac{q}{p}} w(t)dt\right)^{\frac{p}{q}} \le C^p\left(\int_0^\infty h(t)\left(\int_0^t v\right)dt\right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+$$

$$\Updownarrow$$

$$\int_0^\infty h(t)\left(\int_0^t w\right)dt \le C^q\left(\int_0^\infty \left(\int_t^\infty h\right)^{\frac{p}{q}} v(t)dt\right)dt\Big)^{\frac{q}{p}}, \quad h \in \mathcal{M}^+$$

Duality

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$

$$\sup_{f\in\mathcal{M}^+, f\downarrow} \frac{\displaystyle\int_0^\infty f(x)g(x)dx}{\left(\displaystyle\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} \approx \left(\int_0^\infty \left(\int_0^x g\right)^{p'-1} \left(\int_0^x v\right)^{1-p'} g(x)dx\right)^{\frac{1}{p'}}$$

$$\approx \left(\int_0^\infty \left(\int_0^x g\right)^{p'} \frac{v(x)}{\left(\int_0^t v\right)^{p'}} dx\right)^{\frac{1}{p'}} + \frac{\int_0^\infty g}{\left(\int_0^\infty v\right)^{\frac{1}{p}}}.$$

- (Sawyer 1990)

Let $0 < p < 1$

$$\sup_{f\in\mathcal{M}^+, f\downarrow} \frac{\displaystyle\int_0^\infty f(x)g(x)dx}{\left(\displaystyle\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} \approx \sup_{0<x<\infty} \frac{\displaystyle\int_0^x g}{\left(\displaystyle\int_0^x v\right)^{\frac{1}{p}}}$$

Compare with Hölder theorem

$$\sup_{f\in\mathcal{M}^+} \frac{\displaystyle\int_0^\infty f(x)g(x)dx}{\left(\displaystyle\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}} = \left(\int_0^\infty g(t)^{p'} v(t)^{1-p'} dt\right)^{\frac{1}{p'}} \quad 1 \le p < \infty$$

The associate space $(\Lambda^p(v))'$ of $\Lambda^p(v)$ is defined by associate norm

$$\|g\|_{(\Lambda^p(v))'} := \sup_{\|f\|_{\Lambda^p(v)} \le 1} \int_0^\infty f^*(t) g^*(t) dt.$$

$$\|g\|_{(\Lambda^p(v))'} \approx \left(\int_0^\infty (g^{**}(t))^{p'} t^{p'} \left(\int_0^t v \right)^{-p'} v(t) dt \right)^{\frac{1}{p'}} + \frac{\int_0^\infty g^*(t) dt}{\left(\int_0^\infty v(t) dt \right)^{\frac{1}{p}}}.$$

Moreover, when $1 < p < \infty$

$\Lambda^p(v)$ is a Banach space if and only if $\Gamma^p(v) = \Lambda^p(v)$.

Since $f^* \le f^{**}$,

$$\Gamma^p(v) \hookrightarrow \Lambda^p(v)$$

is trivial.
The reverse embedding is not trivial (Ariño-Muckenhoupt 1990).
The embedding $0 < p, q \le \infty$

$$\Lambda^p(v) \hookrightarrow \Gamma^q_u(v)$$

- M. Carro, A. Gogatishvili, M. L. Gol'dman, H. Heinig, L. Pick, E. Sawyer, G. Sinnamon, J. Soria and V.D. Stepanov
- (Carro/Pick/Soria/Stepanov, 2001)

$$\Lambda^p(v) \hookrightarrow \Gamma_u^q(w)$$

$$\Updownarrow$$

$$\left(\int_0^\infty \left(\int_0^t f^*(s)u(s)ds\right)^q \frac{w(t)}{\left(\int_0^t u\right)^q} dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f^*(t))^p v(t)dt\right)^{\frac{1}{p}}, \quad f \in \mathcal{M},$$

$$\Updownarrow$$

$$\left(\int_0^\infty \left(\int_0^t \left(\int_s^\infty h\right)^{\frac{1}{p}} u(s)ds\right)^q \frac{w(t)}{\left(\int_0^t u\right)^q} dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h(t)\left(\int_0^t v\right)dt\right)^{\frac{1}{p}}, \quad h \in \mathcal{M}^+$$

