Net spaces and Holder's inequality

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Holder's inequality

$1 \leq p \leq \infty, \ \frac{1}{p} + \frac{1}{p'} = 1$

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \|f\|_{L_p} \|g\|_{L_{p'}}$$

$1 < p < \infty, \ 1 \leq q \leq \infty$

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \|f\|_{L_{p,q}} \|g\|_{L_{p',q'}}$$

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$1 \leq p \leq \infty, \quad \alpha \in \mathbb{R}$

$$\left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| \leq \|f\|_{B^{\alpha}_{p,q}} \|g\|_{B^{-\alpha}_{p',q'}}$$

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Properties of Fourier coefficients of functions, Fourier transform of functions, norm and spectrum of integral operators depend on not only properties of smoothness and metrics but on "geometrical" properties of corresponding functions and kernels. Here we mean such properties as positional relationship of singularities, their sign orientation. Most of known spaces are not "sensitive" to the pointed out property.

In this talk the net space $N_{p,q}(M)$ is considered.The Net space $N_{p,q}(M)$ (unlike the Lebesgue, Lorentz spaces) is "sensitive" to distribution of functions' singularities and determines some method. The purpose of this paper is to show how methods of net spaces can be used in the different problems.

Let μ be n-dimensional Lebesgue measure in $\mathbb{R}^n$ and let $\mathfrak{M}$ be the collection of all μ-measurable sets of positive measure, i.e.,

$$\mathfrak{M} := \{e \subset \mathbb{R}^n : \quad 0 < |e| := \mu(e) < \infty\}.$$

Then we will call *the net M* a fix subset from $\mathfrak{M}$.
For a function f defined and integrable on each e in M let

$$\bar{f}(t, M) = \sup_{\substack{e \in M \\ |e| \geq t}} \frac{1}{|e|} \left| \int_e f(x)dx \right|, \qquad t > 0,$$

where the least upper bound is taken over all $e \in M$, of measure $|e| \overset{\mathrm{def}}{=} \mu e > t, \;\; t \in (0, \infty)$.
Let $0 < p, q \leq \infty$.

$$N_{p,q}(M) = \left\{ f : \|f\|_{N_{p,q}(M)} = \left(\int_0^\infty \left(t^{\frac{1}{p}} \bar{f}(t, M) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

$$N_{p,\infty}(M) = \left\{ f : \|f\|_{N_{p,\infty}(M)} = \sup_{t>0} t^{\frac{1}{p}} \bar{f}(t, M) < \infty \right\}$$

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Example 1

Let $1 < p < \infty$, $0 < q \leq \infty$, $M = \{e \in \mathfrak{M} : e \subset \Omega\}$ then

$$N_{p,q}(M) = L_{p,q}(\Omega)$$

Example 2 Let $M = \{[a, b] : a < b\}$ and let $1 < p < \infty$, $0 < q < \infty$,

$$f(x) = \sin \alpha x, \qquad \alpha \in \mathbb{R}$$

then $\quad f \in N_{p,q}(M) \quad$ but $\quad |f| \notin N_{p,q}(M)$.

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Example 3 Let $1 < p < \infty, 0 < q \leq \infty$, $\left\{a_k = \left(k \ln^{1/q} k\right)^{p'}\right\}_{k=1}^{\infty}$,

$$f(x) = \chi_{\Omega}(x), \qquad \Omega = \bigcup_{k=1}^{\infty} [a_k, a_k + 1],$$

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Proposition

Assume that $0 < p < \infty$, $0 < q \leq \infty$.

a) *For* $0 < q \leq q_1 \leq \infty$ *we have*

$$N_{p,q}(M) \hookrightarrow N_{p,q_1}(M).$$

b) *If* $p < p_1$ *and* M *- is a net* M *such that* $\sup_{e \in M} |e| = \alpha < \infty$, *then*

$$N_{p_1,q_1}(M) \hookrightarrow N_{p,q}(M)$$

for $q, q_1 \in (0, \infty]$.

c) *If* $M \subset M_1$, *then* $N_{p,q}(M_1) \hookrightarrow N_{p,q}(M)$

Theorem

Let $0 < p_0 < p_1 \leq \infty$, $0 < q_0, q_1, q \leq \infty$, *then*

$$(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} \hookrightarrow N_{p,q}(M),$$

where $0 < \theta < 1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$.

