Extreme Extension of Positive Operators

A. G. Kusraev

North-Caucasus Center for Mathematical Research of the VSC RAS
Southern Mathematical Institute of the VSC RAS

Workshop on approximation theory, theory of extremal problems, and optimal recovery, dedicated to the anniversary of

Professor Georgiĭ Georgievich Magaril-Il'yaev

June 11 – 14, 2024, Moscow — Vladikavkaz

Vector Lattices: Definition

- Definition. A *Vector lattice* (VL for short) is a real vector space E equipped with a partial order $\leq$ for which there exist

  ✓ $x \vee y := \sup\{x, y\}$, the supremum,

  ✓ $x \wedge y := \inf\{x, y\}$, the infimum,

  for all vectors $x, y \in E$ and such that the *positive cone*

  ✓ $E_+ := \{x \in E : \ x \geq 0\}$ of E has the properties

  ✓ $E_+ + E_+ \subset E_+, \quad \mathbb{R}_+ \cdot E_+ \subset E_+$ (*compatibility*).

  ✓ The absolute value (*modulus*) is defined as $|x| := x \vee (-x)$.

Vector Lattices: Definition

- Definition. A *Vector lattice* (VL for short) is a real vector space E equipped with a partial order $\leq$ for which there exist

  ✓ $x \vee y := \sup\{x, y\}$, the supremum,

  ✓ $x \wedge y := \inf\{x, y\}$, the infimum,

  for all vectors $x, y \in E$ and such that the *positive cone*

  ✓ $E_+ := \{x \in E : \ x \geq 0\}$ of E has the properties

  ✓ $E_+ + E_+ \subset E_+, \quad \mathbb{R}_+ \cdot E_+ \subset E_+$ (*compatibility*).

  ✓ The absolute value (*modulus*) is defined as $|x| := x \vee (-x)$.

- Definition. A VL E is called order complete if for any $\varnothing \neq A \subset E$ with $A \subset [a, b]$ there exist $\sup(A)$ and $\inf(A)$.

Vector Lattices: Definition

- Definition. A *Vector lattice* (VL for short) is a real vector space E equipped with a partial order $\leq$ for which there exist

  ✓ $x \vee y := \sup\{x, y\}$, the supremum,

  ✓ $x \wedge y := \inf\{x, y\}$, the infimum,

  for all vectors $x, y \in E$ and such that the *positive cone*

  ✓ $E_+ := \{x \in E : \ x \geq 0\}$ of E has the properties

  ✓ $E_+ + E_+ \subset E_+$, $\mathbb{R}_+ \cdot E_+ \subset E_+$ (*compatibility*).

  ✓ The absolute value (*modulus*) is defined as $|x| := x \vee (-x)$.
- Definition. A VL E is called order complete if for any $\varnothing \neq A \subset E$ with $A \subset [a, b]$ there exist $\sup(A)$ and $\inf(A)$.
- Examples. $C(K)$, $L^p(\Omega, \Sigma, \mu)$, l^p $(0 \leq p \leq \infty)$, c_0, c.

Positive Operators: Definition and Examples

- Definition. An operator $T \in L(E, F)$ is said to be *positive* if $T(E_+) \subset F_+$, i. e. $0 \leq x \in E \implies 0 \leq Tx \in F$.

Positive Operators: Definition and Examples

- Definition. An operator $T \in L(E, F)$ is said to be *positive* if
$$T(E_+) \subset F_+, \text{ i. e. } 0 \leq x \in E \implies 0 \leq Tx \in F.$$
- Example. Let $E \subset L^0(\Omega, \Sigma, \mu)$, $F \subset L^0(\Omega', \Sigma', \mu')$, and let $K \in L(E, F)$ be a kernel operator (with $k \in L^0(\Omega \times \Omega')$):
$$(Kx)(s) = \int_\Omega k(s, t)x(t)\, d\mu(t) \quad (x \in E).$$
$$K \geq 0 \iff k(s, t) \geq 0 \text{ for a. e. } (s, t) \in \Omega \times \Omega'.$$