Q1:

$$\left(\int_0^\infty \left(\int_0^x \left(\int_t^\infty f\right)^q u(t)dt\right)^{\frac{r}{q}} w(x)dx\right)^{\frac{1}{r}} \le c\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \quad f \in \mathcal{M}^+$$

Q2:

$$\left(\int_0^\infty \left(\int_0^x \left(\int_0^t f\right)^q u(t)dt\right)^{\frac{r}{q}} w(x)dx\right)^{\frac{1}{q}} \le c\left(\int_0^\infty f^p v\right)^{\frac{1}{p}}, \quad f \in \mathcal{M}^+$$

- (Carro/Gogatishvili/Martin/Pick 2008)

$$\Lambda^p(v) \hookrightarrow S^q(w)$$

$$\Updownarrow$$

$$\left(\int_0^\infty \left[f^{**}(t) - f^*(t)\right]^q w(t)\,dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty (f^*(t))^p v(t)dt\right)^{\frac{1}{p}},\ f \in \mathfrak{M}$$

$$\Updownarrow$$

$$\left(\int_0^\infty \left(\int_0^t h\right)^q \frac{w(t)}{t^q} dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left(\int_t^\infty \frac{h(s)}{s} ds\right)^p v(t)dt\right)^{\frac{1}{p}},\ h \in \mathcal{M}^+$$

Q3:

$$\left(\int_0^\infty \left(\int_0^t f^{p_2} v_2\right)^{\frac{q_2}{p_2}} u_2(t)dt\right)^{\frac{1}{q_2}} \le c\left(\int_0^\infty \left(\int_t^\infty f^{p_1} v_1\right)^{\frac{q_1}{p_1}} u_1(t)dt\right)^{\frac{1}{q_1}},\ f \in \mathcal{M}^+$$

Q4:

$$\left(\int_0^\infty \left(\int_0^t f^{p_2} v_2\right)^{\frac{q_2}{p_2}} u_2(t)dt\right)^{\frac{1}{q_2}} \le c\left(\int_0^\infty \left(\int_0^t f^{p_1} v_1\right)^{\frac{q_1}{p_1}} u_1(t)dt\right)^{\frac{1}{q_1}},\ f \in \mathcal{M}^+$$

For $f \in \mathcal{M}$, $0 < p, q \leq \infty$, the weighted Cesàro and Copson function spaces $\mathrm{Ces}_{p,q}(u,v)$ and $\mathrm{Cop}_{p,q}(u,v)$

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} := \left(\int_0^\infty \left(\int_0^t |f(s)|^p v(s) ds \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}} < \infty$$

$$\|f\|_{\mathrm{Cop}_{p,q}(u,v)} := \left(\int_0^\infty \left(\int_t^\infty |f(s)|^p v(s) ds \right)^{\frac{q}{p}} u(t) dt \right)^{\frac{1}{q}} < \infty.$$

- $\mathrm{Ces}_{1,p}(x^{-p},1)$, $\mathrm{Cop}_{1,p}(1,x^{-1})$ (G. Bennett 1996)
- (Grosse-Erdmann 1998)

Q3

$$\mathrm{Cop}_{p_1,q_1}(u_1,v_1) \hookrightarrow \mathrm{Ces}_{p_2,q_2}(u_2,v_2)$$

Q4

$$\mathrm{Ces}_{p_1,q_1}(u_1,v_1) \hookrightarrow \mathrm{Ces}_{p_2,q_2}(u_2,v_2)$$

Discretization

Definition 20

Let $M \in \mathbb{Z} \cup \{+\infty\}$ and $\{x_k\}_{k=-\infty}^{M}$ be a strictly increasing sequence in $(0, \infty)$. We say that $\{x_k\}_{k=-\infty}^{M}$ is a covering sequence if $\lim_{k \to -\infty} x_k = 0$ and, either $M = +\infty$ and $\lim_{k \to \infty} x_k = \infty$ or $M \in \mathbb{Z}$ and $x_M := \infty$.

Denote by

$$W(t) := \int_0^t w(s)ds, \quad t \in (0, \infty).$$

Definition 0.2

Let $w \in \mathcal{M}^+(0, \infty)$ such that $0 < W(t) < \infty$, $t > 0$.