Corollary

Let $0 < p_0 < p_1 \leq \infty,\ 0 < q \leq \infty,\ 0 < \theta < 1,$
If a semiadditive operator T *acts as follows:*

$$T : A_i \to N_{p_i,\infty}(M) \ \ c \ \ with \ \ norm \ \ D_i, \ \ i = 0, 1$$

then

$$T : \bar{A}_{\theta q} \to N_{p,q}(M)$$

, *with norm* $\|T\| \leq cD_0^{1-\theta} D_1^{\theta}$, *where* $1/p = (1-\theta)/p_0 + \theta/p_1$.

Theorem

If $p_0 < p_1, \quad M = \{[a,b] : \ a < b\}$ *,then*

$$(N_{p_0,q_0}(M), N_{p_1,q_1}(M))_{\theta,q} = N_{p,q}(M).$$

Theorem

Let $0 < p_0 < p_1 \leq \infty,\ 0 < q_0 < q_1 < \infty \leq \infty,\ 0 < \theta < 1,$ $M = \{B_r(x)\}_{\substack{r>0 \\ x\in\mathbb{R}^n}}$. *If a semiadditive operator* T *acts as follows:*

$$T : N_{p_i,1}(M(x)) \to N_{q_i,\infty}(M(x)) \ \ x \in \mathbb{R}^n, \quad i = 0, 1$$

where $M(x) = \{B_r(x)\}_{r>0}$*. Then*

$$T : N_{p,\tau}(M) \to N_{q,\tau}(M)$$

, where $1/p = (1-\theta)/p_0 + \theta/p_1, 1/q = (1-\theta)/q_0 + \theta/q_1$*.*

Let f be a measurable function. We define a family of sets

$$M_f = \{\{x \in R^n; f(x) > t\}\}_{t>0} \cup \{\{x \in R^n; f(x) < -t\}\}_{t>0}$$

We will call this family of sets M_f the network of the corresponding function f.

Theorem

Let $1 < p < \infty$, $1 \leq q \leq \infty$. Then

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \lesssim \|f\|_{L_{p,q}(\mathbb{R}^n)} \|g\|_{N_{p',q'}(M_f)}$$

In particular,

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Example 4

Let $M = \{[a,b] : a < b\}$ and let $1 < p < \infty$,
$1 < q < \infty$, $f(x) \downarrow 0$, $x \to \pm\infty$

$$\left| \int_{\mathbb{R}} f(x) \sin \alpha x \, dx \right| \lesssim \|f\|_{L_{p,q}(\mathbb{R})} \|\sin \alpha x\|_{N_{p',q'}(M_f)} \asymp \alpha^{-1/p'} \|f\|_{L_{p,q}(\mathbb{R})}$$

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$$\int_0^\infty t^{p-2} \tilde{\hat{f}}^p(t) dt \lesssim \|f\|_{L_p^p(\mathbb{R})},$$

where

$$\tilde{\hat{f}}(t) = \sup_{|\xi| \geq t} |\hat{f}(\xi)|.$$

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Theorem

Let g be a measurable non-negative function, $M(g)$ be a network corresponding to g, and let $1 < p, q, r < \infty$, $\frac{1}{q} + 1 = \frac{1}{p} - \frac{1}{r}$, $1 \leq \tau \leq \infty$..
If $g \in L_{r,\infty}(R^n)$ and $f \in N_{p,\tau}(M) \cap N_{p,\tau}(M(g))$ then $f * g \in N_{q,\tau}(M))$ and the inequality

$$\|f * g\|_{N_{q,\tau}(M)} \lesssim_{p,q,\tau} \|g\|_{L_{r,\infty}(R^n)} \left(\|f\|_{N_{p,\tau}(M)}\right)^{\frac{p}{q}} \left(\|f\|_{N_{p,\tau}(M(g))}\right)^{1-\frac{p}{q}}.$$

