Lorentz boundedness criteria for commutators of maximal operator on spaces of homogeneous type

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The main goal of this work is to study the Lorentz boundedness of the maximal commutator operator M_b, the commutators of the maximal operator $[b, M]$ and the commutators of the sharp maximal operator $[b, M^\sharp]$ on spaces (X, d, μ) of homogeneous type.

We study the Lorentz boundedness properties of the maximal commutator operator M_b on the space of homogeneous type and relate this property to spaces of bounded mean oscillations. We study also the Lorentz boundedness properties of the commutators of maximal operator $[b, M]$ and the commutators of the sharp maximal operator $[b, M^\sharp]$ on the space of homogeneous type and relate this property for certain subclasses of spaces of bounded mean oscillations.

To extend traditional Euclidean space and build a general basic structure for real harmonic analysis, Coifman and Weiss introduced the concept of spaces of homogeneous type [1].

[1] Coifman, R.R., Weiss, G.: Analyse harmonique non-commutative sur certain espaces homogenes. in "Lecture Notes in Math." **242**, Springer-Verlag, Berlin, (1971)

Lorentz boundedness criteria for commutators of maximal operator on spaces of homogeneous type

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a positive measure μ such that

$$d(x,y) \geq 0 \text{ and } d(x,y) = 0 \text{ if and only if } x = y,$$

$$d(x,y) = d(y,x),$$

$$d(x,y) \leq K_1(d(x,z) + d(z,y)), \tag{1}$$

the balls $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x,2r)) \leq K_2\, \mu(B(x,r)) < \infty, \tag{2}$$

where $K_i \geq 1\,(i = 1,2)$ are constants independent of $x, y, z \in X$ and $r > 0$. As usual, the dilation of a ball $B = B(x,r)$ will be denoted by $\lambda B = B(x, \lambda r)$ for every $\lambda > 0$.

Lorentz boundedness criteria for commutators of maximal operator on spaces of homogeneous type

In the sequel, we always assume that $\mu(X) = \infty$, the space of compactly supported continuous function is dense in $L^1(X, \mu)$ and that X is Q-homogeneous ($Q > 0$), i.e.

$$K_3^{-1} r^Q \leq \mu(B(x, r)) \leq K_3 r^Q, \tag{3}$$

where $K_3 \geq 1$ is a constant independent of x and r. The n-dimensional Euclidean space $\mathbb{R}^n$ is n-homogeneous.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the uncentered fractional maximal operator M_η is defined by

$$M_\eta f(x) = \sup_{B \ni x} \mu(B)^{-1+\eta} \int_B |f(y)|\, d\mu(y), \ 0 \le \eta < 1$$

and the sharp maximal function of Fefferman and Stein $M^\sharp f$ is defined by

$$M^\sharp f(x) = \sup_{B \ni x} \mu(B)^{-1} \int_B |f(y) - f_B|\, d\mu(y)$$

where the supremum is taken over all balls $B \subset X$ containing $x \in X$, ${}^{\complement}B$ is its complement and B denotes the μ measure of B. For a fixed $q \in (0,1)$, any suitable function h and $x \in X$, let $M^\sharp_q h(x) = \big(M^\sharp\big(|h|^q\big)(x)\big)^{1/q}$ and $M_q h(x) = \big(M\big(|h|^q\big)(x)\big)^{1/q}$.

Lorentz boundedness criteria for commutators of maximal operator on spaces of homogeneous type

The fractional maximal commutator generated by the operator M and $b \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by

$$M_{b,\eta}f(x) = \sup_{B \ni x} \mu(B)^{-1+\eta} \int_B |b(x) - b(y)||f(y)| \, d\mu(y), \; 0 \le \eta < 1.$$

The commutators generated by the operators M_η, $M^\sharp$ and a suitable function b are defined by

$$[b, M_\eta]f(x) = b(x)M_\eta f(x) - M_\eta(bf)(x)$$

and

$$[b, M^\sharp]f(x) = b(x)M^\sharp f(x) - M^\sharp(bf)(x).$$

Obviously, the operators $M_{b,\eta}$ and $[b, M_\eta]$ essentially differ from each other since M_b is positive and sublinear and $[b, M_\eta]$ is neither positive nor sublinear. The operators M_η, $M_{b,\eta}$, $[b, M_\eta]$ and $[b, M^\sharp]$ play an important role in real and harmonic analysis and applications.

The commutator estimates have many important applications, for example, in studying the regularity and boundedness of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and in characterizing certain function spaces. (see, for instance [2, 3]).

