Boolean Valued Analysis and Positivity

Anatoly G. Kusraev
Southern Mathematical Institute of the VSC RAS

International Workshop on Functional Analysis
(Novosibirsk–Nukus–Vladikavkaz, March 1 – 3, 2023)

Contents

- Historical remarks
- Boolean Valued Analysis
- Maharam Operators
- Injective Banach Lattices
- Some Algebraic Aspects

- Most influential:

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  K. Gödel, P. J. Cohen

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HISTORICAL REMARKS

Background

- Continuum Hypothesis, CH (Cantor, 1878).

  Every $A \subset [0, 1]$ is either finite, or countable, or continual.
- Kantorovich's Heuristic Transfer Principle (Kantorovich, 1935).

  The elements of a Kantorovich space ($\equiv$ Dedekind complete vector lattice) can be considered as generalized reals.
- Theorem (Gödel, 1939).

  ZF is consistent $\Longrightarrow$ $\mathrm{ZFC} + \mathrm{CH}$ is consistent.
- Theorem (Cohen, 1963).

  ZF is consistent $\Longrightarrow \mathrm{ZFC} + \neg\mathrm{CH}$ is consistent.
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  ✓ A comprehensive presentation of the Cohen forcing method.

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  ✓ The vector lattice of (cosets of) measurable functions can be considered as Boolean valued reals.

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BOOLEAN VALUED ANALYSIS

What Is Boolean Valued Analysis?

- Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique and consists in studying the properties of a mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras.
- The von Neumann universe (Cantorian paradise) $\mathbb{V}$ and a specially selected (constructed) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are taken as these models.
- The comparative analysis requires the following operations:

  Ascent $X \mapsto X\uparrow$ (or $X \mapsto X^{\wedge}$) acting from $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$;

  Descent $\mathcal{X} \mapsto \mathcal{X}\downarrow$ acting from $\mathbb{V}^{(\mathbb{B})}$ to $\mathbb{V}$.

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Verification in Boolean Valued Models

- The ascending–descending machinery enables one to carry out the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.
- How to make statements about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$?

  Take a ZF-formula $\varphi = \varphi(u_1, \dots, u_n)$ and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. Then $\varphi(x_1, \dots, x_n)$ is a statement about $x_1, \dots, x_n$.
- How to verify whether or not $\varphi(x_1, \dots, x_n)$ is true in $\mathbb{V}^{(\mathbb{B})}$?

  There is a natural way of assigning to each such statement an element of $\mathbb{B}$, the Boolean truth-value $[\![\varphi(x_1, \dots, x_n)]\!] \in \mathbb{B}$
- Definition. $\mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n) \iff [\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$.

  $\varphi(x_1, \dots, x_n)$ is valid within $\mathbb{V}^{(\mathbb{B})} \iff [\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$.
- The Transfer Principle. All the theorems of ZFC are true in $\mathbb{V}^{(\mathbb{B})}$.

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How does the Boolean valued transfer principle work?

- Let $\mathbf{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ be two classes of mathematical objects. Suppose we are able to prove the result:
- Boolean Valued Representation. *Every $X \in \mathbf{X}$ embeds into an Boolean valued model, becoming an object $\mathcal{X} \in \mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.*
- Boolean Valued Transfer Principle. Every theorem about $\mathcal{X}$ within *ZFC* has its counterpart for the original object X interpreted as a Boolean valued object $\mathcal{X}$.
- Boolean Valued Machinery. Translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ is carried out by the appropriate general operations (ascending–descending) and the principles of Boolean valued analysis.
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Gordon Theorem: Boolean Valued Reals

- **Theorem (Gordon, 1977).** *Let* $\mathbb{B}$ *be a complete Boolean algebra,* $\mathcal{R}$ *be the field of reals within* $\mathbb{V}^{(\mathbb{B})}$*. Then the following hold:*

  (1) $\mathbb{R}^{\wedge}$ *is a dense subfield of* $\mathcal{R}$ within $\mathbb{V}^{(\mathbb{B})}$.

