

Workshop on Operator Theory in
Function Spaces and Their Applications
Dedicated to the Anniversary of
Professor VICTOR I. BURENKOV

BIRKHOFF–JAMES ORTHOGONALITY IN LATTICE–NORMED MODULES

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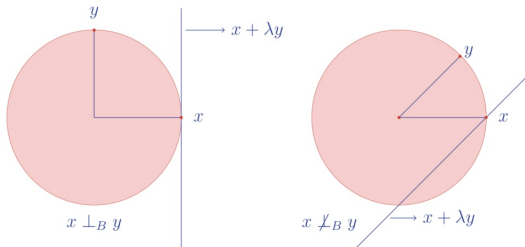
To Victor Burenkov on occasion of his 85th anniversary



with profound respect for his outstanding scientific
achievements and exceptional personal qualities

Birkhoff-James orthogonality

- **Definition.** Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. The members $x, y \in X$ are **orthogonal** if $\langle x, y \rangle = 0$ ($\iff \|x + \lambda y\| \geq \|x\|, \lambda \in \mathbb{F}$).

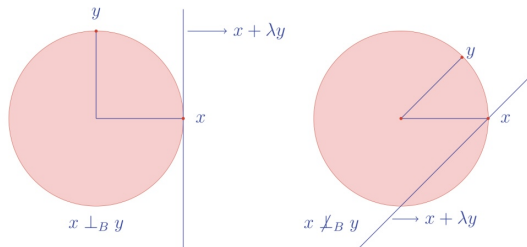


- **Definition** (Birkhoff, 1935). In an arbitrary normed space $(X, \|\cdot\|)$ over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, **Birkhoff-James orthogonality** $\perp^{\text{BJ}}$ (BJ-orthogonality) is defined as follows: given $x, y \in X$, we put

$$x \perp^{\text{BJ}} y \iff \|x + \lambda y\| \geq \|x\| \text{ for all } \lambda \in \mathbb{F}.$$

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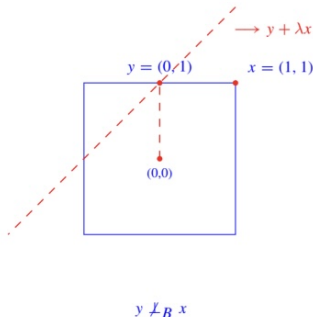
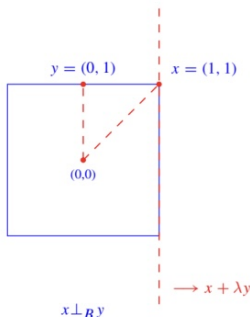
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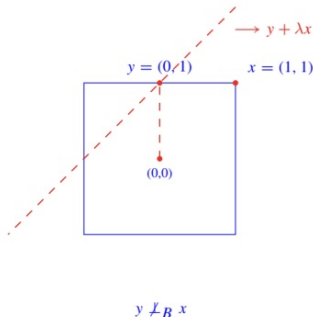
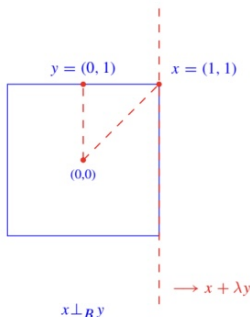
$$x \perp^{\text{BJ}} y \stackrel{\text{df}}{\iff} \|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{F}.$$

Birkhoff-James Orthogonality is not Symmetric!



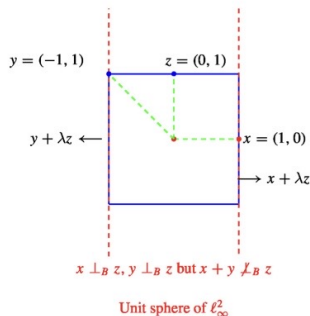
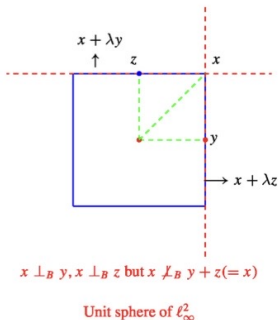
- **Theorem** (Day-James, 1947.) For a real normed space X of dimension ≥ 3 the following assertions are equivalent:
 - (2) $x \perp^{\text{BJ}} y \iff y \perp^{\text{BJ}} x$ for all $x, y \in X$ (i.e., $\perp^{\text{BJ}}$ is symmetric).
 - (2) $\|x\|^2 = \langle x, x \rangle$ ($x \in X$) for some inner product $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}$.