- If $W(\infty) = \infty$, then let $\{x_k\}_{k=-\infty}^{M} \subset (0, \infty)$ such that $W(x_k) = 2^k, k \in \mathbb{Z}$.
- If $W(\infty) < \infty$, then define a sequence $\{x_k\}_{k=-\infty}^{M} \subset (0, \infty)$ such that $W(x_k) = 2^k, k < M, k \in \mathbb{Z}$ and $x_M = +\infty$, where M satisfies

$$2^{M-1} \leq W(\infty) < 2^M.$$

We say that the covering sequence $\{x_k\}_{k=-\infty}^{M} \subset (0, \infty)$ is a discretizing sequence of W.

This technique I learned from the paper

M. L. Gol'dman, Proc. Steklov Inst. Math. 2001, no. 1(232), 109–137; translated from Tr. Mat. Inst. Steklova 232 (2001), 115–143.

Lemma 21

Let $\alpha \geq 0$. *Assume that* w *is a weight on* $(0,\infty)$ *and* $\{x_k\}_{k=-\infty}^{M}$ *is a discretizing sequence of* W. *If* h *is a non-negative, non-increasing function on* $(0,\infty)$.

$$\int_0^\infty W(x)^\alpha w(x) h(x) dx \approx \sum_{k=-\infty}^{M-1} 2^{k(\alpha+1)} h(x_k).$$

Theorem 22

Let $0 < p \le 1$ *and* $0 < q, r < \infty$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then*

$$\left(\int_0^\infty \left(\int_t^\infty \left(\int_0^s f(\tau)^p v(\tau)d\tau\right)^{\frac{q}{p}} u(s)ds\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le C \int_0^\infty f(t)dt. \qquad \text{(Q)}$$

holds for all $f \in \mathcal{M}^+(0,\infty)$ *if and only if there exist positive constants* C' *and* C'' *such that*

$$\left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{x_{k+1}} \left(\int_{x_k}^s f(\tau)^p v(\tau)d\tau\right)^{\frac{q}{p}} u(s)ds\right)^{\frac{r}{q}}\right)^{\frac{1}{r}} \le C' \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} f(t)dt$$

and

$$\left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^\infty u(s)ds\right)^{\frac{r}{q}} \left(\int_0^{x_k} f(\tau)^p v(\tau)d\tau\right)^{\frac{r}{p}}\right)^{\frac{1}{r}} \le C'' \sum_{k=-\infty}^{M-1} \int_{x_k}^{x_{k+1}} f(t)dt$$

hold for all $f \in \mathcal{M}^+(0,\infty)$. *Moreover* $C \approx C' + C''$.

$$V_p(a,b) := \begin{cases} \left(\int_a^b v^{\frac{1}{1-p}}\right)^{\frac{1-p}{p}} & \text{if } 0 < p < 1, \\ \operatorname*{ess\,sup}_{t\in(a,b)} v(t) & \text{if } p = 1. \end{cases}$$

For every $k \in \mathbb{Z}$, $k \le M$, we denote

$$A_k := V_p(x_{k-1}, x_k) \quad \text{and} \quad B_k := \sup_{h \in \mathcal{M}^+(x_{k-1},x_k)} \frac{\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} h(s)^p v(s)\,ds\right)^{\frac{q}{p}} u(t)\,dt\right)^{\frac{1}{q}}}{\int_{x_{k-1}}^{x_k} h(t)\,dt}.$$

Proposition 23

Let $0 < p \le 1$, $0 < q, r < \infty$ *and let* u, v, w *be weights on* $(0, \infty)$. *Assume that* $\{x_k\}_{k=-\infty}^{M}$ *is the discretizing sequence of* W. *Then there exists a positive constant* C *such that the inequality* (Q) *holds for all nonnegative measurable functions* f *on* (a, b) *if and only if there exist positive constants* C', C'' *such that the inequalities*

$$\left(\sum_{k=-\infty}^{M-1} 2^k a_k^r B_{k+1}^r \right)^{\frac{1}{r}} \le C' \sum_{k=-\infty}^{M-1} a_k$$

$$\left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{b} u(t)\, dt \right)^{\frac{r}{q}} \left(\sum_{j=-\infty}^{k} a_j^p A_j^p \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \le C'' \sum_{k=-\infty}^{M-1} a_k$$

hold for every sequence $\{a_k\}_{k=-\infty}^{M-1}$ *of nonnegative numbers. Moreover,* $C \approx C' + C''$.