  (2) *The algebraic structure* $\mathcal{R}{\downarrow}$ *is a universally complete vector lattice.*

  (3) *There is a Boolean isomorphism* χ *from* $\mathbb{B}$ *onto* $\mathbb{P}(\mathcal{R}{\downarrow})$ *such that for all* $x, y \in \mathcal{R}{\downarrow}$; $b \in \mathbb{B}$ *the equivalences hold:*

$$\chi(b)x = \chi(b)y \iff b \le [\![x = y]\!],$$

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Some Problems and Solutions. I

The problem	Raised by Stems from	Reduced to (by means of BA)	Solved by
Intrinsic characterization of subdifferentials	Kutateladze 1976	Weakly compact convex sets of functionals	Kusraev Kutateladze 1982
General desintegration in Kantorovich spaces	Ioffe, Levin Neumann 1972/1977	Hahn–Banach & Radon–Nikodým theorems	Kusraev 1984
Kaplansky Problem: Homogeneity of a type I AW^*-algebra	Kaplansky 1953	Homogeneity of $B(H)$ with H Hilbert space	Ozawa 1984

Some Problems and Solutions. II

The problem	Raised by Stems from	Reduced to (by means of BA)	Solved by
Order boundedness of BP operators, The Wickstead problem	Wickstead 1977	Cauchy type functional equations	Gutman Kusraev 1995, 2006
Maharam extension of a positive operator	Luxemburg Schep 1978	Daniel extension of an elementary integral	Akilov Kolesnikov Kusraev 1988
Goodearl problem 18 in "Von Neumann Regular Rings," RR	Goodearl 1979	Theorem 12.16 in RR	Chupin 1991

Some Problems and Solutions. III

The problem	Raised by Stems from	Reduced to (by means of BA):	Solved by
Description of T with $\lvert T\rvert$ a sum of 2 ℓ-homomorphisms	Grothendieck 1955	Description of functionals with the same property	Kutateladze 2005
Classification of injective Banach lattices	Lotz Cartright 1975	Classification of AL-space (L_1 spaces)	Kusraev 2012
Band preserving isomorphic copies of a VL	Abramovich and Kitover 2000	Extensions of fields	Kusraev 2021

Some Problems and Solutions. IV

The problem	Raised by Stems from	Reduced to (by means of BA)	Solved by
Ando type theorem in the category of $\mathbb{B}$-cyclic BL	Ando 1969	Ando's joint characterization of L^p and c_0	Kusraev Kutateladze 2019
Geometric characterization of preduals of injective Banach lattices	Lindenstrauss 1964	Characterization of L^1-preduals	Kusraev Kutateladze 2020
Geometric Characterization of injective Banach lattices	Ellis 1964	Characterization of L^1 spaces	Kusraev Kutateladze 2021

MAHARAM OPERATORS

- **Definition.** A linear operator $T : X \to Y$ is *order interval preserving* (or enjoys the *Maharam property*) *if* $T[0, x] = [0, Tx]$ $(x \in X_+)$,

$$(\forall\, x \in X_+)\ (\forall\, y \in Y)\ 0 \leq y \leq Tx \to\ (\exists\, 0 \leq u \leq x)\ Tu = y.$$

- **Definition.** A *Maharam operator* is an order continuous linear operator whose modulus has the Maharam property.
- **Definition.** A positive operator $T : X \to Y$ has the *Levi property* if $Y = T(X)^{\perp\perp}$ and $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y.
- The concept of Maharam operator stems from the articles by Dorothy Maharam on the representation of positive operators:

  The representation of abstract integrals, TAMS **75** (1953), 154-184;

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Luxemburg and Maharam

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- Luxemburg was a pioneer and promoter of blending model theory and functional analysis. He pointed out that the Maharam operators may play a fundamental role not only in the theory of positive operators but also in Boolean valued analysis; see, the Maharam anniversary volume:

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Nakano, Hahn, and Radon-Nikodým Theorems

- **Nakano carrier Theorem.** Given two order continuous linear functionals $f, g : X \to \mathbb{R}$, the equivalence holds: $f \perp g \longleftrightarrow C_f \perp C_g$.
- **Radon-Nikodým Theorem.** For a pair of order continuous linear functionals $f, g : X \to \mathbb{R}$ we have $|g| \leq f$ if and only if there exists an orthomorphism $\omega \in \mathrm{Orth}(X)$ such that $g = f \circ \omega$.
- **Hahn Decomposition Theorem.** For any order continuous linear functional $f : X \to \mathbb{R}$ there exists a band projection $\pi \in \mathbb{P}(X)$ such that $f^+ = f \circ \pi$ and $f^- = f \circ \pi^\perp$ with $\pi^\perp = I_X - \pi$.
- **The Claim:** Nakano carrier Theorem, Radon-Nikodým Theorem, and Hahn Decomposition Theorem are valid for Maharam operators.

