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Operators in Vector Lattices: Problems and Solutions

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I. INTRODUCTION

Banach Lattices: Definition

Definition. A *Banach lattice* (BL for short) is a real Banach space E equipped with a partial order $\leq$ for which there exist

✓ $x \vee y := \sup\{x, y\}$, the supremum,

✓ $x \wedge y := \inf\{x, y\}$, the infimum,

for all vectors $x, y \in E$ and such that the *positive cone*

✓ $E_+ := \{x \in E : \ x \geq 0\}$ of E have the properties

✓ $E_+ + E_+ \subset E_+, \quad \mathbb{R}_+ \cdot E_+ \subset E_+$ (*compartability*),

and the order is connected to the norm by the condition that

✓ $|x| \leq |y| \implies \|x\| \leq \|y\|$ for all $x.y \in E$ (*monotonicity*),

where the absolute value (*modulus*) is defined as

✓ $|x| := x \vee (-x)$.

Banach lattices were first considered by Kantorovich in
L. V. Kantorovich, *Mat. Sbornik*, **2(44)** (1937), 121-165.

Banach Lattices: Examples

- **Example 1.** $C(K)$, $L^p(\Omega, \Sigma, \mu)$, l^p $(1 \leq p \leq \infty)$, c_0, c.

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- **Definition.** A BL E is *order continuous* $(E \in (A))$ if the following (a kind of Dominated Convergence Theorem) holds: $x_\alpha \downarrow 0 \implies \|x_\alpha\| \to 0$.
- **Definition.** A BL E is *monotonically complete* $(E \in (B))$ the following (Levi property) holds: $0 \le x_\alpha \uparrow 0$ and $\|x_\alpha\| \le 1$ imply that $\sup x_\alpha$ exists in E.

- **Definition.** Let $S, T : E \to F$ be two operators between VL or BL with S positive, i. e. $x \geq 0$ implies $Sx \geq 0$. We say that T is *dominated* by S (called a *dominant* of T) if

$$|T(x)| \leq S(|x|) \quad (x \in E) \quad (\equiv \ |T| \leq S).$$

In the sequel T is positive, so that $0 \leq T \leq S$, that is,

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- **The Domination Problem (DP):** Let $\mathcal{P}$ denotes a property of a positive operator and $\mathcal{P}(E, F)$ stands for the set of operators $T : E \to F$ having the property $\mathcal{P}$. The DP then asks whether or not the implication holds:

$$0 \leq T \leq S \text{ and } S \in \mathcal{P}(E, F) \Longrightarrow T \in \mathcal{P}(E, F)?$$

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- **General:** What effect does an operator $T : E \to F$ have if its dominant operator $S : E \to F$ has property $\mathcal{P}$?

- **Definition.** An operator $T \in \mathcal{L}(E, F)$ is called *weakly compact*, if the set $T(B_E)$ is relatively weakly compact in F.
- Let $\mathcal{W}(E, F)$ stands for the set of weakly compact operators.

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- **Theorem (Abramovich, 1972).** Let E be a Banach lattice and F be a KB-space. Then for every pair of positive linear operators S, T from E to F the the implication holds:

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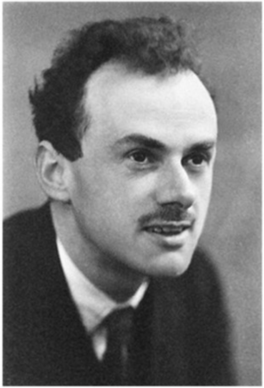
- **Abramovich Y. A.** Weakly compact sets in topological Dedekind complete vector lattices, Teor. Funkcii Funkcional. Anal. i Prilozhen. **15** (1972), 27–35.

II. JOHN VON NEUMANN PROBLEM

Paul Dirac and John von Neumann



Paul Adrien Maurice Dirac

(08.08.1902 – 20.10.1984)
English mathematical and theoretical physicist

John von Neumann

(28.12.1903 – 08.02.1957)
Hungarian and American mathematician and physicist

John von Neumann vs Paul Dirac (1932)

- **Paul Dirac** provides a very elegant and powerful formal framework for quantum mechanics, in which a central role was played by an "improper function", the Dirac delta function, which has the following incompatible properties: it is defined over the real line, is zero everywhere except for one point at which it is infinite, and yields unity when integrated over the real line.

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- **John von Neumann** promotes an alternative framework, which is not merely a refinement of Dirac's; rather, it is a radically different framework that is based on Hilbert's theory of operators. He emphasized that Dirac's theory as being powerful, clear, and unified, but characterized the Dirac delta function as a "mathematical fiction".

