
Sobolev–Morrey Spaces and
Divergence-Form Degenerate
Second-Order Elliptic Equations on
Domains with Higher Co-Dimensional
Boundaries

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1 Introduction

Let $n \geq 2$. Throughout this talk, we *always* assume that $\Gamma \subset \mathbb{R}^n$ is an Ahlfors–David-regular set with dimension $d \in (0, n - 1)$.

Such a set Γ is also called a *d-set*.

For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, we write $\Omega(x, r) := \Omega \cap B(x, r)$.

By this we mean that Γ is closed and there exists a constant $C_0 \in [1, \infty)$ such that, for any $x \in \Gamma$ and $r \in (0, \infty)$,

$$C_0^{-1}r^d \leq \mathcal{H}^d(\Gamma(x, r)) \leq C_0r^d.$$

Here, and thereafter, for any $s \in (0, n]$, $x \in \mathbb{R}^n$, and $r \in (0, \infty)$, $\mathcal{H}^s$ denotes the s -dimensional Hausdorff measure,

$$\Gamma(x, r) := \Gamma \cap B(x, r).$$

Let

$$\Omega := \mathbb{R}^n \setminus \Gamma.$$

Since $d < n - 1$, such a domain Ω is referred to as a *domain with a higher co-dimensional boundary*.

For any $x \in \mathbb{R}^n$, we define the *distance function*

$$\delta(x) := \text{dist}(x, \Gamma) := \inf_{y \in \Gamma} |x - y|$$

and the *distance weight*

$$w(x) := [\delta(x)]^{d+1-n}. \quad (1.1)$$

Obviously, if $x \in \Gamma$, then $\delta(x) = 0$ and $w(x) = \infty$.

Throughout this talk, we always use the *symbol* Ω to denote a domain with a higher co-dimensional boundary; i.e., the d -set Γ .

There exist two typical examples of such d -sets Γ .

- The first one is the *flat lower-dimensional boundary* $\mathcal{F} := \mathbb{R}^{n-2} \times \{(0,0)\}$ with $n \in \mathbb{N} \cap [3, \infty)$, which is an $(n-2)$ -set.
- Another canonical example is the generalized *Cantor dust* in $\mathbb{R}^n$. We can realize any $d \in (0, n-1)$.

In this talk, motivated by the work of David et al. ^{*1} on the real-variable theory of the weighted Sobolev space $\dot{W}^{1,2}(\Omega, w)$, we study the weighted Sobolev–Morrey space $\dot{W}^1 \mathcal{M}_q^p(\Omega, w)$, where w is as in (1.1), $1 < q \leq p < \infty$, and $\mathcal{M}_q^p(\Omega, w)$ denotes the Morrey space in (4.2).

^{*1} G. David, J. Feneuil and S. Mayboroda, Elliptic theory for sets with higher co-dimensional boundaries, Mem. Amer. Math. Soc. 274 (2021), no. 1346, vi+123 pp.

More precisely, we systematically establish a real-variable theory for these spaces, including completeness, embedding theorems, Riesz potential characterizations, continuity, trace and extension theorems, and complex interpolation. Note that $\dot{W}^{1,2}(\Omega, w) = \dot{W}^1 \mathcal{M}_2^2(\Omega, w)$. Thus, these spaces are new when $1 < q \leq p < \infty$ and $p \neq 2 \neq q$. Even in the special case $p \neq 2$ of $\dot{W}^{1,p}(\Omega, w)$ (which is precisely $\dot{W}^1 \mathcal{M}_p^p(\Omega, w)$), our results are also new.

Furthermore, applying the real-variable theory of $\dot{W}^1 \mathcal{M}_q^p(\Omega, w)$, we obtain sharp weighted a priori estimates for solutions to the Dirichlet problem of divergence-form degenerate second-order elliptic equations on Ω in weighted Lebesgue spaces.

2 Fundamental inequality on weights

Lemma 2.1. *Let $p \in (-\infty, 0) \cup (0, \frac{n-d}{n-d-1})$, and w be as in (1.1).*

(i) *For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ satisfying*

$$\delta(x) \geq 2r, \left\{ \int_{B(x,r)} [w(z)]^p dz \right\}^{\frac{1}{p}} \sim w(x),$$

uniformly over x and r .

(ii) *For any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ satisfying*
 $\delta(x) \leq 2r,$

$$\left\{ \int_{B(x,r)} [w(z)]^p dz \right\}^{\frac{1}{p}} \sim r^{d+1-n}, \quad (2.1)$$

where the positive equivalence constants are independent of x and r .