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Strassen Disintegration Theorem

- A range of important questions in convex analysis and probability theory is connected with Strassen-type disintegration theorems. This name was fixed due to the publication:
- V. Strassen, The existence of probability measures with given marginals, Ann. Math. Statist. **36** 1965), 423-439.
- Theorem 1 in this paper states that linear functional dominated by sublinear (convex) integral functional can be obtained by integrating a measurable family of linear functionals, each majorized by the corresponding convex functional ($x' \in X'$, $p_\omega : X \to \mathbb{R}$, $\omega \in \Omega$):

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- **Notation.** $\partial P := \{S \in L(V,X) : \ (\forall\, u \in V)\ Su \le P(u)\}$.
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Abstract Disintegration

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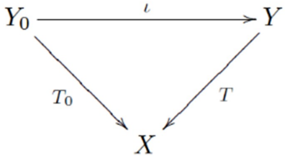
INJECTIVE BANACH LATTICES

- **Definition.** An injective Banach lattice is a real BL X such that:

$$(\forall Y)\,(\forall Y_0)\,(\forall T_0)$$

$$\left.\begin{array}{c} Y_0, Y \in (\mathrm{BL}) \\ Y_0 \text{ is a closed sublattice of } Y \\ 0 \leq T_0 \in L(Y_0, X) \end{array}\right] \implies \left[\begin{array}{c} (\exists T) \\ 0 \leq T \in L(Y, X) \\ T|_{X_0} = T_0 \\ \|T\| = \|T_0\| \end{array}\right.$$

- This amounts to saying that the diagram commutes, i. e. $T_0 = T \circ \iota$ with $\|T_0\| = \|T\|$:



Injective Banach Lattices: Examples

- Lotz was the first who examined the IBL. In his work,
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  he indicated among other things two important classes of IBL.
- **Theorem (Lotz, 1975)** A Dedekind complete AM-space with unit is an IBL. Equivalently, $C(K)$ is an IBL, whenever K is extremally disconnected Hausdorff compact space.
- **Theorem (Lotz, 1975).** Every AL-space is an IBL.
- The first result is not surprising, since $C(K)$ is an injective object in the category $\mathbf{BS}_1$ of Banach lattices and linear contractions.
- The second one shows that there is an essential difference between IBL and IBS, as $C(K)$ is the only (up to isometric isomorphism) injective object in $\mathbf{BS}_1$.

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Injective Banach Lattices: Characterization

- D. I. Cartwright, *Mem. Amer. Math. Soc.*, **164** (1975), 1-48.
  - ◇ The finite order IP $\iff$ The splitting property.
  - ◇ A BL X has the splitting property $\iff$ X'' is an IBL.
- R. Haydon, *Math. Z.* **156** (1977), 19-47.
  - ◇ Discovered that an IBL has a mixed *AM*-*AL*-structure.
  - ◇ Proved three representation theorems for IBL.
- W. Arendt, G. Buskes, J. Lindenstrauss, L. Tzafriri, P. J. Mangheni, A. Wickstead.

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A Boolean-Valued Transfer Principle

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- **Definition.** A positive operator $T : X \to Y$ is said to have the *Levi property* if $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y.
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(1) X is injective.
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A Representation Result

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Representation of AL-Spaces

- **Theorem (Kakutani-Maharam).** Let X be an AL-space. Then there exists a unique cardinal α and a unique family of cardinals $(\mathfrak{m}_\gamma)_{\gamma\in\Gamma}$ with Γ being a set of infinite cardinals such that each $\mathfrak{m}_\gamma$ is either equal to 1, or is uncountable, and

$$X \simeq l^1(\alpha) \oplus \sum_{\gamma\in\Gamma}^{\oplus} \mathfrak{m}_\gamma L^1([0,1]^\gamma),$$

where $\simeq$ denotes lattice isometry, $\oplus$ and $\sum^{\oplus}$ denote l^1-joins, $[0,1]^\gamma$ is product of gamma copies of unit interval with Lebesgue measure.

- Thus $l^1(\alpha)$ and $L^1([0,1]^\gamma)$ are building blocks for any AL-space. Actually, every IBL have a similar representation, so that Dedekind complete AM-spaces with unit ($C(K)$ with extremal compactum K) and AL-spaces (L^1) are the 'building blocks' for general IBL.