Kernel (Integral) Operators: Definition

- **Definition.** Let $E \subset L^0(\Omega, \Sigma, \mu)$, $F \subset L^0(\Omega', \Sigma', \mu')$, and $K \in L(X, Y)$ is a *kernel operator* with *kernel* $k \in L^0(\Omega \times \Omega')$:

$$(Kx)(s) = \int_\Omega k(s,t)x(t)\,d\mu(t) \quad (x \in E).$$

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- **Remarks.**

  ✓ $x : \Omega \to \overline{\mathbb{R}}$ is measurable and the equivalence class $\tilde{x} \in E$.

  ✓ $\int_\Omega |k(s,t)x(t)|\,d\mu(t) < \infty$ for a. e. $s \in \Omega$.

  ✓ $y_x : s \mapsto \int_\Omega |k(s,t)x(t)|\,d\mu(t)$ is measurable for all $\tilde{x} \in E$.

  ✓ $Kx = \widetilde{y_x(\cdot)}$, the equivalence class of $y_x(\cdot)$, belongs to F.

  ✓ $K \geq 0 \iff k(s,t) \geq 0$ for a. e. $(s,t) \in \Omega \times \Omega'$.

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- **John Von Neumann,** Charakterisierung des Spektrums Eines Integraloperators, Actualités Sci. et Ind., Paris, 1935, No. 229.

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- Which operators on an L^2 space are induced by a kernel?
- Which linear operators $T : E \to F$ between ideal function spaces $E \subset L^0(\Omega, \Sigma, \mu)$ and $F \subset L^0(\Omega', \Sigma', \mu')$ admit kernel representation with kernels $k \in L^0(\Omega \times \Omega', \Sigma \otimes \Sigma', \mu \otimes \mu')$?

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- **Definition.** A sequence (x_n) in E is said to

  converge pointwise (i. e., everywhere) to $x \in E$ if

  $$\lim_{n\to\infty} x_n(\omega) = x(\omega)\} = 0 \text{ for all } \omega \in \Omega,$$

  converge almost everywhere to $x \in E$ if

  $$\mu\big(\{\omega \in \Omega : \lim_{n\to\infty} x_n(\omega) \neq x(\omega)\}\big) = 0,$$

  converge in measure to $x \in E$ if $(\forall\, \varepsilon > 0,\ A \in \Sigma)$, $\mu(A) < \infty$,

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- Pointwise $\Longrightarrow$ Almost everywhere $\Longrightarrow$ In measure.
- **Theorem.** If a sequence $x_n \to x$ almost everywhere then there exists a subsequence (x_{n_k}) such that $x_n \to x$ in measure.

John von Neumann Problem: The solution

- **Theorem (Bukhvalov, 1984).** Let E and F be ideal spaces over σ-finite measure spaces. Then for every positive linear operator $T : E \to F$ the following are equivalent:

  (1) T is a kernel operator.

  (2) If a sequence (x_n) in E with $0 \leq x_n \leq x$ converges to zero in measure, then $T(x_n)$ converges to zero almost everywhere.

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- **Corollary.** Assume E, F, and G are ideal function spaces and at least one of the two operators $T \in \mathcal{L}_\sigma^\sim(E, F)$ and $S \in \mathcal{L}_\sigma^\sim(F, G)$ is a kernel operator. Then the composition $S \circ T : E \to G$ is likewise a kernel operator.

- **Notation.** Given ideal function spaces E and F, denote:

  $\mathcal{L}^{\sim}(E,F)$ the space of all regular operators;

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- **Theorem (Lozanovskiĭ, 1966).** Let E and F be ideal function spaces over σ-finite measure spaces.
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$$0 \leq |T| \leq S,\ S \in \mathcal{I}(E, F) \Longrightarrow T \in \mathcal{I}(E, F).$$

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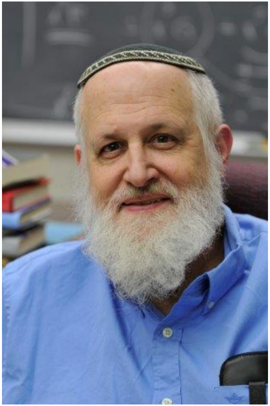
- **Corollary 2.** An operator $T \in \mathcal{L}^{\sim}(E,F)$ is a kernel operator if and only if there exists $0 \leq S \in \mathcal{I}(E,F)$ such that $|T| \leq S$.
- **Corollary 3.** If an increasing sequence (T_n) in $\mathcal{I}(E,F)$ and $S \in \mathcal{L}^{\sim}(E,F)$ are such that $0 \leq T_n \leq S$, then the mapping T defined as $Tx := \sup_{n\in\mathbb{N}} T_n x$ $(x \in E_+)$ is a kernel operator.