(iii) *Let $1 \leq q_0 \leq q_1 < \infty$. For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$, and $f \in L^0(B(x, r))$,*

$$\left[\int_{B(x, r)} |f(y)|^{q_0} dy \right]^{\frac{1}{q_0}} \quad (2.2)$$
$$\lesssim \left[\frac{1}{w(B(x, r))} \int_{B(x, r)} |f(y)|^{q_1} dw(y) \right]^{\frac{1}{q_1}},$$

where the positive implicit constant is independent of x , r , and f .

3 Nakamura's class

Let $1 \leq q \leq p < \infty$. A weight ω belongs to $\mathcal{B}_{p,q}(\mathbb{R}^n)$ if and only if ^{*2}

$$r \mapsto |B(x, r)|^{\frac{1}{p} - \frac{1}{q}} [\omega(B(x, r))]^{\frac{1}{q}}$$

is almost increasing with the constant independent of $x \in \mathbb{R}^n$.

^{*2} S. Nakamura, Generalized weighted Morrey spaces and classical operators, Math. Nachr. 289 (2016), 2235–2262.

Roughly speaking, the class $\mathcal{B}_{p,q}(\mathbb{R}^n)$ is introduced to ensure that the weighted Morrey norm of the characteristic function $\mathbf{1}_{B(x,r)}$ can be estimated efficiently for any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ (see Proposition 4.7 below). This condition also plays an important role in showing the weighted boundedness of classical operators on weighted Morrey spaces.

Proposition 3.1. *Let w be as in (1.1). Let $1 < q \leq p < \infty$.*

• *Assume*

$$\frac{n}{p} - \frac{n - d - 1}{q} > 0. \quad (3.1)$$

Then there exists a positive constant C such that, for any $x \in \mathbb{R}^n$, $r \in (0, \infty)$, and $k \in \mathbb{Z}_+$,
 $w(B(x, r)) \leq C 2^{-k(d+1)} w(B(x, 2^k r)).$

- For any ball $B \subset \mathbb{R}^n$,

$$\int_1^\infty \frac{s^{-1} ds}{|sB|^{\frac{1}{p} - \frac{1}{q}} [w(sB)]^{\frac{1}{q}}} \lesssim \frac{1}{|B|^{\frac{1}{p} - \frac{1}{q}} [w(B)]^{\frac{1}{q}}}$$

holds with the implicit positive constant independent of B if and only if (3.1) holds.

4 Poincaré Inequalities and Trace Operators

Lemma 4.1. *Let $n \geq 2$, $d \in (0, n - 1)$, and $\Gamma \subset \mathbb{R}^n$ be a d -set. Let*

$$G := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : [x, y] \cap \Gamma \neq \emptyset\}.$$

Then $\mathcal{H}^{2n}(G) = 0$.

Lemma 4.2. *Let $B := B(x, r) \subset \mathbb{R}^n$ be a ball and $f \in W_{\text{loc}}^{1,1}(\Omega)$.*

(i) *It holds that*

$$\int_B \int_B |f(y) - f(z)| \, dy dz \lesssim r \int_B |\nabla f(\xi)| \, d\xi.$$

(ii) *For almost every $z \in B$,*

$$\int_B |f(y) - f(z)| \, dy \lesssim \int_B \frac{|\nabla f(\xi)|}{|z - \xi|^{n-1}} \, d\xi.$$

Proposition 4.3. *Let $f \in W_{\text{loc}}^{1,1}(\Omega)$.*

(i) *If $x \in \Gamma$ satisfies*

$$\int_{B(x,r)} \frac{|\nabla f(\xi)|}{|x - \xi|^{n-1}} d\xi < \infty \quad (4.1)$$

for some $r \in (0, \infty)$, then the limit

$$Tf(x) := \lim_{r \rightarrow 0^+} \int_{B(x,r)} f(y) dy$$

exists and is finite.

- (ii) *If $\nabla f \in L^1_{\text{loc}}(\Omega, w)$, then $Tf(x)$ exists for $\mathcal{H}^d$ -almost every $x \in \Gamma$.*
- (iii) *If $x \in \Gamma$ satisfies (4.1), then the Lebesgue differentiation property*

$$\lim_{r \rightarrow 0^+} \int_{B(x,r)} |Tf(x) - f(y)| \, dy = 0$$

holds.

Proposition 4.4. *Let w be as in (1.1) and $f \in \dot{W}_{\text{loc}}^{1,1}(\Omega, w)$. Then, for any $x \in \Gamma$ and $r \in (0, \infty)$ such that $Tf = 0$ on $\Gamma(x, r)$,*

$$\int_{B(x,r)} |f(y)| \, dy \lesssim \frac{1}{r^d} \int_{B(x,r)} |\nabla f(y)| \, dw(y),$$

where the implicit positive constant is independent of f , x , and r .

