

Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups

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Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups

A homogeneous Lie group is a connected simply connected Lie group whose Lie algebra is equipped with dilations.

Let $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ be a homogeneous group, Q is the homogeneous dimension of $\mathbb{G}$, $X_0, X_1, \dots, X_m$ be left invariant real vector fields on $\mathbb{G}$ and satisfy Hörmander's rank condition on $\mathbb{R}^N$. Assume that $X_1, \dots, X_m$ ($m \leq N - 1$) are homogeneous of degree one and X_0 is homogeneous of degree two with respect to the family of dilations $(\delta_\lambda)_{\lambda>0}$. Consider the following hypoelliptic operator with drift on $\mathbb{G}$

$$\mathcal{L} = \sum_{i,j=1}^m a_{ij} X_i X_j + a_0 X_0,$$

where (a_{ij}) is a $m \times m$ constant matrix satisfying the elliptic condition in $\mathbb{R}^m$ and $a_0 \neq 0$.

Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups

By definition a homogeneous Lie group is a Lie group equipped with a family of dilations compatible with the group law. The abelian group $(\mathbb{R}^n, +)$ is the very first example of homogeneous Lie group.

Homogeneous Lie groups have proved to be a natural setting to generalise many questions of Euclidean harmonic analysis. Indeed, having both the group and dilation structures allows one to introduce many notions coming from the Euclidean harmonic analysis. There are several important differences between the Euclidean setting and the one of homogeneous Lie groups.

Carnot groups are distinguished spaces that are rich of structure: they are those Lie groups equipped with a path distance that is invariant by left-translations of the group and admit automorphisms that are dilations with respect to the distance. We present the basic theory of Carnot groups together with several remarks. We consider them as special cases of graded groups and as homogeneous metric spaces. We discuss the regularity of isometries in the general case of Carnot-Carathéodory spaces and of nilpotent metric Lie groups.

Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups

In this talk, for this class of operators, we obtain the generalized weighted Sobolev – Morrey estimates by establishing boundedness of a large class of sublinear operators T_α , $\alpha \in [0, Q)$ generated by Calderón-Zygmund operators ($\alpha = 0$) and generated by fractional integral operator ($\alpha > 0$) on generalized weighted Morrey spaces and proving interpolation results on generalized weighted Sobolev – Morrey spaces on $\mathbb{G}$. The sublinear operators under consideration contain integral operators of harmonic analysis such as Hardy-Littlewood and fractional maximal operators, Calderón-Zygmund operators, fractional integral operators on homogeneous groups, etc.

Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups

Let $\mathbb{G}$ be a homogeneous group on $\mathbb{R}^N$ and $X_0, X_1, \dots, X_m$ ($m < N$) be left invariant real vector fields on $\mathbb{G}$. Assume that $X_1, \dots, X_m$ are homogeneous of degree one and X_0 is homogeneous of degree two satisfying Hörmander's condition

$$\text{rank } L(X_0, X_1, \dots, X_m)(x) = N, \quad x \in \mathbb{G},$$

where $L(X_0, X_1, \dots, X_m)$ denotes the Lie algebra generated by $X_0, X_1, \dots, X_m$. In this paper we are interested in the following hypoelliptic operator with drift

$$\mathcal{L} = \sum_{i,j=1}^m a_{ij} X_i X_j + a_0 X_0, \quad (1)$$

where $a_0 \neq 0$, $(a_{ij})_{i,j=1}^m$ is a constant coefficients matrix satisfying that for some $\mu > 0$,

$$\mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^m a_{ij} \xi_i \xi_j \leq \mu |\xi|^2, \quad \xi \in \mathbb{R}^m.$$

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Since Hörmander's classic work [1] for the operators sum of squares was published, the regularity of hypoelliptic operators structured on Hörmander's vector fields has attracted extensive attention [2, 3, 4, 5].