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Theorem (Kusraev, 2012). *Let X be an arbitrary IBL.*

- $X = X_1 \boxplus X_2$ *with* X_1 *atomic and* X_2 *purely non-atomic.*
- *There exist a set of cardinals* A *and a partition of unity* $(\pi_\alpha)_{\alpha \in \mathrm{A}}$ *in* $\mathbb{M}(X_1)$ *such that* $(\Lambda_\alpha = \pi_\alpha(\Lambda))$:

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Representation of Injective Banach Lattices

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SOME ALGEBRAIC ASPECTS

Abramovich-Kitover Problem

- **Definition.** A linear operator $T : X \to Y$ between vector lattices is disjointness preserving (DP) if T sends disjoint elements in X to disjoint elements in Y and d-isomorphism if T and T^{-1} are DP.
- **Y. A. Abramovich and A. K. Kitover,** Inverses of Disjointness Preserving Operators, Mem. AMS, 143(679), Providence, R. I., 2000.
- **Problem B:** Let X, Y be vector lattices and $T : X \to Y$ a d-isomorphism. Are then X and Y order isomorphic?
- **Theorem 14.17.** In the class of Dedekind complete vector lattices Problem B has an affirmative solution. That is, if $T : X \to Y$ is a d-isomorphism between two Dedekind complete vector lattices, then these vector lattices are order isomorphic.

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- The answer to Problem B is negative in general.
- **Theorem 13.4 (Abramovich, Kitover).** There exist a universally complete vector lattice W and a vector sublattice W_0 of W such that W_0 and W are d-isomorphic but are not order isomorphic.
- **Definition.** A linear operator $T : X \to Y$ is called band preserving (BP for short), if $T(L) \subset L$ for every band $L \subset X$ and b-isomorphism if both T and T^{-1} are band preserving.
- **Corollary.** In Theorem 13.4, a vector sublattice $W_0 \subset W$ can be chosen to be b-isomorphic to the ambient vector lattice W.
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Boolean Valued Representation

- **Theorem.** Assume that Let $\mathbb{R}^\wedge \subset \mathcal{X} \subset \mathcal{R}$, $\mathcal{X}$ is a subfield of $\mathcal{R}$, $X := \mathcal{X}{\downarrow}$, and $Y := \mathcal{R}{\downarrow}$. Every BP operator $T : X \to Y$ is representable as (the descent of) a $\mathbb{R}^\wedge$-linear function $\tau : \mathcal{X} \to \mathcal{R}$.
- **Query.** It is important to know whether $\mathcal{R} = \mathbb{R}^\wedge$ is true.
- **Definition.** A Boolean algebra $\mathbb{B}$ is said to be *σ-distributive* if, for any double sequence $(b_m^n)_{n,m\in\mathbb{N}}$ in $\mathbb{B}$, the equality holds

$$\bigwedge_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} b_m^n = \bigvee_{m\in\mathbb{N}^\mathbb{N}} \bigwedge_{n\in\mathbb{N}} b_{m(n)}^n$$

- **Definition.** A universally complete VL X with order unit $\mathbb{1}$ is *locally one-dimensional* if every $x \in X_+$ has the form $x = \sum_\xi \lambda_\xi \pi_\xi \mathbb{1}$, where $(\lambda_\xi) \subset \mathbb{R}_+$ and (π_ξ) a family of pairwise band projections.

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σ-Distributivity and Locally One-dimensionality

- **Theorem (Gutman, 1995).** Let $\mathbb{B}$ be a complete Boolean algebra and $\mathcal{R}$ the field of reals within $\mathbb{V}^{(\mathbb{B})}$. The following are equivalent:

  (1) $\mathbb{B}$ is σ-distributive.

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- **Lemma.** The field of reals $\mathbb{R}$ has no proper subfield $\mathbb{P}$ of which it is a finite extension. Consequently, if $\mathbb{R} \neq \mathbb{P}$ then $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{P}$.
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Solution to Problem B′

- A. G. Kusraev and S. S. Kutateladze, Two applications of Boolean valued analysis, Siberian Math. J., 2019, **60**:5, 902-910.
- **Theorem 3.5.** Let X be a universally complete vector lattice not containing nonzero locally one-dimensional bands. Then there are component-wise closed laterally complete vector sublattices $X_1 \subset X$ and $X_2 \subset X$ and linear bijections $T_1 : X_1 \to X$ and $T_2 : X_2 \to X$ s. th. (1) $X = X_1 \oplus X_2$ and $X = X_1^{\perp\perp} = X_2^{\perp\perp}$.
(2) The canonical projections $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$ are BP.
(3) T_k and T_k^{-1} are BP for $k = 1, 2$.
(4) None of the sublattices X_1 and X_2 is order complete and so is not lattice isomorphic to X.