- Definition. An operator $T \in L(E, F)$ is said to be *positive* if
$$T(E_+) \subset F_+, \text{ i. e. } 0 \leq x \in E \implies 0 \leq Tx \in F.$$
- Example. Let $E \subset L^0(\Omega, \Sigma, \mu)$, $F \subset L^0(\Omega', \Sigma', \mu')$, and let $K \in L(E, F)$ be a kernel operator (with $k \in L^0(\Omega \times \Omega')$):
$$(Kx)(s) = \int_\Omega k(s, t)x(t)\, d\mu(t) \quad (x \in E).$$
$$K \geq 0 \iff k(s, t) \geq 0 \text{ for a. e. } (s, t) \in \Omega \times \Omega'.$$
- Example. Let $M_\phi \in L(E, E)$ be a multiplication operator:
$$M_\phi x := \phi x \quad (x \in E) \quad \text{with } \phi \in L^0(\Omega, \Sigma, \mu)\ .$$
$$M_\phi \geq 0 \iff \phi(s) \geq 0 \text{ for a. e. } s \in \Omega.$$

Notation

- X Vector space.
- E Vector lattice (Riesz space).
- F, G Order complete vector lattices (Kantorovich spaces).
- D Majorizing vector subspace of E.

$$D \subset E \text{ is majorizing} \iff (\forall x \in E)\,(\exists y \in D)\ x \leq y.$$

Notation

- X Vector space.
- E Vector lattice (Riesz space).
- F, G Order complete vector lattices (Kantorovich spaces).
- D Majorizing vector subspace of E.

$$D \subset E \text{ is majorizing} \iff (\forall x \in E)(\exists y \in D)\ x \leq y.$$

- $L(E, F)$ Vector space of all linear operators from E to F.
- $L^+(E, F)$ Part of $L(E, F)$ consisting of positive operators.

Notation

- X Vector space.
- E Vector lattice (Riesz space).
- F, G Order complete vector lattices (Kantorovich spaces).
- D Majorizing vector subspace of E.

$$D \subset E \text{ is majorizing} \iff (\forall x \in E)\,(\exists y \in D)\ x \leq y.$$

- $L(E, F)$ Vector space of all linear operators from E to F.
- $L^{+}(E, F)$ Part of $L(E, F)$ consisting of positive operators.
- $\mathcal{E}(S)$ Collection of all positive extension of $S : D \to F$ to E:

$$\mathcal{E}(S) := \{R \in L(E, F) : \ R \geq 0 \text{ and } R|_D = S\}.$$

Notation

- X Vector space.
- E Vector lattice (Riesz space).
- F, G Order complete vector lattices (Kantorovich spaces).
- D Majorizing vector subspace of E.

$$D \subset E \text{ is majorizing} \iff (\forall\, x \in E)\,(\exists y \in D)\ x \leq y.$$

- $L(E, F)$ Vector space of all linear operators from E to F.
- $L^{+}(E, F)$ Part of $L(E, F)$ consisting of positive operators.
- $\mathcal{E}(S)$ Collection of all positive extension of $S : D \to F$ to E:

$$\mathcal{E}(S) := \{R \in L(E, F) : \ R \geq 0 \text{ and } R|_D = S\}.$$

- $\mathrm{ext}(\Omega)$ Collection of extreme points of a convex set Ω:

$$R \in \mathrm{ext}(\Omega) \iff (\forall\, R_1, R_2 \in \Omega)\,(\forall\, 0 < \alpha \in \mathbb{R})$$
$$R = \alpha R_1 + (1 - \alpha) R_2 \implies R = R_1 = R_2.$$

- Theorem (Kantorovich: 1937). *If D is a majorizing vector subspace of E and $S : D \to F$ is a positive operator, then S has a positive linear extension to all of E; in symbols,*

$$\mathcal{E}(S) \neq \varnothing.$$

[1] Lipecki, Z., Plachky, D. and Thomsen, W. Extension of Positive Operators and Extreme Points. I, *Colloquium Mathematicum*, **42**, (1979), 279–284.

- Theorem (Kantorovich: 1937). *If D is a majorizing vector subspace of E and $S : D \to F$ is a positive operator, then S has a positive linear extension to all of E; in symbols,*

$$\mathcal{E}(S) \neq \varnothing.$$

- Theorem (Lipecki, Plachky, and Thomsen; 1979)[1]. *Under the same assumptions a positive operator S admits extreme extensions; in symbols,*

$$\mathrm{ext}(\mathcal{E}(S)) \neq \varnothing.$$

[1] Lipecki, Z., Plachky, D. and Thomsen, W. Extension of Positive Operators and Extreme Points. I, *Colloquium Mathematicum*, **42**, (1979), 279–284.