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 Bramanti, M., Brandolini, L.: L^p estimates for nonvariational hypoelliptic operators with VMO coefficients. Trans. Amer. Math. Soc. **352**(2), 781-822 (2000)

 Bramanti, M., Brandolini, L.: Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDES. Rev. Mat. Iberoam. **21**(2), 511-556 (2005)

 Kogoj, A.E., Lanconelli, E.: An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations. Mediterr. J. Math. **1**, 

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The relative of properties of weak generalized solutions to elliptic equations constructed by Hörmander's vector fields was studied in [1, 2]. Folland [3] proved that any Hörmander type operator like (1) has a homogeneous fundamental solution.

 Borrello, F.: On degenerate elliptic equations in Morrey spaces. Matematiche (Catania) **61**(1), 13-26 (2006)

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For the further properties of the fundamental solutions, see Bramanti and Brandolini [1]. The authors of [1, 2, 3] considered a priori estimates for the operator $\mathcal{L}$.

 Bramanti, M., Brandolini, L.: L^p estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups. **Rend. Semin. Mat. Univ. Politec. Torino** **58**, 389-433 (2000)

 Gutierrez, C.E., Lanconelli, E.: Schauder estimates for sub-elliptic equations. **J. Evol. Equ.** **9**, 707-726 (2009)

 Niu, P.C., Feng, X.J.: Global Sobolev-Morrey estimates for hypoelliptic operators with drift on homogeneous group. (in Chinese) **Sci. Sin. Math.** **42**(9), 905-920 (2012)

 Hou, Y., Niu, P.C.: Weighted Sobolev-Morrey estimates for hypoelliptic operators with drift on homogeneous group. **J. Math. Anal. Appl.** **428**, 1319-1338 (2015).

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The operator $\mathcal{L}$ contains many particular cases. When

$$X_0 = \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t, \quad X_i = \partial_{x_i}, \quad i = 1, 2, \dots, m, \quad \mathcal{L} \text{ is a}$$

Kolmogorov-Fokker-Planck ultraparabolic operator of the kind

$$\mathcal{L}_1 u = \sum_{i,j=1}^m a_{ij} \partial_{x_i x_j}^2 u + \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} u - \partial_t u,$$

where $(x, t) \in \mathbb{R}^{n+1}$, $(a_{ij})_{i,j=1}^m$ is a positive definite matrix, $(b_{ij})_{i,j=1}^n$ is a constant coefficients matrix with a suitable upper triangular structure. It is clear that $\mathcal{L}_1$ is a heat operator, when $m = n$, $(b_{ij})_{i,j=1}^n = (0)_{i,j=1}^n$.

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For more details see [2, 2].

-  Lanconelli, E., Pascucci, A., Polidoro, S.: Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance. in: Nonlinear Problems in Mathematical Physics and Related Topics, II, Kluwer/Plenum, New York, 243-265 (2002)
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The operator $\mathcal{L}_1$ arises in many research fields, for instance, stochastic processes and kinetic models [1, 2, 3],

 Chandrasekhar, S.: Stochastic problems in physics and astronomy. Rev. Modern Phys. **15**, 1-89 (1943)

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mathematical finance theory [1, 2, 3] etc.

 Barucci, E., Polidoro, S., Vespri, V.: Some results on partial differential equations and Asian options. Math. Models Methods Appl. Sci. **11**, 475-497 (2001)

 Lanconelli, E., Pascucci, A., Polidoro, S.: Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance. in: Nonlinear Problems in Mathematical Physics and Related Topics, II, Kluwer/Plenum, New York, 243-265 (2002)

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Since $\mathcal{L}_1$ owns a homogeneous fundamental solution with good properties, many authors still pay attention to it up to now [1, 2, 3].

 Bramanti, M., Cerutti, M.C.: L^p estimates for some ultraparabolic operators with discontinuous coefficients. J. Math. Anal. Appl. **200**, 332-354 (1996)

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In addition, other examples of (1) can see in [1, 2].