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Another solution to problem B′

- **A. G. Kusraev,** Some Algebraic Aspects of Boolean Valued Analysis, In: Operator Theory and Harmonic Analysis Springer, 2021, 333-344.
- **Notation.** $[A]_\sigma := \left\{ \sum\limits_{n=1}^{\infty} \pi_k a_k : \ (a_k) \subset A, \ (\pi_k) \in \mathrm{Prt}\left(\mathbb{P}(X)\right)\right\}$.
- **Theorem 3.8.** Assume that a real universally complete vector lattice X is strictly Hamel $\varkappa$-homogeneous for some infinite cardinal $\varkappa$. Then there exists a family $(X_\alpha)_{\alpha\leq\varkappa}$ of component-wise closed and laterally complete vector sublattices $X_\alpha \subset X$ satisfying the conditions:

  (1) $X = \left[\bigoplus_{\alpha\leq\varkappa} X_\alpha\right]_\sigma$ and $X = X_\alpha^{\perp\perp}$ for all $\alpha \leq \varkappa$.

  (2) The canonical projection $\pi_\alpha : X \to X_\alpha$ are all band preserving.

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  (4) X_α is not Dedekind complete and hence not lattice isomorphic to X for all $\alpha \leq \varkappa$.

Another solution to problem B′

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- **Notation.** $[A]_\sigma := \Big\{ \sum_{n=1}^{\infty} \pi_k a_k : \ (a_k) \subset A, \ (\pi_k) \in \mathrm{Prt}\,(\mathbb{P}(X)) \Big\}$.
- **Theorem 3.8.** Assume that a real universally complete vector lattice X is strictly Hamel $\varkappa$-homogeneous for some infinite cardinal $\varkappa$. Then there exists a family $(X_\alpha)_{\alpha \leq \varkappa}$ of component-wise closed and laterally complete vector sublattices $X_\alpha \subset X$ satisfying the conditions:
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Reduction to Field Extension

- **Theorem.** Let $\mathbb{P} \subsetneq \mathbb{R}$. There exists an infinite cardinal $\varkappa$ and a family $(\mathcal{X}_\alpha)_{\alpha<\varkappa}$ of $\mathbb{P}$-linear subspace in $\mathbb{R}$ such that $\mathbb{R} = \bigoplus_{\alpha<\varkappa} \mathcal{X}_\alpha$ and, for every $\alpha < \varkappa$, the $\mathbb{P}$-vector spaces $\mathcal{X}_\alpha$ and $\mathbb{R}$ are isomorphic, whilst they are not isomorphic as ordered vector spaces over $\mathbb{P}$.
- **Proof.** Let $\mathcal{E}$ be a Hamel basis of a $\mathbb{P}$-vector space $\mathbb{R}$ and $\varkappa := |\mathcal{E}|$. Since $\varkappa$ is an infinite cardinal, we have

$$\varkappa = \sum_{\alpha<\varkappa} \varkappa_\alpha, \quad \varkappa_\alpha = \varkappa \ \ (\alpha < \varkappa).$$

It follows that there is a family of subsets $\mathcal{E}_\alpha \subset \mathcal{E}$ $(\alpha < \varkappa)$ such that

$$\mathcal{E} = \bigcup_{\alpha<\varkappa} \mathcal{E}_\alpha, \quad |\mathcal{E}_\alpha| = |\mathcal{E}|, \quad \mathcal{E}_\alpha \cap \mathcal{E}_\beta = \varnothing \quad (\alpha \neq \beta).$$

If $\mathcal{X}_\alpha \subset \mathbb{R}$ is the $\mathbb{P}$-subspace spanned by $\mathcal{E}_\alpha$, then $\mathcal{X}_\alpha \subsetneq \mathbb{R}$, $\mathcal{X}_\alpha \simeq_{\mathbb{P}} \mathbb{R}$.
- If $\mathcal{X}_\alpha$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$, then $\mathcal{X}$ would be order complete and we would have $\mathcal{X}_\alpha = \mathbb{R}$; a contradiction.

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The End

THANK YOU FOR ATTENTION!