Proposition 4.5. *Let $p \in (1, n)$, $p^* := \frac{np}{n-p}$, $q \in [1, p^*]$, and w be as in (1.1). Then, for any $f \in \dot{W}_{\text{loc}}^{1, \max\{p, q\}}(\Omega, w)$, $x \in \mathbb{R}^n$, and $r \in (0, \infty)$, two quantites*

$$\left[\int_{B(x, r)} |f(y) - f_{B(x, r)}|^q dw(y) \right]^{\frac{1}{q}},$$

$$\left[\int_{B(x, r)} \left| f(y) - \int_{B(x, r)} f(z) dw(z) \right|^q dw(y) \right]^{\frac{1}{q}}$$

are equivalent and is dominated by

$$r \left[\int_{B(x,r)} |\nabla f(y)|^p dw(y) \right]^{\frac{1}{p}},$$

where the implicit positive constants are independent of f , x , and r .

The *weighted Morrey space* $\mathcal{M}_q^p(\Omega, w)$ adapted to Γ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\mathcal{M}_q^p(\Omega, w)} \tag{4.2}$$

$$:= \sup_{\substack{x \in \mathbb{R}^n, \\ r \in (0, \infty)}} |B(x, r)|^{\frac{1}{p} - \frac{1}{q}} \left[\int_{\Omega(x, r)} |f(y)|^q dw(y) \right]^{\frac{1}{q}}$$

is finite.

This space can be seen as a natural extension of the weighted Lebesgue space $L^p(\Omega, w)$. In particular, for any $p \in [1, \infty)$,

$$\mathcal{M}_p^p(\Omega, w) = L^p(\Omega, w) = L^p(\mathbb{R}^n, w).$$

Proposition 4.6. *Let $1 \leq q \leq p < \infty$ and w be as in (1.1). Then*

$$\mathcal{M}_q^p(\Omega, w) \hookrightarrow L_{\text{loc}}^1(\Omega, w) \hookrightarrow L_{\text{loc}}^1(\Omega).$$

Next, we discuss conditions under which $L_c^\infty(\mathbb{R}^n)$ is embedded into $\mathcal{M}_q^p(\Omega, w)$, where a new restriction appears.

Proposition 4.7. *Let $1 \leq q \leq p < \infty$ and w be as in (1.1). Then the following assertions hold.*

- (i) $\mathbf{1}_{B(x,r)} \in \mathcal{M}_q^p(\Omega, w)$ for any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ if and only if

$$\frac{n}{p} - \frac{n - d - 1}{q} \geq 0. \quad (4.3)$$

In this case, $\|\mathbf{1}_{B(x,r)}\|_{\mathcal{M}_q^p(\Omega,w)} \sim r^{\frac{n}{p} - \frac{n-d-1}{q}}$, where the positive equivalence constants are independent of both x and r .

(ii) *The constant function $1 \in \mathcal{M}_q^p(\Omega, w)$ if and only if $\frac{n}{p} = \frac{n-d-1}{q}$.*

Theorem 4.8. *Let $1 \leq q \leq p < \infty$, $\theta \in (0, 1)$, and w be as in (1.1). Then*

$$\|f\|_{\mathcal{M}_q^p(\Omega, w)} \sim \|f\|_{\widetilde{\mathcal{M}}_q^p(\Omega, w)} \quad (4.4)$$

$$:= \sup_{\substack{x \in \Omega \\ r \in (0, \infty)}} r^{\frac{n}{p} - \frac{n}{q}} \|f [M_w(\mathbf{1}_{\Omega(x, r)})(y)]^{\frac{\theta}{q}}\|_{L^q(\Omega)}$$

holds for any $f \in \mathcal{M}_q^p(\Omega, w)$ if and only if

$$\frac{n}{p} - \frac{n}{q} + \frac{(d+1)\theta}{q} > 0. \quad (4.5)$$

Proposition 4.9. *Let $1 < q \leq p < \infty$ satisfy (3.1), $r \in (1, \infty)$, and w be as in (1.1). For any sequence $\{f_k\}_{k \in \mathbb{N}}$ in $\mathcal{M}_q^p(\Omega, w)$,*

$$\left\| \left(\sum_{k \in \mathbb{N}} |M f_k|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p(\Omega, w)} \lesssim \left\| \left(\sum_{k \in \mathbb{N}} |f_k|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p(\Omega, w)}.$$

5 Noi's theorem

Let X be a ball Banach function space. Then the vector-valued inequality

$$\left\| \left(\sum_{j=1}^{\infty} (M f_j)^r \right)^{\frac{1}{r}} \right\|_X \lesssim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_X$$

implies the boundedness of M on X' . The method is to resort to sparse operators.