 Bramanti, M., Brandolini, L.: L^p estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups. Rend. Semin. Mat. Univ. Politec. Torino **58**, 389-433 (2000)

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Morrey spaces and their properties play an important role in the study of local behavior of solutions to elliptic partial differential equations, refer to [1, 2]. In [3, 4] the authors showed the boundedness in Morrey spaces for some important operators in harmonic analysis such as Hardy-Littlewood operators, Calderón-Zygmund singular integral operators and fractional integral operators.

 Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. **43**, 126-166 (1938)

 Peetre, J.: On the theory of space. J. Funct. Anal. **4**, 71-87 (1969)

 Adams, D.R.: A note on Riesz potentials. Duke Math. J. **42**, 765-778 (1975)

 Chiarenza, F., Frasca, M.: Morrey spaces and Hardy-Littlewood maximal function. Rend. Mat. Appl. **7**, 273-279 (1987)

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Moreover, various Morrey spaces are defined in the process of study. In [1, 2, 3] the authors introduced and studied the boundedness of the classical operators in generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$

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 Nakai, E. : Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. Math. Nachr. **166**, 95-103 (1994)

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(see, also [1, 2, 3]) and etc.

 Guliyev, V.S. : Function spaces, integral operators and two weighted inequalities on homogeneous groups, some applications. (Russian) Baku: Casioglu; p. 332. (1999)

 Guliyev, V.S. : Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal Appl. Art. ID 503948: 1-20 (2009)

 Sawano, Y. : A thought on generalized Morrey spaces. J. Indonesian Math. Soc. **25**(3), 210-281 (2019)

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Recently, Komori and Shirai [1] defined the weighted Morrey spaces $L_w^{p,\kappa}(\mathbb{R}^n)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev in [2] first introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}(\mathbb{R}^n)$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces. Note that, Guliyev [2] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M_w^{p,\varphi}(\mathbb{R}^n)$ and $L_w^{p,\kappa}(\mathbb{R}^n)$.



Komori, Y., Shirai, S.: Weighted Morrey spaces and a singular integral operator. Math. Nachr. **282**(2), 219-231 (2009)



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In this talk motivated by these articles, we will establish the boundedness of sublinear integral operators on generalized weighted Morrey spaces in the framework of homogeneous groups. The sublinear operators under consideration contain integral operators of harmonic analysis such as Hardy-Littlewood and fractional maximal operators, Calderón-Zygmund operators, potential operators on homogeneous groups, etc. Homogeneous groups include the Euclidean space, the Heisenberg group, the Carnot group, see [1, 2].

 Bonfiglioli, A., Lanconelli, E., Uguzzoni, F. : Stratified Lie groups and potential theory for their sub-Laplacians. Springer Monographs in Mathematics. Springer, Berlin (2007)

 Capogna, L., Danielli, D., Pauls, S.D., Tyson, J.T.: An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem. Progress in Mathematics, vol. 259, Birkhauser-Verlag, Basel (2007)

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Furthermore, applications to generalized weighted Sobolev - Morrey estimates for hypoelliptic operators with drift on homogeneous groups are given. Also, generalized weighted Morrey estimates for the sublinear operators generated by fractional integral operators on the homogeneous group and an application are obtained. Recall that the local Morrey-type space was introduced and proved the boundedness in this spaces of the fractional integral operator and singular integral operators defined on homogeneous Lie groups by author in [1], see also [2].