- Theorem (Kantorovich: 1937). *If D is a majorizing vector subspace of E and $S : D \to F$ is a positive operator, then S has a positive linear extension to all of E; in symbols,*

$$\mathcal{E}(S) \neq \varnothing.$$

- Theorem (Lipecki, Plachky, and Thomsen; 1979)[1]. *Under the same assumptions a positive operator S admits extreme extensions; in symbols,*

$$\operatorname{ext}(\mathcal{E}(S)) \neq \varnothing.$$

- Theorem (Lipecki, Plachky, and Thomsen; 1979)[1]. For an operator $R \in \mathcal{E}(S)$ the following equivalence holds:

$$R \in \operatorname{ext}(\mathcal{E}(S)) \iff (\forall x \in E)\ \inf\{R(|x - y|) : \ y \in D\} = 0.$$

[1] Lipecki, Z., Plachky, D. and Thomsen, W. Extension of Positive Operators and Extreme Points. I, *Colloquium Mathematicum*, **42**, (1979), 279–284.

Main Result

- Definition. An operator $T : F \to G$ is said to be:
  - ✓ interval preserving if $T([0, x]) = [0, Tx]$ for all $x \in F_+$;
  - ✓ order continuous if $\inf_\alpha Tx_\alpha = 0$ in G for any $x_\alpha \downarrow 0$ in F;
  - ✓ Maharam if T is order continuous and interval preserving;
  - ✓ strictly positive whenever $T(|x|) = 0$ implies $x = 0$.

[2] Kusraev A. G. On Extreme Extension of Positive Operators, *Vladikavkaz Math. J.* **26**:3 (20024), 48–53.

[3] Lipecki, Z. Compactness and Extreme Points of the Set of Quasi-Measure Extensions of a Quasi-Measure, *Dissertationes Mathematicae*, **493**, 2013.

Main Result

- Definition. An operator $T : F \to G$ is said to be:
  - ✓ interval preserving if $T([0,x]) = [0, Tx]$ for all $x \in F_+$;
  - ✓ order continuous if $\inf_\alpha Tx_\alpha = 0$ in G for any $x_\alpha \downarrow 0$ in F;
  - ✓ Maharam if T is order continuous and interval preserving;
  - ✓ strictly positive whenever $T(|x|) = 0$ implies $x = 0$.
- Theorem (Kusraev; 2024)[2] *Let $S \in L^+(D, F)$ and $T : F \to G$ be strictly positive Maharam operators. The following hold:*

$$\mathcal{E}(T \circ S) = T \circ \mathcal{E}(S);$$
$$\operatorname{ext} \mathcal{E}(T \circ S) \subset T \circ \operatorname{ext} \mathcal{E}(S).$$

[2] Kusraev A. G. On Extreme Extension of Positive Operators, *Vladikavkaz Math. J.* **26**:3 (20024), 48–53.

[3] Lipecki, Z. Compactness and Extreme Points of the Set of Quasi-Measure Extensions of a Quasi-Measure, *Dissertationes Mathematicae*, **493**, 2013.

- Definition. An operator $T : F \to G$ is said to be:
  - ✓ interval preserving if $T([0, x]) = [0, Tx]$ for all $x \in F_+$;
  - ✓ order continuous if $\inf_\alpha Tx_\alpha = 0$ in G for any $x_\alpha \downarrow 0$ in F;
  - ✓ Maharam if T is order continuous and interval preserving;
  - ✓ strictly positive whenever $T(|x|) = 0$ implies $x = 0$.
- Theorem (Kusraev; 2024)[2] *Let $S \in L^+(D, F)$ and $T : F \to G$ be strictly positive Maharam operators. The following hold:*

$$\mathcal{E}(T \circ S) = T \circ \mathcal{E}(S);$$
$$\operatorname{ext} \mathcal{E}(T \circ S) \subset T \circ \operatorname{ext} \mathcal{E}(S).$$

- A very special case was obtained by Z. Lipecki[3].

[2] Kusraev A. G. On Extreme Extension of Positive Operators, *Vladikavkaz Math. J.* **26**:3 (20024), 48–53.

[3] Lipecki, Z. Compactness and Extreme Points of the Set of Quasi-Measure Extensions of a Quasi-Measure, *Dissertationes Mathematicae*, **493**, 2013.