Corollary 24

(i) Let $0 < p \le 1 \le r, q$ *,* $u, v, w \in \mathcal{M}^+(0,\infty)$*. Then the best constant* C *in (Q) satisfies* $C \approx A_1^* + B_1^*$*, where*

$$\mathcal{A}_1^* = \sup_{k \le M-1} 2^{\frac{k}{r}} \sup_{t \in (x_{k-1}, x_k)} \left(\int_{x_{k-1}}^{t} v^{\frac{1}{1-p}} \right)^{\frac{1-p}{p}} \left(\int_t^{x_k} u \right)^{\frac{1}{q}}$$

$$\mathcal{B}_1^* = \sup_{k \le M-1} \left(\sum_{i=k}^{M-1} 2^i \left(\int_{x_i}^{\infty} u(t)dt \right)^{\frac{r}{q}} \right)^{\frac{1}{r}} V_p(a, x_k).$$

(ii) Let $0 < p \le 1 \le r$*,* $q < 1$ *,* $u, v, w \in \mathcal{M}^+(0,\infty)$*. Then the best constant* C *in (Q) satisfies* $C \approx A_2^* + B_1^*$*, where*

$$A_2^* := \sup_{k \le M-1} 2^{\frac{k}{r}} \left(\int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} u(s)\, ds \right)^{\frac{q}{1-q}} u(t) V_p(x_k, t)^{\frac{q}{1-q}}\, dt \right)^{\frac{1-q}{q}}.$$

Corollary 24

(iii) Let $0 < p \le 1 \le q$, $r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$. *Then the best constant* C *in* (Q) *satisfies* $C \approx A_3^* + B_2^*$, *where*

$$A_3^* := \left(\sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \operatorname*{ess\,sup}_{t \in (x_k, x_{k+1})} \left(\int_t^{x_{k+1}} u(s)\,ds \right)^{\frac{r}{q(1-r)}} V_p(x_k, t)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}},$$

and

$$B_2^* := \left(\sum_{k=-\infty}^{M-1} 2^k \left(\int_{x_k}^{b} u(t)\,dt \right)^{\frac{r}{q}} \left(\sum_{i=k}^{M-1} 2^i \left(\int_{x_i}^{b} u(t)\,dt \right)^{\frac{r}{q}} \right)^{\frac{r}{1-r}} V_p(a, x_k)^{\frac{r}{1-r}} \right)^{\frac{1-r}{r}}.$$

(iv) Let $0 < p \le 1$, $r, q < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$. *Then the best constant* C *in* (Q) *satisfies* $C \approx A_4^* + B_2^*$, *where* (iv) $r < 1$, $q < 1$, $B_2^* < \infty$ and

$$A_4^* := \left(\sum_{k=-\infty}^{M-1} 2^{\frac{k}{1-r}} \left(\int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} u(s)\,ds \right)^{\frac{q}{1-q}} u(t) V_p(x_k, t)^{\frac{q}{1-q}}\,dt \right)^{\frac{r(1-q)}{q(1-r)}} \right)^{\frac{1-r}{r}}.$$

Theorem 25

(i) Let $0 < p \le 1 \le r, q$, $u, v, w \in \mathcal{M}^+(0,\infty)$. *Then the best constant* C *in* (Q) *satisfies* $C \approx C_1 + C_2$, *where*

$$C_1 := \sup_{t\in(a,b)} \left(\int_a^t w(s)\,ds\right)^{\frac{1}{r}} \operatorname*{ess\,sup}_{s\in(t,b)} \left(\int_s^b u(\tau)\,d\tau\right)^{\frac{1}{q}} V_p(a,s)$$

and

$$C_2 := \sup_{t\in(a,b)} \left(\int_t^b w(s)\left(\int_s^b u(\tau)\,d\tau\right)^{\frac{r}{q}} ds\right)^{\frac{1}{r}} V_p(a,t).$$