Guliyev, V.S. : Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$, [Russian Doctor's degree dissertation]. Moscow: Mat. Inst. Steklov; p. 329 (1994)



Guliyev, V.S. : Function spaces, integral operators and two weighted inequalities on homogeneous groups, some applications. (Russian) Baku: Casioglu; p. 332. (1999)

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We now recall some basic notions concerning homogeneous Lie groups.
Given a pair of smooth mappings

$$[(x, y) \rightarrow x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N; \quad [x \mapsto x^{-1}] : \mathbb{R}^N \mapsto \mathbb{R}^N,$$

the space $\mathbb{R}^N$ with these mappings forms a group, in which the identity is the origin. If there exist $0 < w_1 \leq w_2 \leq \dots \leq w_N$, such that the dilations

$$\delta_\lambda : (x_1, \dots, x_N) \mapsto (\lambda^{w_1} x_1, \dots, \lambda^{w_N} x_N), \quad \lambda > 0$$

are group automorphisms, then the space $\mathbb{R}^N$ with this structure is called a homogeneous group, denoted by $\mathbb{G}$. A homogeneous norm $\|\cdot\|$ on $\mathbb{G}$ is defined in the following way: if for any $x \in \mathbb{G}$, $x \neq 0$, it holds

$$\|x\| = \rho \Leftrightarrow |\delta_{1/\rho} x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm; also, let $\|0\| = 0$.

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It is not difficult to verify that the homogeneous norm satisfies

1. $\|\delta_\lambda x\| = \lambda \|x\|$ for every $x \in \mathbb{G}$, $\lambda > 0$;
2. there exists $c_0 \equiv c(\mathbb{G}) \geq 1$, such that for every $x, y \in \mathbb{G}$,

$$\|x^{-1}\| \leq c_0 \|x\| \quad \text{and} \quad \|x \circ y\| \leq c_0 (\|x\| + \|y\|). \quad (2)$$

In view of the above properties, it is natural to define the quasi distance d :

$$d(x, y) = \|x \circ y^{-1}\|.$$

The ball with respect to d is defined by

$B(x, r) \equiv B_r(x) = \{y \in \mathbb{G} : d(x, y) < r\}$. Note that $B(0, r) = \delta_r B(0, 1)$, therefore

$$|B(x, r)| = r^Q |B(0, 1)|, \quad x \in \mathbb{G}, r > 0,$$

where

$$Q = w_1 + \dots + w_N.$$

Integral operators on homogeneous groups

Let $f \in L^1_{\text{loc}}(\mathbb{G})$. The maximal operator M and the fractional maximal operator M_α are defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x, r)|^{-1+\alpha/Q} \int_{B(x, r)} |f(y)| dy, \quad 0 \leq \alpha < Q.$$

If $\alpha = 0$, then $M_0 \equiv M$ is the Hardy-Littlewood maximal operator. In the talk we consider fractional integral operator

$$I_\alpha f(x) = \int_{\mathbb{G}} \|y^{-1}x\|^{\alpha-Q} f(y) dy, \quad 0 < \alpha < Q$$

in generalized weighted Morrey spaces.

Integral operators on homogeneous groups

For any $f \in L_1^{\text{loc}}(\mathbb{G})$, we say that T is a Calderón-Zygmund operator on $\mathbb{G}$ if

$$Tf(x) = \lim_{\varepsilon \rightarrow 0} \int_{\{y \in \mathbb{G}: \|x \circ y^{-1}\| > \varepsilon\}} K(x \circ y^{-1}) f(y) dy = V.P. \int_{\mathbb{G}} K(x \circ y^{-1}) f(y) dy,$$

where K satisfies

$$|K(x)| \leq \frac{c}{\|x\|^Q}; \quad |\nabla K(x)| \leq \frac{c}{\|x\|^{Q+1}}, \quad x \neq 0.$$

Suppose that T_α , $\alpha \in [0, Q)$ represents a linear or a sublinear operator, which satisfies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$, the inequality

$$|T_\alpha f(x)| \leq c_1 \int_{\mathbb{G}} \frac{|f(y)|}{\|x \circ y^{-1}\|^{Q-\alpha}} dy, \quad (3)$$

where c_1 is independent of f and x .