III. BARRY SIMON PROBLEM



Barry Simon, born 16.04.1946

known for his contributions
in spectral theory, functional analysis,
and nonrelativistic quantum mechanics

Domination: Schrödinger Operator

- The *Schrödinger Operator* with magnetic potential is:

$$H(\mathbf{a}) := (i\nabla + \mathbf{a})^2 + V,$$

$\mathbf{a} := (a_1, \dots, a_m) : \mathbb{R}^m \to \mathbb{R}^m$ is the magnetic potential,
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- Further assumptions about **a** and V imply that $e^{-H(a)}$ is a self adjoint operator in $L^2(\mathbb{R}^m)$.
- **Simon's inequality:** If $H = H(0) = -\Delta + V$, with $\Delta = -(i\nabla)^2$ being the Laplace operator, then

$$|e^{-tH(\mathbf{a})}| \leq e^{tH} = e^{t(-\Delta+V)} \quad (0 \leq t \in \mathbb{R}).$$

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- **J. Avron, I. Herbst, B. Simon.** Schrödinger operators with magnetic fields. I. General interactions, Duke Math. J. **45**(4) (1978), 847–883.

- **Theorem.** Let T be a compact operator in a Hilbert space H. There exist two orthonormal sequences (e_k) and (u_k) in H and a sequence (s_k) in $\mathbb{R}$ with $0 < s_k = s_k(T)$, $s_k \downarrow$, $\lim\limits_{n\to\infty} s_n = 0$,

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- **Definition.** $(s_k(T))_{k\in\mathbb{N}}$ is the sequence of s-numbers of T. Define the Banach space $(\mathfrak{S}_p(L^2), \|\cdot\|_p)$ $\quad (1 \leq p < \infty)$:

$$\|T\|_p := \left(\sum_{k=1}^{\infty} s_k(T)^p\right)^{\frac{1}{p}} \leq \infty \quad (T \in \mathfrak{S}_p(L^2)).$$

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- **Schatten-von Neumann classes $\mathfrak{S}_p(H)$:**
  $p = 1 \quad \mathfrak{S}_1(L^2) \equiv$ Trace class operators;
  $p = 2 \quad \mathfrak{S}_2(L^2) \equiv$ Hilbert-Schmidt operators;
  $p = \infty \quad \mathfrak{S}_\infty(L^2) \equiv$ Compact operators.

Domination: Simon's Problem

- **Simon's problem:** If $S, T \in \mathcal{L}(L^2)$, is it true that

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- **Yes,** if $p = 2n$ $(n \in \mathbb{N})$ (B. Simon, 1976).
- **No,** if $1 \leq p < \infty$, $p \neq 2n$ (V. V. Peller, 1980).
- **Yes,** if $p = \infty$ (P. Doods and D. Fremlin, 1979).

- **Simon's problem:** If $S, T \in \mathcal{L}(L^2)$, is it true that

$$|T| \leq S \text{ and } S \in \mathfrak{S}_p(L^2) \Longrightarrow T \in \mathfrak{S}_p(L^2)?$$

- **B. Simon,** Analysis with weak trace ideals and the number of bounded states of Schrödinger operators. Trans. Amer.Math. Soc. **224**(4) (1976), 367–380.
- **Yes,** if $p = 2n$ $(n \in \mathbb{N})$ (B. Simon, 1976).
- **No,** if $1 \leq p < \infty$, $p \neq 2n$ (V. V. Peller, 1980).
- **Yes,** if $p = \infty$ (P. Doods and D. Fremlin, 1979).
- **An application.** If H has a compact resolvent then $H(\mathbf{a})$ has also a compact resolvent:

$$|(\lambda I - H(\mathbf{a}))^{-1}| \leq -((\operatorname{Re}\lambda)I - H)^{-1} \quad (\operatorname{Re}\lambda < \inf \sigma(H)).$$

- **Definition.** An operator $T \in \mathcal{L}(E, F)$ is called *compact*, if the set $T(B_E)$ is relatively compact in F. Let $\mathcal{K}(E, F)$ stands for the set of all compact operators in $\mathcal{L}(E, F)$.

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- **Dodds–Fremlin Theorem.** Let E and F be BL with $E' \in (A)$ and $F \in (A)$. Then for any pair $S, T \in \mathcal{L}(E, F)$ the the implication holds:

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- **Peter Dodds, David Fremlin.** Compact operators in Banach lattices. Israel J. Math. **34**(4) (1979), 287–320.

Compact Domination: Wickstead Theorem

- **Definition.** A member $a \in E_+$ is called an atom if $[0, a] = [0, 1]a$. i. e., $0 \leq b \leq a$ implies that $b = \lambda a$ for some $0 \leq \lambda \leq 1$. A Banach lattice is said to be atomic if every members of E_+ majorizes at least one nonzero atom.