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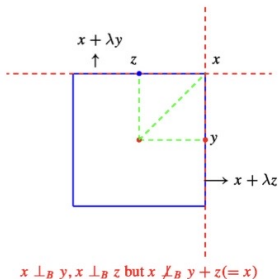
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Birkhoff-James Orthogonality is not Additive!

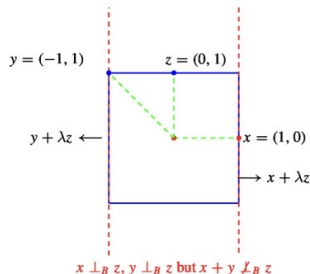


- **Theorem.** A normed space X , with $\dim(X) \geq 3$, is an inner product space if and only if BJ orthogonality is additive on the left in X .
- **Theorem.** A normed space X is smooth if and only if Birkhoff-James orthogonality is additive on the right in X .

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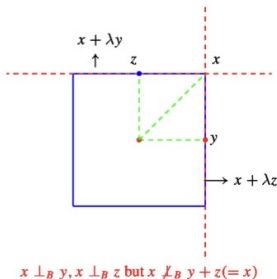
Unit sphere of ℓ_∞^2



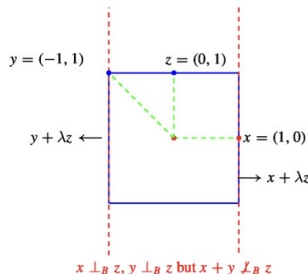
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Birkhoff–James orthogonality: Major Research Themes

- **Geometry of Banach Spaces:** Left- and right-symmetry points of $\perp_B$ are often used to characterize smooth points, extreme points, etc.
- **Operator Theory:** Studies extend $\perp_B$ to bounded linear operators, examining when operators achieve their norms.
- **Approximate Orthogonality:** Exact orthogonality is relaxed into ε -approximate BJ orthogonality.
- **Preserver Problems:** Operators preserving Birkhoff–James (ε -)orthogonality often induce structural isomorphisms or isometric isometries between spaces under consideration.
- **Stability of Mappings:** To study whether a map that approximately preserves orthogonality is close to an exact isometry.

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Birkhoff-James Orthogonality: References

There is an extensive literature devoted to Birkhoff–James orthogonality. We mention some of these works: a book¹, book chapters^{2,3}, survey articles⁴, articles on orthogonality in modules⁵, articles on preservers^{6,7}.

¹A. Mal, K. Paul, D. Sain, Birkhoff-James Orthogonality and Geometry of Operator Spaces, Springer Nature, 2024.

²J. Alonso, H. Martini, and S. Wu, Orthogonality Types in Normed Linear Spaces, In: Surveys in Geometry I (A. Papadopoulos Ed.), Springer, Cham (2022), 97-170.

³P. Grover and S. Singla, Birkhoff–James Orthogonality and Applications: A Survey, In: Bastos, M. A., Castro, L., Karlovich, A.Y. (eds) Operator Theory, Functional Analysis and Applications, Birkhäuser, (2021), 293-315.

⁴Chmieliński, J., Orthogonality preserving property and its Ulam stability, In: Rassias, Th.M., Brzdek, J. (eds.) Functional Equations in Mathematical Analysis, 33-58, (2012).

⁵L. Arambašić and R. Rajić, A strong version of the Birkhoff–James orthogonality in Hilbert C^* -modules, Ann. Funct. Anal., **5**(1) (2014), 109-120.

⁶A. Koldobsky, Operators preserving orthogonality are isometries, Proc. Royal. Soc. Edinb. A123 (1993), 835-837.

⁷A. Blanco and A. Turnšek, On maps that preserve orthogonality in normed spaces, Proc. of the Royal Soc. of Edinburgh, 136A, 709-716, 2006.