Integral operators on homogeneous groups

We point out that the condition (3) with $\alpha = 0$ was first introduced by Soria and Weiss in [1] in the case $\mathbb{G} = \mathbb{R}^n$. Condition (3) is satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operator, Carleson’s maximal operator, Hardy–Littlewood maximal operators, fractional maximal operator, C. Fefferman’s singular multipliers, R. Fefferman’s singular integrals, Riesz potentials, Ricci–Stein’s oscillatory singular integrals, Bochner–Riesz means and so on (see [2, 3, 1] for details).

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Integral operators on homogeneous groups

Note that, the maximal operator M , and the Calderón-Zygmund operator T satisfy the condition (3) with $\alpha = 0$, and the fractional maximal operator M_α , and the fractional integral operator I_α satisfy the condition (3) with $0 < \alpha < Q$.

Let $0 < \alpha < Q$, $1 \leq p < \frac{Q}{\alpha}$ and $f \in L_p(\mathbb{G})$. Then the integral $I_\alpha f(x)$ converges absolutely for almost every $x \in \mathbb{G}$, see [2, Theorem 3.2.1]. The Hardy-Littlewood-Sobolev result states that (see [2, Theorem 3.2.1]) the operator I_α is bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ if and only if $1 < p < q < \infty$ and $\alpha = Q/p - Q/q$. Also I_α is bounded from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$ if and only if $1 < q < \infty$ and $\alpha = Q - Q/q$.

 Guliyev, V.S. : Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$, [Russian Doctor's degree dissertation]. Moscow: Mat. Inst. Steklov; p. 329 (1994)

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Function Spaces Under Consideration

We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

Generalized weighted Morrey spaces

Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{G} \times (0, \infty)$ and w be non-negative measurable function on $\mathbb{G}$. We denote by $M_{p,\varphi}(\mathbb{G}, w)$ the generalized weighted Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{G}, w)$ with finite norm

$$\|f\|_{M_{p,\varphi}(\mathbb{G}, w)} = \sup_{x \in \mathbb{G}, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{L_p(B(x, r), w)},$$

Also the weak generalized weighted Morrey space $WM_{p,\varphi}(\mathbb{G}, w)$ is defined of all functions $f \in L_p^{\text{loc}}(\mathbb{G}, w)$ by the finite norm

$$\|f\|_{WM_{p,\varphi}(\mathbb{G}, w)} = \sup_{x \in \mathbb{G}, r > 0} \varphi(x, r)^{-1} w(B(x, r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x, r), w)} < \infty,$$

Function Spaces Under Consideration

Remark

- (1) If $w \equiv 1$, then $M_{p,\varphi}(\mathbb{G}, 1) = M_{p,\varphi}(\mathbb{G})$ is the generalized Morrey space and $WM_{p,\varphi}(\mathbb{G}, 1) = WM_{p,\varphi}(\mathbb{G})$ is the weak generalized Morrey space.
- (2) If $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\kappa}(\mathbb{G}, w)$ is the weighted Morrey space and $WM_{p,\varphi}(\mathbb{G}, w) = WL_{p,\kappa}(\mathbb{G}, w)$ is the weak weighted Morrey space.
- (3) If $\varphi(x, r) \equiv v(B(x, r))^{\frac{\kappa}{p}} w(B(x, r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\kappa}(\mathbb{G}, v, w)$ is the two weighted Morrey space.
- (4) If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{\lambda-Q}{p}}$ with $0 < \lambda < Q$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\lambda}(\mathbb{G})$ is the classical Morrey space and $WM_{p,\varphi}(\mathbb{G}, w) = WL_{p,\lambda}(\mathbb{G})$ is the weak Morrey space.
- (5) If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\mathbb{G}, w) = L_p(\mathbb{G}, w)$ is the weighted Lebesgue space and $WM_{p,\varphi}(\mathbb{G}, w) = WL_p(\mathbb{G}, w)$ is the weak weighted Lebesgue space.