(ii) Let $0 < p \le 1 \le r$, $q < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$. *Then the best constant* C *in* (Q) *satisfies* $C \approx C_2 + C_3$, *where*

$$C_3 := \sup_{t\in(a,b)} \left(\int_a^t w(s)\,ds\right)^{\frac{1}{r}} \left(\int_t^b \left(\int_s^b u(\tau)\,d\tau\right)^{\frac{q}{1-q}} u(s) V_p(a,s)^{\frac{q}{1-q}}\,ds\right)^{\frac{1-q}{q}}.$$

Theorem 25

(iii) Let $0 < p \le 1 \le q$, $r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$. *Then the best constant* C *in* (Q) *satisfies* $C \approx C_4 + C_5$, *where*

$$C_4 := \left(\int_a^b \left(\int_a^t w(s)\,ds \right)^{\frac{r}{1-r}} w(t) \operatorname*{ess\,sup}_{s\in(t,b)} \left(\int_s^b u(\tau)\,d\tau \right)^{\frac{r}{q(1-r)}} V_p(a,s)^{\frac{r}{1-r}}\,dt \right)^{\frac{1-r}{r}}$$

and

$$C_5 := \left(\int_a^b \left(\int_t^b w(s) \left(\int_s^b u(\tau)\,d\tau \right)^{\frac{r}{q}} ds \right)^{\frac{r}{1-r}} w(t) \left(\int_t^b u(\tau)\,d\tau \right)^{\frac{r}{q}} V_p(a,t)^{\frac{r}{1-r}}\,dt \right)^{\frac{1-r}{r}};$$

(iv) Let $0 < p \le 1$, $q, r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$. *Then the best constant* C *in* (Q) *satisfies* $C \approx C_5 + C_6$, *where*

$$C_6 := \left(\int_a^b \left(\int_a^t w(s)\,ds \right)^{\frac{r}{1-r}} w(t) \left(\int_t^b \left(\int_s^b u(\tau)\,d\tau \right)^{\frac{q}{1-q}} u(s) V_p(a,s)^{\frac{q}{1-q}}\,ds \right)^{\frac{r(1-q)}{q(1-r)}} dt \right)^{\frac{1-r}{r}}$$

Theorem 26

Let $0 < r \le 1$ *and* $0 < p, q < \infty$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then*

$$\left(\int_0^\infty \left(\int_0^t f(s)^r v(s)ds\right)^{\frac{q}{r}} u(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left(\int_t^\infty f(s)ds\right)^p w(t)dt\right)^{\frac{1}{p}}. \quad \text{(QQ)}$$

holds for all $f \in \mathcal{M}^+(0,\infty)$ *if and only if there exist positive constants* $\mathcal{C}'$ *and* $\mathcal{C}''$ *such that*

$$\left(\sum_{k\in\mathbb{Z}} \left(\int_0^{x_k} f^r v\right)^{\frac{q}{r}} \int_{x_k}^{x_{k+1}} u(t)dt\right)^{\frac{1}{q}} \le \mathcal{C}'\left(\sum_{k\in\mathbb{Z}} 2^k \left(\int_{x_k}^{x_{k+1}} f(s)ds\right)^p\right)^{\frac{1}{p}},$$

and

$$\left(\sum_{k\in\mathbb{Z}} \int_{x_k}^{x_{k+1}} \left(\int_{x_k}^t f^r v\right)^{\frac{q}{r}} u(t)dt\right)^{\frac{1}{q}} \le \mathcal{C}''\left(\sum_{k\in\mathbb{Z}} 2^k \left(\int_{x_k}^{x_{k+1}} f(s)ds\right)^p\right)^{\frac{1}{p}},$$

hold for all $f \in \mathcal{M}^+(0,\infty)$. *Moreover* $C \approx C' + C''$.

Theorem 27

Let $0 < r < 1$ *and* $0 < p, q < \infty$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then* (*QQ*) *holds for all* $f \in \mathcal{M}^+(0,\infty)$ *if there exist positive constants* C' *and* C'' *such that*

$$\left(\sum_{k\in\mathbb{Z}}\left(\sum_{i=-\infty}^{k} 2^{-i\frac{r}{p}} V_r(x_{i-1},x_i)^r a_i^r\right)^{\frac{q}{r}} \int_{x_k}^{x_{k+1}} u\right)^{\frac{1}{q}} \le C'\left(\sum_{k\in\mathbb{Z}} a_k^p\right)^{\frac{1}{p}}, \tag{37}$$