Orthogonality Preserving Operators

- **Defintion.** A mapping $T : X \rightarrow Y$ between Banach spaces is said to be **orthogonality preserving** if for all $x_1, x_2 \in X$ the implication holds:

$$x_1 \perp^{\text{BJ}} x_2 \implies T(x_1) \perp^{\text{BJ}} T(x_2).$$

- One of the most important achievements of the isometric theory of Banach spaces is the following fundamental result.
- **Theorem** (Koldobsky–Blanco–Turnšek). Let X and Y be real or complex normed spaces. For an arbitrary linear operator $T : X \rightarrow Y$ the following are equivalent:
 - (1) T is BJ-orthogonality preserver.
 - (2) $T = kU$, where $k \in \mathbb{F}$, $U \in L(X, Y)$ and $\|Ux\| = \|x\|$ for all $x \in X$, i.e. T is a scalar multiple of a linear isometry.

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Birkhoff-James Approximate Orthogonality

- **Definition**⁸. Given $0 \leq \varepsilon < 1$ and $x, y \in X$, we say that x is ε -Birkhoff-James orthogonal to y and write $x \perp_{\varepsilon}^{\text{BJ}} y$ if

$$\|x + \gamma y\| \geq \|x\| - 2\varepsilon \|\gamma y\| \quad \text{for all } \gamma \in \mathbb{F}.$$

- **Theorem.** (Chmieliński–Stypka). Let X and Y be real normed spaces and let $0 \leq \varepsilon < 1$. If $T : X \rightarrow Y$ is additive and $x \perp_{\varepsilon}^{\text{BJ}} y$ implies $T(x) \perp_{\varepsilon}^{\text{BJ}} T(y)$, then T is a linear and bounded operator, as well as:

$$(1 - 8\varepsilon) \|T\| \|x\| \leq \|Tx\| \leq \|T\| \|x\| \quad (x \in X).$$

⁸Chmieliński, J., On an ε -Birkhoff orthogonality, J. Inequal. Pure and Appl. Math., 6:3 (2005), Article 79.

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Birkhoff-James Approximate Orthogonality in Modules

- The aim of the present study is to extend the theory of Birkhoff–James orthogonality to modules over Stone algebras.
Banach-Kantorovich modules and **Kaplansky-Hilber modules**.
- By definition, a **Stone algebras** is a commutative AW^* -algebra.
- Every Stone algebras Λ is linearly isometric and order isomorphic to the space of continuous functions $C(Q, \mathbb{R})$ or $C(Q, \mathbb{C})$ on some extremally discontinuous compact space Q (Kreĭns-Kakutani Theorem).
- **Example.** $L^\infty(\Omega, \Sigma, \mu)$ with σ -finite or localizable measure space.
- **Boolean Algebras:** The projections, self-adjoint idempotents, $(p = p^* = p^2)$ of a Stone algebra Λ form a complete BA $\mathbb{B} = \mathbb{B}(\Lambda)$.
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Kantorovich Heuristic

- Kantorovich indicated⁹ an important class of ordered vector spaces, Dedekind complete vector lattices, often called **Kantorovich spaces**.
- Kantorovich wrote in this note: *"In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals"*.
- Following this heuristic principle, in 1936 Kantorovich introduced the concept of lattice-normed space; **Banach-Kantorovich modules** constitute the most important class of lattice-normed spaces¹⁰.

⁹Kantorovich L. V., On semiordered linear spaces and their applications to the theory of linear operations // Dokl. Akad. Nauk SSSR, 4:1–2 (1935), 11-14.

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Banach-Kantorovich Spaces

- **Definition** A mapping $|\cdot| : X \rightarrow E_+$ is a **vector** (**E -valued**) *norm* if it satisfies the following conditions:

- (1) $|x| = 0$ if and only if $x = 0$ ($x \in X$);
- (2) $|\lambda x| = |\lambda||x|$ for all $\lambda \in \mathbb{R}(\mathbb{C})$ and $x \in X$;
- (3) $|x + y| \leq |x| + |y|$ for all $x, y \in X$.

A vector norm is called *decomposable* if

- (4) given $e_1, e_2 \in E_+$ and $x \in X$ such that $|x| = e_1 + e_2$, there are $x_1, x_2 \in X$ with $x = x_1 + x_2$ and $|x_k| = e_k$ ($k := 1, 2$).

- **Definition** A decomposable *bo*-complete lattice-normed space is called a **Banach-Kantorovich space**. (***bo*-completeness** means that for any net (x_α) such that $(|x_\alpha - x_\beta|)$ *o*-converges to zero in E , there exists an element $x \in X$ for which $(|x_\alpha - x|)$ *o*-converges to zero in E .)