Function Spaces Under Consideration

We use the following simplified notation later:

$$\|Du\|_{M_{p,\varphi}(\mathbb{G},w)} = \sum_{i=1}^m \|X_i u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

$$\|D^2 u\|_{M_{p,\varphi}(\mathbb{G},w)} = \sum_{i,j=1}^m \|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|X_0 u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

and generally,

$$\|D^k u\|_{M_{p,\varphi}(\mathbb{G},w)} = \sum \|X_{ji} \dots X_{jl} u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

where $X_{ji} \dots X_{jl}$ is homogeneous of degree k (let us note that X_0 is homogeneous of degree two while the remaining $X_1, \dots, X_m$ are homogeneous of degree one).

Function Spaces Under Consideration

Definition

For $p \in [1, \infty)$, a nonnegative integer k and a weight w , the weighted Sobolev-Morrey space $S_{k,p,\varphi}(\mathbb{G}, w)$ consists of all $M_{p,\varphi}(\mathbb{G}, w)$ functions such that

$$\|u\|_{S_{k,p,\varphi}(\mathbb{G}, w)} = \sum_{h=0}^k \|D^h u\|_{M_{p,\varphi}(\mathbb{G}, w)}$$

is finite.

The space $S_{k,p,\varphi}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$ consists of all functions $u \in S_{k,p}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$ with $D^h u \in M_{p,\varphi}(\mathbb{G}, w)$, and is endowed by the same norm. Recall that $S_p^{1,0}(\mathbb{G}, w)$ is the closure of $C_0^\infty(\mathbb{G})$ with respect to the norm in $S_p^1(\mathbb{G}, w)$.

Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups

Theorem

(Generalized weighted Sobolev-Morrey estimate). Let $1 < p < \infty$, $Q > 4$, $w \in A_p(\mathbb{G})$ and φ satisfy the condition

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi(x, s) w(B(x, s))^{\frac{1}{p}}}{w(B(x, t))^{\frac{1}{p}}} \frac{dt}{t} \lesssim \varphi(x, r). \quad (4)$$

Let also $u \in S_{2,p,\varphi}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$. Then there exists a constant $C > 0$ such that

$$\|u\|_{S_{2,p,\varphi}(\mathbb{G}, w)} \leq C \left(\|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G}, w)} + \|u\|_{M_{p,\varphi}(\mathbb{G}, w)} \right), \quad (5)$$

where $A_p(\mathbb{G})$ is Muckenhoupt class.

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$$\begin{aligned}\|u\|_{S_{1,p}(\mathbb{G},w)} &= \|u\|_{L_p(\mathbb{G},w)} + \sum_{i=1}^m \|X_i u\|_{L_p(\mathbb{G},w)}, \\ \|u\|_{S_{2,p,\varphi}(\mathbb{G},w)} &= \|u\|_{M_{p,\varphi}(\mathbb{G},w)} + \sum_{i=1}^m \|X_i u\|_{M_{p,\varphi}(\mathbb{G},w)} \\ &\quad + \sum_{i,j=1}^m \|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|X_0 u\|_{M_{p,\varphi}(\mathbb{G},w)}.\end{aligned}$$

The space $S_{2,p,\varphi}(\mathbb{G},w) \cap S_{1,p}^0(\mathbb{G},w)$ consists of all functions $u \in S_{2,p}(\mathbb{G},w) \cap S_{1,p}^0(\mathbb{G},w)$ with $D^2 u \in M_{p,\varphi}(\mathbb{G},w)$, and is endowed by the same norm (see Definition 1). Recall that $S_{1,p}^0(\mathbb{G},w)$ is the closure of $C_0^\infty(\mathbb{G})$ with respect to the norm in $S_{1,p}(\mathbb{G},w)$.