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- **Theorem (Wickstead, 1996).** For every pair od Banach lattices E and F the following assertions are equivalent:

  (1) One of the following (non-exclusive) conditions holds:

  ✓ Both E' and F have an order continuous norm.

  ✓ F is an atomic BL with an order continuous norm.

  ✓ E' is an atomic BL with an order continuous norm.

  (2) If $S, T : E \to F$, $0 \leq S \leq T$, and T is compact then S is compact.

IV. WHAT NEXT?

Another Domination Results

Operators $E \to F$	**Restrictions**	**Author(s)**	**Year**
Compact	$E', F \in (A)$ complete description	P. Dodds, D. Fremlin A. W. Wickstead	1979 1996
Weakly compact	$F \in (KB)$ $E' \in (A)$ or $F \in (A)$	Y.A. Abramovich A. W. Wickstead	1972 1981
AM-compact	$F \in (A)$ and E' is discrete	B. Aqzzouz, R. Nouira, L. Zraoula	2007
Dunford–Pettis	$F \in (A)$ complete description	N. Kalton, P. Saab A. W Wickstead	1985 1996
Disjointly strictly singular	$F \in (A)$	J. Flores, F. L. Hernández	2001
Banach–Saks	$F \in (A)$	J. Flores, C. Ruiz	2006

Another Domination Results

Operators $E \to F$	Restrictions	Author(s)	Year
Radon–Nikodým	$E, F \in (P)$; $F \in (A)$ $E \in (B)$, $F \in (KB)$	C. C. A. Labuschagne A. G. Kusraev	2006 2011
Asplund	$E \in (P)$, $E' \in (A)$ $E' \in (A)$	C. C. A. Labuschagne A. G. Kusraev	2006 2011
Strictly singular	$E \in (SSP)$ and $F \in (A)$	J. Flores F. L. Hernández P. Tradacete	2008
Narrow	$E, F \in (A)$ and E is atomless	O. D. Maslynchenko V. V. Mikhaylyuk M. M. Popov	2009
p-Summing	E and F are of cotype 2	C. Parazuelos E. A. Sánches-Perez P. Tradacete	2010

- **Definition.** Let $E_1, \ldots, E_n$ и F be BL. A multilinear operator $S : E_1 \times \ldots \times E_n \to F$ is called *positive* (in symbols $S \geq 0$), if

$$0 \leq x_1 \in E_1, \ldots, 0 \leq x_n \in E_n \implies S(x_1, \ldots, x_n).$$

Open Problems: Multilinear Domination

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- Denote by $L_+(E_1, \dots, E_n; F)$ the set of all positive multilinear operators from $E_1 \times \dots \times E_n$ to F.
- **Problem 1.** Let $\Phi(E_1, \dots, E_n; F)$ denotes the set of positive multilinear $T : E_1 \times \cdots \times E_n \to F$ with the property Φ. The multilinear domination problem asks whether or not

$$0 \leq S \leq T \text{ and } T \in \Phi(E, F) \Longrightarrow S \in \Phi(E, F)?$$

Open Problems: Polynomial Domination

- **Definition.** Let E and F be Banach lattices. A map $P : E \to F$ is called *n-homogeneous polynomial* if for some symmetric n-linear operator $\check{P} : E^n \to F$ we have

$$P(x) = \check{P}(x, \dots, x) \quad (x \in E).$$

(Such $\check{P}$ is unique). The polynomial P is said to be *positive* if

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- **Problem 2.** Let $\Phi(E, F)$ denotes the set of positive homogeneous polynomials $P : E \to F$ with the property Φ. The polynomial domination problem asks whether or not

$$0 \leq Q \leq P \text{ and } P \in \Phi(E, F) \Longrightarrow Q \in \Phi(E, F)?$$

- **Definition.** An operator $P : E \to F$ is called *sublinear*, if

$$P(x+y) \leq P(x) + P(y) \quad (x, y \in E),$$
$$P(\lambda x) = \lambda P(x) \quad (\lambda \in \mathbb{R}_+;\ x, y \in E),$$

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- **Proposition.** If F is Dedekind complete, then:

$$P \in \mathrm{Sbl}^+(E, F) \text{ if and only if } \partial P \subset L^+(E, F).$$

Open Problems: Sublinear Domination

- Let Ψ stands for a property of an increasing sublinear operator and let $\Psi(E, F)$ denotes the set of all $P \in \mathrm{Sbl}^{+}(E, F)$ with the property Ψ.

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- **Problem 3.** The *sublinear domination problem:* Under what conditions the implication holds:

$$P \in \Psi(E,F) \implies \partial P \subset \Phi(E,F)?$$

The properties Φ and Ψ may differ but they are, of course, correlated.

THANK YOU FOR ATTENTION!