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Kaplansky Heuristic

- The concept of a **Kaplansky-Hilbert module** was introduced by Kaplansky in¹¹ under the name AW^* -module. In this paper he wrote:
- *“... the new idea is to generalize Hilbert space by allowing the inner product to take values in a more general ring than the complex numbers. After the appropriate preliminary theory of these AW^* -modules has been developed, one can operate with a general AW^* -algebra of type I in almost the same manner as with the factor.”*
- In other words, a Kaplansky-Hilbert module H over a Stone algebra Λ can be taken as a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$, AW^* -algebra of bounded Λ -linear operators $\mathcal{L}_\Lambda(H)$ can be taken as the space of bounded linear operators $\mathcal{L}(\mathcal{H})$. Needless to say, this is a version of Kantorovich's heuristic principle.

¹¹Kaplansky I., Modules over operator algebras, Amer. J. Math., **75** (1953):4, 839-858.

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- *“... the new idea is to generalize Hilbert space by allowing the inner product to take values in a more general ring than the complex numbers. After the appropriate preliminary theory of these AW^* -modules has been developed, one can operate with a general AW^* -algebra of type I in almost the same manner as with the factor.”*
- In other words, a Kaplansky-Hilbert module H over a Stone algebra Λ can be taken as a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$, AW^* -algebra of bounded Λ -linear operators $\mathcal{L}_\Lambda(H)$ can be taken as the space of bounded linear operators $\mathcal{L}(\mathcal{H})$. Needless to say, this is a version of Kantorovich's heuristic principle.

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Kaplansky-Hilbert Module

- **Definition** Let A be a Stone algebra ($A = C(Q, \mathbb{C})$) and let X be a unital A -module. The mapping $\langle \cdot | \cdot \rangle : X \times X \rightarrow A$ is an **A -valued inner product**, if for all $x, y, z \in X$ and $a \in A$ the following hold:
 - (1) $\langle x | x \rangle \geq \mathbf{0}$; $\langle x | x \rangle = \mathbf{0} \Leftrightarrow x = \mathbf{0}$;
 - (2) $\langle x | y \rangle = \langle y | x \rangle^*$;
 - (3) $\langle ax | y \rangle = a \langle x | y \rangle$;
 - (4) $\langle x + y | z \rangle = \langle x | z \rangle + \langle y | z \rangle$.
- **Definition** Say that $(X, \langle \cdot | \cdot \rangle)$ is a **Kaplansky-Hilbert module over Λ** if $(X, |\cdot|)$ is a Banach-Kantorovich space with respect to the Λ -valued norm defined as

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$\mathbb{B}$ -Orthogonality in Banach-Kantorovich Modules

- **Definition.** Let X be a Banach-Kantorovich Λ -module over E and $x, y \in X$. Say that x is **$\mathbb{B}$ -orthogonal** to y ($\equiv x \perp^{\mathbb{B}} y$) if

$$|x| \leq |x + \lambda y| \quad \text{for all } \lambda \in \Lambda.$$

The relation $x \perp^{\mathbb{B}} y$ is said to be **symmetric** if $x \perp^{\mathbb{B}} y \iff y \perp^{\mathbb{B}} x$.

- **Lemma.** Let $(E, \|\cdot\|_E)$ be an order complete Banach lattice and define the mixed norm $\|x\| := \| |x| \|_E$ ($x \in X$). The following are equivalent:
 - (1) x is $\mathbb{B}$ -orthogonal to y ;
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Applications of $\mathbb{B}$ -Orthogonality

- **Definition.** Say that $x, y \in X$ are **orthogonal** ($x \perp^{\mathbb{B}} y$) if $\langle x|y \rangle = 0$.
- **Theorem.** Assume that Λ is a Stone algebra and $\mathbb{B} = \mathbb{P}(\Lambda)$.
Let $(X, |\cdot|)$ be a real Banach-Kantorovich Λ -module of $\mathbb{B}$ -dimension at least three over Λ . Then the following are equivalent:
 - (1) $\mathbb{B}$ -orthogonality is symmetric in X .
 - (2) $|x|^2 = \langle x|x \rangle$ ($x \in X$) for some A -valued inner product $\langle \cdot | \cdot \rangle : X \times X \rightarrow A$ (i.e. X is a Kaplansky-Hilbert the module).