Subdifferentials: Definition

- Motivation:
  - ✓ Lipecki's memoir[3] on extensions of a given quasi-measure
  - ✓ The authors article[4] on disintegration in Kantorovich spaces.

[4] A. G. Kusraev, Abstract disintegration in Kantorovich spaces, Sibirsk. Mat. Zh., , **25**:5 (1984), 79-89.

Subdifferentials: Definition

- Motivation:
  - ✓ Lipecki's memoir[3] on extensions of a given quasi-measure
  - ✓ The authors article[4] on disintegration in Kantorovich spaces.
- Definition. An operator $P : X \to E$ is said to be sublinear if

$$P(\lambda x) = \lambda P(x) \text{ for all } x \in X \text{ and } 0 \leq \lambda \in \mathbb{R};$$
$$P(x + y) \leq P(x) + P(y) \text{ for all } x, y \in X.$$

- Notation. $\mathrm{Sbl}(X, E)$ is the set of sublinear operators $X \to E$.

[4] A. G. Kusraev, Abstract disintegration in Kantorovich spaces, Sibirsk. Mat. Zh., , **25**:5 (1984), 79-89.

Subdifferentials: Definition

- Motivation:
  - ✓ Lipecki's memoir[3] on extensions of a given quasi-measure
  - ✓ The authors article[4] on disintegration in Kantorovich spaces.
- Definition. An operator $P : X \to E$ is said to be sublinear if

$$P(\lambda x) = \lambda P(x) \ \text{ for all } x \in X \text{ and } 0 \leq \lambda \in \mathbb{R};$$
$$P(x + y) \leq P(x) + P(y) \ \text{ for all } x, y \in X.$$

- Notation. $\mathrm{Sbl}(X, E)$ is the set of sublinear operators $X \to E$.
- Definition. A support set or a subdifferential (at zero) ∂P of a sublinear operator P is defined as $T \in \partial P \iff T \leq P$:

$$\partial P := \{T \in L(X, E) : \ (\forall x \in X) \ Tx \leq P(x)\}$$

[4] A. G. Kusraev, Abstract disintegration in Kantorovich spaces, Sibirsk. Mat. Zh., , **25**:5 (1984), 79-89.

- **Definition.** Say that E has the dominated extension property if

$(\forall X)\,(\forall X_0)\,(\forall T_0)\,(\forall P)$

$$\left.\begin{array}{c} X_0, X \in (\mathbf{VS}) \\ X_0 \subset X \\ T_0 \in L(X_0, E) \\ P \in \mathrm{Sbl}(X, E) \\ T_0 \in \partial(P|_{X_0}) \end{array}\right] \implies \left[\begin{array}{c} (\exists T) \\ T \in L(X, E) \\ T|_{X_0} = T_0 \\ T \in \partial P \end{array}\right.$$

Dominated Extension Property

- Definition. Say that E has the dominated extension property if

  $(\forall X)\,(\forall X_0)\,(\forall T_0)\,(\forall P)$

$$\left.\begin{matrix} X_0, X \in (\mathbf{VS}) \\ X_0 \subset X \\ T_0 \in L(X_0, E) \\ P \in \mathrm{Sbl}(X, E) \\ T_0 \in \partial(P|_{X_0}) \end{matrix}\right] \implies \left[\begin{matrix} (\exists T) \\ T \in L(X, E) \\ T|_{X_0} = T_0 \\ T \in \partial P \end{matrix}\right.$$

- Hahn–Banach–Kantorovich Theorem (Kantorovich; 1935).
  Every Kantorovich space has the dominated extension property.

Dominated Extension Property

- Definition. Say that E has the dominated extension property if

$(\forall X)\,(\forall X_0)\,(\forall T_0)\,(\forall P)$

$$\left.\begin{array}{c} X_0, X \in (\mathbf{VS}) \\ X_0 \subset X \\ T_0 \in L(X_0, E) \\ P \in \mathrm{Sbl}(X, E) \\ T_0 \in \partial(P|_{X_0}) \end{array}\right] \implies \left[\begin{array}{c} (\exists T) \\ T \in L(X, E) \\ T|_{X_0} = T_0 \\ T \in \partial P \end{array}\right.$$

- Hahn–Banach–Kantorovich Theorem (Kantorovich; 1935).
  Every Kantorovich space has the dominated extension property.
- Theorem (Bonnice and Silverman, To; 1967, 1970).
  If an ordered vector space has the dominated extension property, then it is a Kantorovich space.