Proposition 5.1. *Let $1 < q \leq p < \infty$ satisfy (3.1), $r \in (1, \infty)$, and w be as in (1.1). For any $j \in \mathbb{N} \cap [1, n]$ and any $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{M}_q^p(\Omega, w)$,*

$$\left\| \left(\sum_{k \in \mathbb{N}} |R_j f_k|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p(\Omega, w)} \lesssim \left\| \left(\sum_{k \in \mathbb{N}} |f_k|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_q^p(\Omega, w)}.$$

Proposition 5.2. *Let $1 < u \leq s < \infty$, $1 < q \leq p < \infty$ with $\frac{s}{u} = \frac{p}{q}$ and $u < q$, $\alpha \in (0, n)$ satisfy $\alpha = (n - d - 1)(\frac{1}{u} - \frac{1}{q}) - n(\frac{1}{s} - \frac{1}{p})$, and w be as in (1.1). Then*

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

is bounded from $\mathcal{M}_q^p(\Omega, w)$ to $\mathcal{M}_u^s(\Omega, w)$.

Proposition 5.3. *Let $1 \leq q \leq p < \infty$ and w be as in (1.1). Then $\mathcal{M}_q^p(\Omega, w) \subset \mathcal{S}'(\mathbb{R}^n)$.*

We move on to the Littlewood–Paley decomposition: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function such that

$$\sum_{j \in \mathbb{Z}} \varphi_j = \mathbf{1}_{\mathbb{R}^n \setminus \{\mathbf{0}\}}.$$

Proposition 5.4. *Let $1 < q \leq p < \infty$ satisfy (3.1) and w be as in (1.1).*

(i) *If $f \in \mathcal{M}_q^p(\Omega, w)$, then $f = \sum_{j \in \mathbb{Z}} \varphi_j(D) f$ in the weak-* topology of $\mathcal{M}_q^p(\Omega, w)$ and*

$$\|f\|_{\mathcal{M}_q^p(\Omega, w)} \sim \left\| \left[\sum_{j \in \mathbb{Z}} |\varphi_j(D) f|^2 \right]^{\frac{1}{2}} \right\|_{\mathcal{M}_q^p(\Omega, w)},$$

where the positive equivalence constants are independent of f .

(ii) *If $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$$\left\| \left[\sum_{j \in \mathbb{Z}} |\varphi_j(D)f|^2 \right]^{\frac{1}{2}} \right\|_{\mathcal{M}_q^p(\Omega, w)} < \infty,$$

then there exists a decomposition $f = F + P$ in $\mathcal{S}'(\mathbb{R}^n)$, where $F := \sum_{j \in \mathbb{Z}} \varphi_j(D)f$ pointwise and P is a polynomial on $\mathbb{R}^n$.

Moreover,

$$\|F\|_{\mathcal{M}_q^p(\Omega, w)} \sim \left\| \left[\sum_{j \in \mathbb{Z}} |\varphi_j(D)f|^2 \right]^{\frac{1}{2}} \right\|_{\mathcal{M}_q^p(\Omega, w)},$$

where the positive equivalence constants are independent of f .

Proposition 5.5. *Let $1 \leq q_0 \leq p_0 < \infty$,
 $1 \leq q_1 \leq p_1 < \infty$ with*

$$\frac{q_0}{p_0} = \frac{q_1}{p_1}, \quad (5.1)$$

$\theta \in (0, 1)$, $1 \leq q \leq p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad (5.2)$$

and w be as in (1.1). Then

$$\left[\mathcal{M}_{q_0}^{p_0}(\Omega, w), \mathcal{M}_{q_1}^{p_1}(\Omega, w) \right]^\theta = \mathcal{M}_q^p(\Omega, w). \quad (5.3)$$

5.1 Weighted Sobolev–Morrey Spaces Adapted to Γ

Definition 1. Let $1 \leq q \leq p < \infty$ and w be as in (1.1). The *weighted Sobolev–Morrey space* $\dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ adapted to Γ is defined to be the set of all $f \in L_{\text{loc}}^1(\Omega)$, modulo constant functions, such that $\nabla f \in \mathcal{M}_q^p(\Omega, w)$. This space is equipped with the norm

$$\|f\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)} := \|\nabla f\|_{\mathcal{M}_q^p(\Omega, w)}.$$

Lemma 5.6. *Let $1 \leq q \leq p < \infty$ and w be as in (1.1). Then $\dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ is complete.*