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- **Definition.** A mapping $T : X \rightarrow Y$ between lattice normed modules is said to be **$\mathbb{B}$ -orthogonality preserving** if for all $x_1, x_2 \in X$ we have

$$x_1 \perp^{\mathbb{B}} x_2 \implies T(x_1) \perp^{\mathbb{B}} T(x_2).$$

- **Theorem.** Let X and Y be real or complex Banach Kantorovich Λ -modules over E and $\Lambda = \mathcal{Z}(E)$. A Λ -linear operator $T : X \rightarrow Y$ is $\mathbb{B}$ -orthogonality preserving if and only the following hold:
 - (1) $T = k \circ U$ for some $k \in \text{Orth}(\mathcal{I}|X|)_+$ and $U \in \mathbb{L}_{\Lambda}(X, Y)$;
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- **Theorem.** Let H and K be two Hilbert spaces and let $S, T \in \mathcal{L}(H, K)$. Then $S \perp^{\mathbb{B}J} T$ if and only if there exists a state $\phi : \mathcal{L}(H) \rightarrow \mathbb{C}$ such that $\phi(S^*S) = \|S\|^2$ and $\phi(S^*T) = 0$.
- **Definition.** A mapping $\Phi : \mathcal{L}_\Lambda(H) \rightarrow \Lambda$ is said to be a Λ -valued state of $\mathcal{L}_\Lambda(H)$ whenever Φ is
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- **Theorem.** Let H and K be two Kaplansky-Hilbert modules over a Stone algebra Λ , $\mathbb{B} = \mathbb{P}(\Lambda)$, and let $S, T \in \mathcal{L}_\Lambda(H, K)$. Then $S \perp^{\mathbb{B}} T$ if and only if there exists a Λ -valued state $\Phi : \mathcal{L}_\Lambda(H) \rightarrow \Lambda$ with $\Phi(S^*S) = |S|^2$ and $\Phi(S^*T) = 0$.

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What is Boolean Valued Analysis?

- **Boolean valued analysis** is the study of the properties of a mathematical object by comparing its representations in two distinct set-theoretic models: in the von Neumann universe $\mathbb{V}$ and in a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ built over a complete Boolean algebra $\mathbb{B}$.
- There is a natural way of assigning to such a statement $\varphi(x_1, \dots, x_n)$ ($x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$) the **Boolean truth value** $\llbracket \varphi(x_1, \dots, x_n) \rrbracket \in \mathbb{B}$. By definition, $\varphi(x_1, \dots, x_n)$ is **true within $\mathbb{V}^{(\mathbb{B})}$** (in symbols, $\mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n)$) if $\llbracket \varphi(x_1, \dots, x_n) \rrbracket = 1$.
- **The Boolean-Valued Transfer Principle** states that all the theorems of ZFC (Zermelo–Fraenkel set theory with Choice) are true within $\mathbb{V}^{(\mathbb{B})}$.
- **The ascent and descent machinery** allows us to carry out the interplay between the standard universe $\mathbb{V}$ and a Boolean-valued universe $\mathbb{V}^{(\mathbb{B})}$: The descent functor $\mathcal{X} \mapsto \mathcal{X} \downarrow$. The ascent functor $X \mapsto X \uparrow$.

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Banach-Kantorovich and Kaplansky-Hilbert Modules

- **Theorem** (Ozawa¹²). The bounded descent of an arbitrary Hilbert space in $\mathbb{V}(\mathbb{B})$ is a Kaplansky-Hilbert module over the Stone algebra $\overline{\Lambda}(\mathbb{B})$. Conversely, if X is a Kaplansky-Hilbert module over $\overline{\Lambda}(\mathbb{B})$, then there is a Hilbert space $\mathcal{X}$ in $\mathbb{V}(\mathbb{B})$ whose bounded descent is unitarily equivalent with X .
- **Theorem** (Kusraev¹³). The bounded descent of an arbitrary Banach space within $\mathbb{V}(\mathbb{B})$ is a Banach-Kantorovich space over $\Lambda = \Lambda(\mathbb{B})$ and, conversely, every Banach-Kantorovich space over Λ is Λ -linearly isometric to the bounded descent of a Banach space within $\mathbb{V}(\mathbb{B})$ with $\mathbb{B} = \mathbb{P}(\Lambda)$.

¹²Ozawa M., Boolean valued interpretation of Hilbert space theory, J. Math. Soc. Japan, **35**:4 (1983), 609-627.

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