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VG, *Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups. Journal of Mathematical Inequalities* **16** (1), 219-245 (2022). (IF 1.225) **Q2**

Remark

For a non-negative measurable function w defined on $(0, \infty)$, we denote by $\mathcal{G}_w^p$ the set of all almost decreasing functions $\varphi : \mathbb{G} \times (0, \infty) \rightarrow (0, \infty)$ such that

$$\inf_{D \in \mathcal{B}: r_D \leq r_{B_0}} \varphi(D) \gtrsim \varphi(B_0) \text{ for all } B_0 \in \mathcal{B}$$

and

$$\inf_{B \in \mathcal{B}: r_B \geq r_{B_0}} \varphi(B) w^p(B)^{\frac{1}{p}} \gtrsim \varphi(B_0) w^p(B_0)^{\frac{1}{p}},$$

where r_B and r_{B_0} denote the radius of the d -balls B and B_0 , respectively.

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Remark

Then for $\varphi \in \mathcal{G}_p$ the condition (4) stays the following form

$$\int_r^\infty \varphi(t) \frac{dt}{t} \leq C \varphi(r), \quad (6)$$

where C does not depend on r . For the nontriviality of generalized Morrey spaces $M_{p,\varphi}(\mathbb{G})$ we assumed in Theorem 2 and in the sequel that $\varphi \in \Omega_p$.

Note that the condition (4) is weaker than (6).

Corollary

Let $1 < p < \infty$, $Q > 4$, $w \in A_p(\mathbb{G})$ and $\varphi \in \mathcal{G}_w^p$ satisfy the condition (6). If $u \in S_{2,p,\varphi}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$, then inequality (5) is valid.

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If in Theorem 2 take $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$ with $0 < \kappa < 1$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\kappa}(\mathbb{G}, w)$ is the weighted Morrey space and we get the following corollary, which were proved in

 Hou, Y., Niu, P.C.: Weighted Sobolev-Morrey estimates for hypoelliptic operators with drift on homogeneous group. **J. Math. Anal. Appl.** **428**, 1319-1338 (2015).

Corollary

[1] If $1 < p < \infty$, $Q > 4$, $w \in A_p$, and $0 < k < 1$. If $u \in S_{2,p,\kappa}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$, then there exists a constant $C > 0$ such that

$$\|u\|_{S_{2,p,\kappa}(\mathbb{G}, w)} \leq C \left(\|\mathcal{L}u\|_{L_{p,\kappa}(\mathbb{G}, w)} + \|u\|_{L_{p,\kappa}(\mathbb{G}, w)} \right),$$

where $\|u\|_{S_{2,p,\kappa}(\mathbb{G}, w)} = \|u\|_{L_{p,\kappa}(\mathbb{G}, w)} + \sum_{i=1}^m \|X_i u\|_{L_{p,\kappa}(\mathbb{G}, w)} + \sum_{i,j=1}^m \|X_i X_j u\|_{L_{p,\kappa}(\mathbb{G}, w)} + \|X_0 u\|_{L_{p,\kappa}(\mathbb{G}, w)}$.

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Theorem

(Higher order generalized weighted Sobolev-Morrey estimate). Let $1 < p < \infty$, $Q > 4$, $w \in A_p(\mathbb{G})$ and φ satisfy the condition (4). If k is a positive integer and $u \in S_{2k,p,\varphi}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$, then there exists a constant $C > 0$ such that

$$\|u\|_{S_{2k+2,p,\varphi}(\mathbb{G}, w)} \leq C \left(\|\mathcal{L}u\|_{S_{2k,p,\varphi}(\mathbb{G}, w)} + \|u\|_{M_{p,\varphi}(\mathbb{G}, w)} \right), \quad (7)$$

where $\|u\|_{S_{2k,p,\varphi}(\mathbb{G}, w)} = \sum_{h=1}^{2k} \|D^h u\|_{M_{p,\varphi}(\mathbb{G}, w)}$ (see Definition 1),

$$\|D^h u\|_{M_{p,\varphi}(\mathbb{G}, w)} = \sum \|X_{j_1} \dots X_{j_l} u\|_{M_{p,\varphi}(\mathbb{G}, w)},$$

where $X_{j_1} \dots X_{j_l}$ is homogeneous of degree h (let us note that X_0 is homogeneous of degree two while the remaining $X_1, \dots, X_m$ are homogeneous of degree one).