[2] R.R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), no. 3, 611-635.
[3] L. Grafakos, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009.

The boundedness of the Hardy-Littlewood maximal operator M on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of M with BMO functions b. These turn out to be L^p bounded for $1 < p < \infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b, 0\} \in L^\infty(\mathbb{R}^n)$ [4]. This is useful, for instance, when studying the product of an H^1 function with a BMO function [5]. Note that, the boundedness of the operator M_b on L^p spaces was proved by Garcia-Cuerva et al. [6].

[4] J. Bastero, M. Milman, F.J. Ruiz, *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc., **128** (2000), no. 11, 3329-3334 (electronic).

[5] A. Bonami, T. Iwaniec, P. Jones, M. Zinsmeister, *On the product of functions in BMO and H_1*, Ann. Inst. Fourier Grenoble, **57** (2007), no. 5, 1405-1439.

[6] J. Garcia-Cuerva, E. Harboure, C. Segovia, J.L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J., **40** (1991), 1397-1420.

Lorentz spaces on spaces of homogeneous type

In this work we obtain necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b, the commutators of the maximal operator $[b, M]$ and the commutators of the sharp maximal operator $[b, M^\sharp]$ on the Lorentz spaces $L^{p,q}(X)$. We give some new characterizations for certain subclasses of $BMO(X)$.

Lorentz spaces on spaces of homogeneous type

We start with the definition of Lorentz spaces. Lorentz spaces are introduced by Lorentz in the 1950. These spaces are Banach spaces and generalizations of the more familiar L^p spaces, also they are appear to be useful in the general interpolation theory.
Suppose that f is a measurable function on X, then we define

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\},$$

where

$$d_f(s) := \mu(\{x \in X : |f(x)| > s\}), \quad \forall s > 0.$$

The Lorentz space $L^{p,q} \equiv L^{p,q}(X)$, $0 < p, q \le \infty$ is the collection of all measurable functions f on X such the quantity

$$\|f\|_{L^{p,q}(X)} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} \tag{4}$$

is finite. Clearly $L^{p,p}(X) \equiv L^p(X)$ and $L^{p,\infty}(X) \equiv WL^p(X)$. The functional $\|\cdot\|_{L^{p,q}(X)}$ is a norm if and only if either $1 \le q \le p$ or $p = q = \infty$.

Lorentz spaces on spaces of homogeneous type

Lemma

[7, Proposition 2.11] Let $0 < q_1, q_2 < \infty$*, and* $0 < q_1, q_2 < \infty$*. Suppose that* $f \in L^{q_1,r_1}(X)$ *and* $g \in L^{q_2,r_2}(X)$*. Then*

$$\|fg\|_{L^{q,r}(X)} \leq 2\|f\|_{L^{q_1,r_1}(X)} \|g\|_{L^{q_2,r_2}(X)}$$

where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$*, and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$*.*

[7] N.A. Dao, S.G. Krantz, *Lorentz boundedness and compactness characterization of integral commutators on spaces of homogeneous type*, Nonl. Anal. **203** (2021) 112-162.

Examples

We give several examples of spaces of homogeneous type.

1 $X = \mathbb{R}^n,\ \rho(x,y) = |x-y| = \left(\sum_{j=1}^{n}(x_j - y_j)^2\right)^{\frac{1}{2}}$ and μ equals Lebesgue measure.

2 $X = \mathbb{R}^n,\ \rho(x,y) = \sum_{j=1}^{n}(x_j - y_j)^{\alpha_j}$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive numbers, not necessarily equal, and μ equals Lebesgue measure (this distance is called nonisotropic).

3 $X = [0,1),\ \rho(x,y)$ is the length of the smallest dyadic interval containing x and y, and μ is Lebesgue measure.

4 Any C^∞ compact Riemannian manifold with the Riemannian metric and volume.

5 Let G be a nilpotent Lie group with a left-invariant Riemannian metric and μ is the induced measure.

6 When X is the boundary of a smooth and bounded pseudo-convex domain in $\mathbb{C}^n$ one can introduce a nonisotropic quasi-distance that is related to the complex structure in such a way that we obtain a space of homogeneous type by using Lebesgue surface measure. For example, if X is the surface of the unit sphere

$$\sigma_{2n-1} = \Big\{ z \in \mathbb{C}^n : z \cdot \overline{z} = \sum_{j=1}^{n} z_j \overline{z_j} = 1 \Big\},$$

the nonisotropic distance is given by $d(z,w) = |1 - z \cdot \overline{w}|^{\frac{1}{2}}$.