- Lemma. *Let $S : L^+(D, F)$ define a mapping $P_S : E \to F$ as*

$$P_S(x) := \inf\{Sx' : \ x' \in D, \, x \leq x'\} \quad (x \in E).$$

Then $P_S : E \to F$ is a sublinear operator and $\partial(P_S) = \mathcal{E}(S)$.

[5] Kutateladze, S. S. Extreme points of subdifferentials, Dokl. Akad. Nauk SSSR, vol. 242, no. 5 (1978), pp. 1001-1003.

- Lemma. *Let* $S : L^{+}(D, F)$ *define a mapping* $P_S : E \to F$ *as*

$$P_S(x) := \inf\{Sx' : \ x' \in D,\, x \le x'\} \quad (x \in E).$$

Then $P_S : E \to F$ *is a sublinear operator and* $\partial(P_S) = \mathcal{E}(S)$.

- Hahn–Banach–Kantorovich $\implies \partial(P) \neq \varnothing \implies \mathcal{E}(S) \neq \varnothing$.

[5] Kutateladze, S. S. Extreme points of subdifferentials, Dokl. Akad. Nauk SSSR, vol. 242, no. 5 (1978), pp. 1001-1003.

- Lemma. *Let* $S : L^+(D, F)$ *define a mapping* $P_S : E \to F$ *as*

$$P_S(x) := \inf\{Sx' : \ x' \in D,\, x \leq x'\} \quad (x \in E).$$

Then $P_S : E \to F$ *is a sublinear operator and* $\partial(P_S) = \mathcal{E}(S)$.

- Hahn–Banach–Kantorovich $\implies \partial(P) \neq \varnothing \implies \mathcal{E}(S) \neq \varnothing$.
- Theorem (Kutateladze; 1978)[5] *Every sublinear operator is the upper envelope of the set of extreme points of its support set:*

$$P(x) = \sup\{Tx : \ T \in \mathrm{ext}(\partial(P))\} \quad (x \in X).$$

[5] Kutateladze, S. S. Extreme points of subdifferentials, Dokl. Akad. Nauk SSSR, vol. 242, no. 5 (1978), pp. 1001-1003.

- Lemma. *Let* $S : L^{+}(D, F)$ *define a mapping* $P_S : E \to F$ *as*

$$P_S(x) := \inf\{Sx' : \ x' \in D,\, x \le x'\} \quad (x \in E).$$

Then $P_S : E \to F$ *is a sublinear operator and* $\partial(P_S) = \mathcal{E}(S)$.

- Hahn–Banach–Kantorovich $\implies \partial(P) \neq \varnothing \implies \mathcal{E}(S) \neq \varnothing$.
- Theorem (Kutateladze; 1978)[5] *Every sublinear operator is the upper envelope of the set of extreme points of its support set:*

$$P(x) = \sup\{Tx : \ T \in \operatorname{ext}(\partial(P))\} \quad (x \in X).$$

- Kutateladze $\implies \operatorname{ext}(\partial(P)) \neq \varnothing \implies \operatorname{ext}(\mathcal{E}(S)) \neq \varnothing$.

[5] Kutateladze, S. S. Extreme points of subdifferentials, Dokl. Akad. Nauk SSSR, vol. 242, no. 5 (1978), pp. 1001-1003.

Strassen's Disintegration Theorem: Entourage

Strassen's seminal disintegration theorem was established in the following environment:

- X Separable Banach space.
- (Ω, Σ, μ) Measure space with a complete finite measure.

Strassen's Disintegration Theorem: Entourage

Strassen's seminal disintegration theorem was established in the following environment:

- X Separable Banach space.
- (Ω, Σ, μ) Measure space with a complete finite measure.
- $p_\omega : X \to \mathbb{R}$ Continuous sublinear functional for all $\omega \in \Omega$.
- $\omega \mapsto p_\omega(x)$ Integrable function for all $x \in X$.
- $\omega \mapsto \|p_\omega\| := \sup\{|p_\omega(x) : x \in X, \|x\| \leq 1\}$ Integrable.