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Corollary

Let $1 < p < \infty$, $Q > 4$, $w \in A_p(\mathbb{G})$, k be a positive integer and $\varphi \in \mathcal{G}_w^p$ satisfy the condition (6). If $u \in S_{2k,p,\varphi}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$, then inequality (7) is valid.

Corollary

(Higher order weighted Sobolev-Morrey estimate) [1] Let $1 < p < \infty$, $Q > 4$, $w \in A_p$, $0 < k < 1$ and k be a positive integer. If $u \in S_{2,p,\kappa}(\mathbb{G}, w) \cap S_{1,p}^0(\mathbb{G}, w)$, then there exists a constant $C > 0$ such that

$$\|u\|_{S_{2k+2,p,\kappa}(\mathbb{G},w)} \leq C \left(\|\mathcal{L}u\|_{S_{2k,p,\kappa}(\mathbb{G},w)} + \|u\|_{L_{p,\kappa}(\mathbb{G},w)} \right).$$

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Theorem

(Generalized weighted Morrey estimate). Let $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, $w \in A_{p,q}(\mathbb{G})$, and (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) w^p(B(x, s))^{\frac{1}{p}}}{w^q(B(x, t))^{\frac{1}{q}}} \frac{dt}{t} \lesssim \varphi_2(x, r). \quad (8)$$

Then there exists a constant $C > 0$ such that for every $\mathcal{L}u \in M_{p,\varphi_1}(\mathbb{G}, w^p)$, we have

$$\|X_i u\|_{M_{q,\varphi_2}(\mathbb{G}, w^q)} \leq C \|\mathcal{L}u\|_{M_{p,\varphi_1}(\mathbb{G}, w^p)}, \quad i = 1, 2, \dots, m.$$



VG, Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups. **Journal of Mathematical Inequalities** 16 (1), 219-245 (2022). (IF 1.225) **Q2**

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If in Theorem 8 take $\varphi_1 \equiv \varphi \in \mathcal{G}_w^p$, $\varphi_2(r) = r\varphi(r)$, then we get the following new corollary.

Corollary

Let $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, $w \in A_{p,q}(\mathbb{G})$, and $\varphi \in \mathcal{G}_w^p$ satisfy the condition

$$\int_r^\infty \varphi(t) dt \lesssim r \varphi(r). \quad (9)$$

Then there exists a constant $C > 0$ such that for every $\mathcal{L}u \in M_{p,\varphi}(\mathbb{G}, w^p)$, we have

$$\|X_i u\|_{M_{q,r\varphi(r)}(\mathbb{G}, w^q)} \leq C \|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G}, w^p)}, \quad i = 1, 2, \dots, m.$$

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If in Theorem 8 take $\varphi_1 \equiv \varphi \in \mathcal{G}_w^p$, $\varphi_2(r) = r\varphi(r)$, then we get the following new corollary.

Corollary

(Morrey estimate with two weights). [1] If $1 < p < q < \infty$, $1/q = 1/p - 1/Q$, $w \in A_{p,q}(\mathbb{G})$, and $0 < \kappa < p/q$, then there exists a constant $C > 0$ such that for every $\mathcal{L}u \in L_{p,\kappa}(\mathbb{G}, w^p, w^q)$, we have

$$\|X_i u\|_{L_{q,\kappa q/p}(\mathbb{G}, w^q)} \leq C \|\mathcal{L}u\|_{L_{p,\kappa}(\mathbb{G}, w^p, w^q)}, \quad i = 1, 2, \dots, m.$$

 **VG**, Generalized weighted Sobolev – Morrey estimates for hypoelliptic operators on homogeneous groups. **Journal of Mathematical Inequalities** **16** (1), 219-245 (2022). (IF 1.225) **Q2**

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THANKS A LOT FOR YOUR ATTENTION

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