The following result completely characterizes the boundedness of M on Lorentz spaces.

Lemma

[7, Proposition 2.11] Let $1 \leq p, q \leq \infty$.
(i) If $1 < p \leq \infty$, *then the operator* M *is bounded on the Lorentz spaces* $L^{p,q}(X)$.
(ii) If $p = 1$, *then the operator* M *is bounded on the Lorentz spaces* $L^{1,q}(X)$ *to* $WL^1(X)$.

[7] N.A. Dao, S.G. Krantz, *Lorentz boundedness and compactness characterization of integral commutators on spaces of homogeneous type*, Nonl. Anal. **203** (2021) 112-162.

The following result completely characterizes the boundedness of M_η on Lorentz spaces.

Lemma

Let $0 \le \eta < 1$, $1 \le p < \frac{1}{\eta}$ and $p \le q < \infty$.
(i) If $1 < p < \frac{1}{\eta}$, $1 \le r \le s \le \infty$, then the condition $\frac{1}{p} - \frac{1}{q} = \eta$ is necessary and sufficient for the boundedness of the operator M_η from the Lorentz spaces $L^{p,r}(\mathbb{R}^n)$ to $L^{q,s}(\mathbb{R}^n)$.
(ii) If $p = 1$, $1 \le r \le \infty$, then the condition $1 - \frac{1}{q} = \eta$ is necessary and sufficient for the boundedness of the operator M_η from the Lorentz spaces $L^{1,r}(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$.

Function Spaces Under Consideration

BMO spaces

We define the space $BMO(X)$ as the set of all locally integrable functions f with finite norm

$$\|f\|_* = \sup_{x\in X, t>0} \mu(B(x,t))^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}|\, d\mu(y) < \infty,$$

where $f_{B(x,t)} = \mu(B(x,r))^{-1} \int_{B(x,t)} f(y)\, d\mu(y)$.

Lemma

[8, Lemma 1] Let $0 < \beta < 1$ and $1 \le q < \infty$. Then If $b \in BMO(X)$, then for any $q \in (0,1)$, there exists a positive constant C such that

$$M_q^\sharp\big(M_b f\big)(x) \le C\|b\|_* M\big(Mf\big)(x) \tag{5}$$

for every $x \in X$ and for all $f \in L^1_{\rm loc}(X)$.

[8] G. Hu, D. Yang, *Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type*, J. Math. Anal. Appl., **354** (2009), 249-262.

$L^{p,q}$-boundedness of the commutator of maximal operator $[b, M]$

For a function b defined on X, we denote

$$b^{-}(x) := \begin{cases} 0\ , & \text{if } b(x) \geq 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^{+}(x) := |b(x)| - b^{-}(x)$. Obviously, $b^{+}(x) - b^{-}(x) = b(x)$.
The following relations between $[b, M_\eta]$ and $M_{b,\eta}$ are valid :
Let b be any non-negative locally integrable function. Then for all $f \in L^1_{\rm loc}(X)$ and $x \in X$ the following inequality is valid

$$\big|[b, M_\eta]f(x)\big| = \big|b(x)M_\eta f(x) - M_\eta(bf)(x)\big|$$
$$= \big|M_\eta(b(x)f)(x) - M_\eta(bf)(x)\big| \leq M_\eta(|b(x) - b|f)(x) = M_{b,\eta}f(x).$$

$L^{p,q}$-boundedness of the commutator of maximal operator $[b, M]$

If b is any locally integrable function on X, then

$$|[b, M_\eta]f(x)| \le M_{b,\eta}f(x) + 2b^-(x)\, M_\eta f(x), \qquad x \in X \qquad (6)$$

holds for all $f \in L^1_{\rm loc}(X)$.
Denote by $M_b f$ the local maximal function of f:

$$M_B f(x) := \sup_{B' \ni x:\, B' \subset B} \frac{1}{\mu(B')} \int_{B'} |f(y)|\, d\mu(y),\ x \in X.$$

Characterization of maximal commutator operator on Lorentz spaces

Theorem

Let $p, q \in (1, \infty)$. The following assertions are equivalent:
(i) $b \in BMO(X)$.
(ii) *The operator* M_b *is bounded on* $L^{p,q}(X)$.
(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{\left\| \big(b(\cdot) - b_B(\cdot)\big)\chi_B \right\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \le C.$$

(iv) *There exist a constant* $C > 0$ *such that*

$$\sup_B \mu(B)^{-1} \left\| \big(b(\cdot) - b_B(\cdot)\big)\chi_B \right\|_{L^1(X)} \le C.$$