Theorem

Let $1 < p < q < \infty$, $\gamma = n/p - n/q$ *and let* $M = \{B_r(x)\}_{x \in \mathbb{R}^n, r>0}$. *Then operator*

$$I_\gamma(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}$$

bounded from $N_{p,\tau}(M)$ *in* $N_{q,\tau}(M)$,

Morrey space

Let $0 \le \lambda \le \frac{n}{p}$ and $0 < p < \infty$. A set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ is called the Morrey space if

$$\|f\|_{M_p^\lambda} \equiv \|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B_r(x))} < \infty.$$

Here $B_r(x)$ is the ball centered at the point x and with radius $r > 0$.

In 1938 Morrey introduced the function spaces now bearing his name. These spaces were studied as a consequence of questions of regular solutions of nonlinear elliptic equations and systems.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, and $0 < \lambda < \infty$. We define generalized Morrey-type spaces $M^{\alpha}_{p,q}$ as the set of all Lebesgue measurable functions $f \in L^{loc}_p(\mathbb{R}^n)$, such that

$$\|f\|_{M^{\lambda}_{p,q}} = \left(\sum_{k \in \mathbb{Z}} \left(2^{-k\lambda} \sup_{x \in \mathbb{R}^n} \|f\|_{L_p(B_{2^k}(x))} \right)^q \right)^{\frac{1}{q}} < \infty$$

Let us note that the spaces introduced in this way are a generalization of the classical Morrey spaces and for $q = \infty$ we have

$$M^{\alpha}_{p,\infty}(\mathbb{R}^n) = M^{\lambda}_p. \tag{1}$$

Theorem

Let $1 < p < \infty$, $0 < q \leq \infty$, $\lambda = \frac{1}{r} - \frac{1}{p}$ *and let* $M = \{[a, b]\}$. *Then*

$$\|\hat{f}\|_{N_{p,q}(M)} \lesssim \|f\|_{M^{\lambda}_{r,q}}$$

in $N_{q,\tau}(M)$,

Let $\mathbb{R}^n$ – be n - dimensional Euclidean space. Let

$$(Af)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy = K * f \tag{2}$$

be the integral convolution operator, acting from L_p in L_q, where $L_p = L_p(\mathbb{R}^n)$ – Lebesgue space.
Later O' Neil proved the inequality $1 < p < q \leq +\infty$

$$\|A\|_{L_p \to L_q} \leq C \cdot \|K\|_{L_{r,\infty}} \tag{2}$$

where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$, $L_{r,\infty}$ – Marcinkiewich – Lorentz space.

Let us consider a problem of lower estimate of convolution operator (1). For solution of this problem we will write down the inequality (2) in the terms of net spaces.

$$\|A\|_{L_p\to L_q} \leq C\|K\|_{N_{r,\infty}(M)} \asymp \sup_{e\in M} \frac{1}{|e|^{1/p-1/q}} \left|\int_e K(x)dx\right|, \quad (3)$$

where $M = \{e \subset \mathbb{R}^n : \ \ 0 < |e| < \infty\}$.
Thus, the new parameter M takes part in the inequality (3) and it let us make hypothesis: we can find such net M_0 that a reverse inequality takes place.

Theorem

Assume that $1 < p < q < +\infty$. If operator (1) bounded acting from L_p in L_q, then exist the constant $C(p,q,n)$ such that

$$C \sup_{e \in M_0} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)dx \right| \leq \|A\|_{L_p \to L_q} \leq$$

$$\leq C_2 \sup_{e \in M} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)dx \right|$$

Where M_0 –is the set all n-dimensional parallelepipeds from $\mathbb{R}^n$, $M = \{e \subset \mathbb{R}^n : \ 0 < |e| < \infty\}$.

Thank you for your attention!