and

$$\left(\sum_{k\in\mathbb{Z}} 2^{-k\frac{q}{p}} B_k^q a_k^q\right)^{\frac{1}{q}} \le C''\left(\sum_{k\in\mathbb{Z}} a_k^p\right)^{\frac{1}{p}}, \tag{38}$$

hold for every sequence of non-negative numbers $\{a_k\}_{k\in\mathbb{Z}}$, *where* $\{x_k\}_{k\in\mathbb{Z}}$ *is the discretizing sequence of* W, B_k *is the best constant of the inequality*

$$\left(\int_{x_k}^{x_{k+1}}\left(\int_{x_k}^{t} h^r v\right)^{\frac{q}{r}} u(t)dt\right)^{\frac{1}{q}} \le c\int_{x_k}^{x_{k+1}} h, \quad h \in \mathcal{M}^+(x_k,x_{k+1}), \quad k \in \mathbb{Z}$$

Moreover $C \approx C' + C''$.

Corollary 28

Let $0 < p \le q < 1$, $0 < p \le r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then the best constant* C *in* (QQ) *satisfies* $C \approx \mathcal{A}_1 + \mathcal{B}_1$, *where*

$$\mathcal{A}_1 = \sup_{k\in\mathbb{Z}} 2^{-\frac{k}{p}} \left(\int_{x_{k-1}}^{x_k} v^{\frac{1}{1-r}}\right)^{\frac{1-r}{r}} \left(\int_{x_k}^{\infty} u\right)^{\frac{1}{q}} < \infty$$

$$\mathcal{B}_1 = \sup_{k\in\mathbb{Z}} 2^{-\frac{k}{p}} \left(\int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} u\right)^{\frac{q}{1-q}} u(t) \left(\int_{x_k}^{t} v^{\frac{1}{1-r}}\right)^{\frac{q(1-r)}{r(1-q)}} dt\right)^{\frac{1-q}{q}} < \infty.$$

Theorem 29

Let $0 < p \le q < 1$, $0 < p \le r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then the best constant* C *in* (QQ) *satisfies* $C \approx C_1 + C_2$, *where*

$$C_1 = \sup_{x\in(0,\infty)} \left(\int_0^x w\right)^{-\frac{1}{p}} \left(\int_0^x v^{\frac{1}{1-r}}\right)^{\frac{1-r}{r}} \left(\int_x^{\infty} u\right)^{\frac{1}{q}} < \infty$$

$$C_2 = \sup_{x\in(0,\infty)} \left(\int_0^x w\right)^{-\frac{1}{p}} \left(\int_0^x \left(\int_t^{\infty} u\right)^{\frac{q}{1-q}} u(t) \left(\int_0^t v^{\frac{1}{1-r}}\right)^{\frac{q(1-r)}{r(1-q)}} dt\right)^{\frac{1-q}{q}} < \infty.$$

Corollary 30

Let $0 < q < p$, $q < 1$, $0 < r < p$ *and* $r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then the best constant* C *in (QQ) satisfies* $C \approx \mathcal{A}_2 + \mathcal{B}_2$, *where*

$$\mathcal{A}_2 = \left(\sum_{k\in\mathbb{Z}} 2^{-k\frac{q}{p-q}} \left(\int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} u \right)^{\frac{q}{1-q}} u(t) \left(\int_{x_k}^t v^{\frac{1}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} dt \right)^{\frac{p(1-q)}{p-q}} \right)^{\frac{p-q}{pq}} < \infty$$

$$\mathcal{B}_2 = \left(\sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} u \right) \left(\int_{x_k}^{\infty} u \right)^{\frac{q}{p-q}} \left(\sum_{i=-\infty}^{k} 2^{-i\frac{r}{p-r}} \left(\int_{x_{i-1}}^{x_i} v^{\frac{1}{1-r}} \right)^{\frac{p(1-r)}{p-r}} \right)^{\frac{q(p-r)}{r(p-q)}} \right)^{\frac{p-q}{pq}} < \infty.$$