Strassen's Disintegration Theorem: Entourage

Strassen's seminal disintegration theorem was established in the following environment:

- X Separable Banach space.
- (Ω, Σ, μ) Measure space with a complete finite measure.
- $p_\omega : X \to \mathbb{R}$ Continuous sublinear functional for all $\omega \in \Omega$.
- $\omega \mapsto p_\omega(x)$ Integrable function for all $x \in X$.
- $\omega \mapsto \|p_\omega\| := \sup\{|p_\omega(x) : x \in X, \|x\| \leq 1\}$ Integrable.
- Lemma. A sublinear function $p : X \to \mathbb{R}$ is correctly defined by

$$p(x) := \int_\Omega p_\omega(x)\, d\mu(\omega) \quad (x \in X).$$

Strassen's Disintegration Theorem: Entourage

Strassen's seminal disintegration theorem was established in the following environment:

- X Separable Banach space.
- (Ω, Σ, μ) Measure space with a complete finite measure.
- $p_\omega : X \to \mathbb{R}$ Continuous sublinear functional for all $\omega \in \Omega$.
- $\omega \mapsto p_\omega(x)$ Integrable function for all $x \in X$.
- $\omega \mapsto \|p_\omega\| := \sup\{|p_\omega(x)| : x \in X, \|x\| \leq 1\}$ Integrable.
- Lemma. A sublinear function $p : X \to \mathbb{R}$ is correctly defined by

$$p(x) := \int_\Omega p_\omega(x)\, d\mu(\omega) \quad (x \in X).$$

- Strassen's disintegration theorem gives a description of $\partial(p)$.

Strassen's Disintegration Theorem

- Theorem (Strassen; 1965)[6]. *For every* $x^* \in \partial p$ *there exists a mapping* $\Omega \ni \omega \mapsto x^*_\omega \in X^*$ *such that the following hold:*

  (1) $\omega \mapsto \langle x, x^*_\omega \rangle \in L_1(\Omega, \Sigma, \mu)$ *for all* $x \in X$*;*

  (2) $x^*_\omega \in \partial p_\omega$ for all $(\omega \in \Omega)$;

  (3) the representation holds:

$$\langle x, x^*_\omega \rangle = \int_\Omega \langle x, x^*_\omega \rangle \, d\mu(\omega) \quad (x \in X).$$

[6] V. Strassen. *The existence of probability measures with given marginals* Ann. Math. Stat., **36** (1965), 423–439.

Strassen's Disintegration Theorem

- Theorem (Strassen; 1965)[6]. *For every $x^* \in \partial p$ there exists a mapping $\Omega \ni \omega \mapsto x^*_\omega \in X^*$ such that the following hold:*
  (1) $\omega \mapsto \langle x, x^*_\omega \rangle \in L_1(\Omega, \Sigma, \mu)$ *for all* $x \in X$;
  (2) $x^*_\omega \in \partial p_\omega$ for all $(\omega \in \Omega)$;
  (3) the representation holds:

$$\langle x, x^*_\omega \rangle = \int_\Omega \langle x, x^*_\omega \rangle \, d\mu(\omega) \quad (x \in X).$$

- Interpretation. Define $I_\mu : L^1(\mu) \to \mathbb{R}$ and $P : X \to L^1(\mu)$ as

$$I_\mu := \int_\Omega u(s) d\mu(s), \quad P(x) : \omega \mapsto p_\omega(x)$$

  Then the representations hold:

$$p = I_\mu \circ P, \quad \partial(I_\mu \circ P) = I_\mu \circ \partial P.$$

[6] V. Strassen. *The existence of probability measures with given marginals* Ann. Math. Stat., **36** (1965), 423–439.

- Theorem (Levin; 1972)[7]. If $\Phi \in L^+(F, \mathbb{R})$ is order continuous and $P : X \to F$ is sublinear then the following holds:

$$\partial(\Phi \circ P) = \Phi \circ \partial(P).$$

[7] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

[8] S. S. Kutateladze, *Russian Math. Surveys*, **34**:1 (1979), 181-214.

[9] M. Neumann. On the Strassen disintegration theorem, *Arch. Math.*, **29**:4 (1977), 413-420.