V.S. Guliyev, *Maximal commutator and commutator of maximal operator on Lorentz spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci **44**(4) Mathematics, 43-49 (2024). in the case $X = \mathbb{R}^n$

Characterization of commutator of maximal operator on Lorentz spaces

Theorem

Let $p, q \in (1, \infty)$. The following assertions are equivalent:
(i) $b \in BMO(X)$ and $b^- \in L^\infty(X)$.
(ii) The operator $[b, M]$ is bounded on $L^{p,q}(X)$.
(iii) There exist a constant $C > 0$ such that

$$\sup_B \frac{\left\| \big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B \right\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \le C. \tag{7}$$

(iv) There exist a constant $C > 0$ such that

$$\sup_B \mu(B)^{-1} \left\| \big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B \right\|_{L^1(X)} \le C. \tag{8}$$

V.S. Guliyev, *Maximal commutator and commutator of maximal operator on Lorentz spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci **44**(4) Mathematics, 43-49 (2024). in the case $X = \mathbb{R}^n$

Characterization of commutator of sharp maximal operator on Lorentz spaces

Theorem

Let $p, q \in (1, \infty)$. The following assertions are equivalent:
(i) $b \in BMO(X)$ and $b^- \in L^\infty(X)$.
(ii) The operator $[b, M^\sharp]$ is bounded on $L^{p,q}(X)$.
(iii) There exist a constant $C > 0$ such that

$$\sup_B \frac{\left\| \left(b(\cdot) - 2M^\sharp(b\,\chi_B)\right)\chi_B \right\|_{L^{p,q}(X)}}{\|\chi_B\|_{L^{p,q}(X)}} \leq C. \tag{9}$$

(iv) There exist a constant $C > 0$ such that

$$\sup_B \mu(B)^{-1} \left\| \left(b(\cdot) - 2M^\sharp(b\,\chi_B)\right)\chi_B \right\|_{L^1(X)} \leq C. \tag{10}$$

Characterization of fractional maximal commutator operator on Lorentz spaces

Theorem

Let $0 \le \eta < 1$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$, $1 < p < \frac{1}{\eta}$, $1 \le r \le s \le \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \eta$. *The following assertions are equivalent:*

(i) $b \in BMO(X)$.

(ii) *The operator* $M_{b,\eta}$ *is bounded from* $L^{p,r}(X)$ *to* $L^{q,s}(X)$.

(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - b_B\big)\chi_B\big\|_{L^{q,s}(X)}}{\|\chi_B\|_{L^{q,s}(X)}} \le C. \tag{11}$$

(iv) *There exist a constant* $C > 0$ *such that*

$$\sup_B \mu(B)^{-1} \big\|\big(b(\cdot) - b_B\big)\chi_B\big\|_{L^1(X)} \le C. \tag{12}$$

Guliyev, V.S.: *Commutator of fractional maximal function on Lorentz spaces*, Socar Proceedings No. 3, 113-117 (2024). in the case $X = \mathbb{R}^n$

Characterization of commutator of fractional maximal operator on Lorentz spaces

Theorem

Let $0 \le \eta < 1$, $b \in L^1_{\rm loc}(\mathbb{R}^n)$, $1 < p < \frac{1}{\eta}$, $1 \le r \le s \le \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \eta$.
The following assertions are equivalent:
(i) $b \in BMO(X)$ *and* $b^- \in L^\infty(X)$.
(ii) *The operator* $[b, M_\eta]$ *is bounded from* $L^{p,r}(X)$ *to* $L^{q,s}(X)$.
(iii) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{\big\|\big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B\big\|_{L^{q,s}(X)}}{\|\chi_B\|_{L^{q,s}(X)}} \le C. \tag{13}$$

(iv) *There exist a constant* $C > 0$ *such that*

$$\sup_B \frac{1}{\mu(B)} \big\|\big(b(\cdot) - M_B(b)(\cdot)\big)\chi_B\big\|_{L^1(X)} \le C. \tag{14}$$

Guliyev, V.S.: *Commutator of fractional maximal function on Lorentz spaces*, Socar Proceedings No. 3, 113-117 (2024). in the case $X = \mathbb{R}^n$

Characterization of commutator of maximal operator on Lorentz spaces

- V.S. Guliyev, *Some characterizations of BMO space via commutators in Orlicz spaces on stratified Lie groups*. **Results Math.** 77 (1) (2022), 1-18.
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- Guliyev, V.S.: *Commutator of fractional maximal function on Lorentz spaces*, Socar Proceedings No. 3, 113-117 (2024).

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