Theorem 31

Let $0 < r \le q < p < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$ such that $W(\infty) = \infty$. Then the best constant C in (QQ) satisfies $C \approx C_3 + C_4$, where

$$C_3 = \left(\int_0^\infty \left(\int_0^x w \right)^{-\frac{p}{p-q}} w(x) \left(\int_0^x \left(\int_t^x u \right)^{\frac{q}{1-q}} u(t) \left(\int_0^t v^{\frac{1}{1-r}} \right)^{\frac{q(1-r)}{r(1-q)}} dt \right)^{\frac{p(1-q)}{p-q}} dx \right)^{\frac{p-q}{pq}} <$$

$$C_4 = \left(\int_0^\infty \left(\int_t^\infty u \right)^{\frac{q}{p-q}} u(t) \left(\int_0^t \left(\int_0^s w \right)^{-\frac{p}{p-r}} w(s) \left(\int_0^s v^{\frac{1}{1-r}} \right)^{\frac{p(1-r)}{p-r}} ds \right)^{\frac{q(p-r)}{r(p-q)}} dt \right)^{\frac{p-q}{pq}} <$$

Theorem 32

Let $0 < q < p$, $q < 1$, $0 < r < p$ *and* $r < 1$, $u, v, w \in \mathcal{M}^+(0,\infty)$ *such that* $W(\infty) = \infty$. *Then the best constant* C *in* (QQ) *satisfies* $C \approx C_5 + C_6$, *where*

$$C_5 = \left(\int_0^\infty \left(\int_0^x w\right)^{-\frac{p}{p-q}} w(x) \left(\int_0^x \left(\int_t^x u\right)^{\frac{q}{1-q}} u(t) \left(\int_0^t v^{\frac{1}{1-r}}\right)^{\frac{q(1-r)}{r(1-q)}} dt\right)^{\frac{p(1-q)}{p-q}} dx\right)^{\frac{p-q}{pq}} <$$

$$C_6 = \left(\int_0^\infty \left(\int_0^x w\right)^{-2} w(x) \sup_{y\in(0,x)} \left(\int_0^y w\right) \left(\int_y^x u(s) \left(\int_s^\infty u\right)^{\frac{q}{p-q}} ds\right) \times \right.$$
$$\left. \times \left(\int_0^y \left(\int_0^s w\right)^{-\frac{p}{p-r}} w(s) \left(\int_0^s v^{\frac{1}{1-r}}\right)^{\frac{p(1-r)}{p-r}} ds\right)^{\frac{q(p-r)}{r(p-q)}} dx\right)^{\frac{p-q}{pq}} < \infty.$$

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ЖОҚ, КЕДЕЙ ХАЛЫҚТЫ КӨП КӨТЕРДІМ, КЕДЕЙ ХАЛЫҚТЫ БАЙ ҚЫЛДЫМ,
АЗ ХАЛЫҚТЫ КӨП ҚЫЛДЫМ. ОСЫ СӨЗІМДЕ ӨТІРІК БАР МА?
ТҮРК БЕКТЕРІ, ХАЛҚЫ, БҰНЫ ТЫҢДАҢДАР! ТҮРК ХАЛҚЫН ЖИЫП,
ЕЛ БОЛҒАНДАРЫҢДЫ МҰНДА БАСТЫМ.
ЖАҢЫЛЫП, КЕШКЕНДЕРІҢДІ ДЕ МҰНДА БАСТЫМ.
БАРЛЫҚ СӨЗІМДІ АЙТАР, МӘҢГІ ТАСҚА БАСТЫМ.
БҰҒАН ҚАРАП БІЛІҢДЕР, ТҮРК ХАЛҚЫНЫҢ ҚАЗІРГІ БЕКТЕРІ –
ТАҚҚА КІРІПТАР БЕКТЕР, СЕНДЕР ЖАҢЫЛҒЫШСЫҢДАР ҒОЙ?!
МЕН МӘҢГІ ТАС...(ОРНАТТЫМ), ТАБҒАШ ҚАҒАННАН БӘДІЗШІ АЛДЫРДЫМ,
ЖАЗУ ЖАЗҒЫЗДЫМ, МЕНІҢ СӨЗІМДІ БҰЗБАДЫ.
КҮЛТЕГІН (КІШІ ЖАЗУ)

Happy Birthday to Mikhail Lvovich (Misha)
I wish you good health
and a long and happy life
I wish we could meet in person in Prague
next year

Thank you very much for your attention!