- Theorem (Levin; 1972)[7]. If $\Phi \in L^{+}(F, \mathbb{R})$ is order continuous and $P : X \to F$ is sublinear then the following holds:

$$\partial(\Phi \circ P) = \Phi \circ \partial(P).$$

- Theorem (Kutatelaze; 1979)[8]. If $M := M_{\phi} \in \mathrm{Orth}^{+}(F)$ is a multiplication operator and $P \in \mathrm{Sbl}(X, F)$ then we have:

$$\partial(M \circ P) = M \circ \partial(P).$$

[7] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

[8] S. S. Kutateladze, *Russian Math. Surveys*, **34**:1 (1979), 181-214.

[9] M. Neumann. On the Strassen disintegration theorem, *Arch. Math.*, **29**:4 (1977), 413-420.

- Theorem (Levin; 1972)[7]. If $\Phi \in L^+(F, \mathbb{R})$ is order continuous and $P : X \to F$ is sublinear then the following holds:

$$\partial(\Phi \circ P) = \Phi \circ \partial(P).$$

- Theorem (Kutatelaze; 1979)[8]. If $M := M_\phi \in \mathrm{Orth}^+(F)$ is a multiplication operator and $P \in \mathrm{Sbl}(X, F)$ then we have:

$$\partial(M \circ P) = M \circ \partial(P).$$

- Theorem (Neumann; 1977)[9]. Let $(P_\alpha)_{\alpha \in \mathrm{A}}$ be a pointwise summable family in $\mathrm{Sbl}(X, F)$ and define $P(x) := (P_\alpha)_{\alpha \in \mathrm{A}}$. Put $\Sigma : (f_\alpha)_{\alpha \in \mathrm{A}} \mapsto \sum_{\alpha \in \mathrm{A}} f_\alpha$ for any $(f_\alpha)_{\alpha \in \mathrm{A}} \in l_1(\mathrm{A}, F)$. Then

$$\partial(\Sigma \circ P) = \Sigma \circ \partial(P).$$

---

[7] V. L. Levin. Subdifferentials of convex mappings and of composite functions, *Siberian Math. J.*, **13**:6 (1972), 903-909.

[8] S. S. Kutateladze, *Russian Math. Surveys*, **34**:1 (1979), 181-214.

[9] M. Neumann. On the Strassen disintegration theorem, *Arch. Math.*, **29**:4 (1977), 413-420.

Abstract Desintegration: General

- Theorem (Kusraev; 1982)[10] *Let* $T : F \to G$ *be a Maharam operator and let* $P : X \to F$ *be sublinear. Then*

$$\partial(T \circ P) = T \circ \partial(P).$$

---

[10] A. G. Kusraev. General disintegration formulas, Dokl. Akad. Nauk SSSR, **265**:6 (1982), 1312-1316.

[11] A. G. Kusraev and S. S. Kutateladze, *Subdifferential Calculus. Theory and Applications*, Moscow: Nauka, 2007.

[12] S. S. Kutateladze, The Krein–Mil'man Theorem and Its Inverse, *Siberian Math. J.*, **21**: 1 (1980), 97-103.

- Theorem (Kusraev; 1982)[10] *Let $T : F \to G$ be a Maharam operator and let $P : X \to F$ be sublinear. Then*

$$\partial(T \circ P) = T \circ \partial(P).$$

- There are a number of disintegration formulas that unify in a conventional form of calculus, various facts of the theory of Kantorovich spaces based on the Radon–Nikodym theorem [11]

[10] A. G. Kusraev. General disintegration formulas, Dokl. Akad. Nauk SSSR, **265**:6 (1982), 1312-1316.

[11] A. G. Kusraev and S. S. Kutateladze, *Subdifferential Calculus. Theory and Applications*, Moscow: Nauka, 2007.

[12] S. S. Kutateladze, The Krein–Mil'man Theorem and Its Inverse, *Siberian Math. J.*, **21**: 1 (1980), 97-103.

Abstract Desintegration: General

- Theorem (Kusraev; 1982)[10] *Let $T : F \to G$ be a Maharam operator and let $P : X \to F$ be sublinear. Then*

$$\partial(T \circ P) = T \circ \partial(P).$$

- There are a number of disintegration formulas that unify in a conventional form of calculus, various facts of the theory of Kantorovich spaces based on the Radon–Nikodym theorem [11]
- Is it true that $\operatorname{ext} \partial(T \circ P) \subset T \circ \operatorname{ext} \partial(P)$ with P sublinear?

---

[10] A. G. Kusraev. General disintegration formulas, Dokl. Akad. Nauk SSSR, **265**:6 (1982), 1312-1316.

[11] A. G. Kusraev and S. S. Kutateladze, *Subdifferential Calculus. Theory and Applications*, Moscow: Nauka, 2007.

[12] S. S. Kutateladze, The Krein–Mil'man Theorem and Its Inverse, *Siberian Math. J.*, **21**: 1 (1980), 97-103.

Abstract Desintegration: General

- Theorem (Kusraev; 1982)[10] *Let $T : F \to G$ be a Maharam operator and let $P : X \to F$ be sublinear. Then*

$$\partial(T \circ P) = T \circ \partial(P).$$

- There are a number of disintegration formulas that unify in a conventional form of calculus, various facts of the theory of Kantorovich spaces based on the Radon–Nikodym theorem [11]
- Is it true that $\operatorname{ext}\partial(T \circ P) \subset T \circ \operatorname{ext}\partial(P)$ with P sublinear?
- Mil'man Theorem (Kutateladze; 1980)[12]. If $S \in L(Y, X)$ and $P \in \mathrm{Sbl}(X, F)$, then the relation holds:

$$\operatorname{ext}\partial(P \circ S) \subset \operatorname{ext}\partial(P) \circ S.$$

---

[10] A. G. Kusraev. General disintegration formulas, Dokl. Akad. Nauk SSSR, **265**:6 (1982), 1312-1316.

[11] A. G. Kusraev and S. S. Kutateladze, *Subdifferential Calculus. Theory and Applications*, Moscow: Nauka, 2007.

[12] S. S. Kutateladze, The Krein–Mil'man Theorem and Its Inverse, *Siberian Math. J.*, **21**: 1 (1980), 97-103.

- Theorem. If $P : X \to F$ is a sublinear operator and $T : F \to G$ is a Maharam operator, then the inclusion holds:

$$\operatorname{ext}\partial(T \circ P) \subset T \circ \operatorname{ext}\partial(P).$$

[13] S. S. Kutateladze, Extreme points of subdifferentials, *Dokl. Akad. Nauk SSSR*, **242**:5 (1978), 1001-1003.

[14] R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Springer-Verlag, Berlin etc., 1975.

Extreme points of a composition

- Theorem. If $P : X \to F$ is a sublinear operator and $T : F \to G$ is a Maharam operator, then the inclusion holds:

$$\operatorname{ext}\partial(T \circ P) \subset T \circ \operatorname{ext}\partial(P).$$

- Theorem (Kutateladze; 1978)[13]. Let $T \in L^+(F, G)$ and $R \in \partial P$ with $P : X \to F$ sublinear. Then $T \circ R \in \operatorname{ext}\partial(T \circ P)$ if and only if for any $x \in X$, $y \in E$ the equality holds:

$$Ty^+ = \inf\left\{T\big((P(u)-Ru)\vee(P(u-x)-R(u-x)+y)\big) : u \in X\right\}.$$

[13] S. S. Kutateladze, Extreme points of subdifferentials, *Dokl. Akad. Nauk SSSR*, **242**:5 (1978), 1001-1003.

[14] R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Springer-Verlag, Berlin etc., 1975.

Extreme points of a composition

- Theorem. If $P : X \to F$ is a sublinear operator and $T : F \to G$ is a Maharam operator, then the inclusion holds:

$$\operatorname{ext}\partial(T \circ P) \subset T \circ \operatorname{ext}\partial(P).$$

- Theorem (Kutateladze; 1978)[13]. Let $T \in L^{+}(F, G)$ and $R \in \partial P$ with $P : X \to F$ sublinear. Then $T \circ R \in \operatorname{ext}\partial(T \circ P)$ if and only if for any $x \in X$, $y \in E$ the equality holds:

$$Ty^{+} = \inf\left\{T\big((P(u)-Ru)\vee(P(u-x)-R(u-x)+y)\big) :\ u \in X\right\}.$$

- In scalar case ($F = G = \mathbb{R}$) this fact is known as the Buck–Phelps theorem[14].

[13] S. S. Kutateladze, Extreme points of subdifferentials, *Dokl. Akad. Nauk SSSR*, **242**:5 (1978), 1001-1003.

[14] R. B. Holmes, *Geometric Functional Analysis and Its Applications*, Springer-Verlag, Berlin etc., 1975.

Thank you for attention