Theorem 5.7. *Let $1 < q \leq p < \infty$ satisfy (3.1), w be as in (1.1), and $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function such that*

$$\sum_{j \in \mathbb{Z}} \varphi_j = \mathbf{1}_{\mathbb{R}^n \setminus \{\mathbf{0}\}}.$$

- (i) *If $f \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$, then $(-\Delta)^{\frac{1}{2}} f \in \mathcal{M}_q^p(\Omega, w)$ and $\|f\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)} \sim \|(-\Delta)^{\frac{1}{2}} f\|_{\mathcal{M}_q^p(\Omega, w)}$, where the positive equivalence constants are independent of f .*
- (ii) *Conversely, if $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$(-\Delta)^{\frac{1}{2}} f \in \mathcal{M}_q^p(\Omega, w)$, then there exist
 $F \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ and a polynomial P on $\mathbb{R}^n$
such that $f = F + P$ in $\mathcal{S}'(\mathbb{R}^n)$. Moreover,
 $\|F\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)} \sim \|(-\Delta)^{\frac{1}{2}} f\|_{\mathcal{M}_q^p(\Omega, w)}$, where
the positive equivalence constants are
independent of f .

In particular, $(-\Delta)^{\frac{1}{2}}$ is an isomorphism from
 $\dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ to $\mathcal{M}_q^p(\Omega, w)$.

Corollary 5.8. *Let $1 < u \leq s < \infty$,
 $1 < q \leq p < \infty$ with $\frac{s}{u} = \frac{p}{q}$, $u < q$, and
 $(n - d - 1)(\frac{1}{u} - \frac{1}{q}) - n(\frac{1}{s} - \frac{1}{p}) = 1$. Then
 $\dot{W}^1 \mathcal{M}_q^p(\Omega, w) \hookrightarrow \mathcal{M}_s^u(\Omega, w)$.*

Lemma 5.9. *Let $1 \leq q \leq p < \infty$ satisfy*

$$\frac{n}{p} - \frac{n - d - 1}{q} > 1 \quad (5.4)$$

and w be as in (1.1). Fix $x_0 \in \Gamma$. Then, for any $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$,

$$u^0(x_0) := \lim_{j \rightarrow \infty} \int_{B(x_0, 2^j)} u(z) \, dz$$

exists and is finite.

Lemma 5.10. *Let $1 \leq q \leq p < \infty$ satisfy*

$$\frac{n}{p} - \frac{n-d-1}{q} < 1 \quad (5.5)$$

and w be as in (1.1). Then, for any $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$, $x \in \Gamma$, and $r \in (0, \infty)$,

$$\begin{aligned} & \int_{B(x,r)} \frac{|\nabla u(\xi)|}{|x - \xi|^{n-1}} d\xi \\ & \lesssim r^{1 - \frac{n}{p} + \frac{n-d-1}{q}} \|u\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)}. \end{aligned}$$

In particular, for any $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ and $x \in \Gamma$, $Tu(x)$ exists and, for any $x, y \in \Gamma$,

$$\begin{aligned} & |Tu(x) - Tu(y)| \\ & \lesssim |x - y|^{1 - \frac{n}{p} + \frac{n-d-1}{q}} \|u\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)}, \end{aligned}$$

where the implicit positive constant is independent of u , x , and y .

Lemma 5.11. *Let $1 \leq q \leq p < \infty$ satisfy $p \in (n, \infty)$ and w be as in (1.1). Then, for any $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ and any Lebesgue points $x, y \in \Omega$ such that $B(x, 4|x - y|) \subset \Omega$,*

$$\begin{aligned} & |u(x) - u(y)| \\ & \lesssim |x - y|^{1 - \frac{n}{p}} [w(x)]^{-\frac{1}{q}} \|u\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)}, \end{aligned}$$

where the implicit positive constant is independent of u , x , and y .

Theorem 5.12. *Let $1 \leq q \leq p < \infty$ satisfy $p \in (n, \infty)$, and let w be as in (1.1). Then, for any $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$, any Lebesgue point $x \in \Omega$, and any $y \in \Gamma$,*

$$|u(x) - Tu(y)| \tag{5.6}$$

$$\lesssim |x - y|^{1 - \frac{n}{p} + \frac{n-d-1}{q}} \|u\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)} + |x - y|^{1 - \frac{n}{p}} [w(x)]^{-\frac{1}{q}} \|u\|_{\dot{W}^1 \mathcal{M}_q^p(\Omega, w)},$$

where the implicit positive constant is independent of u , x , and y .

Lemma 5.13. *Let $1 \leq q \leq p < \infty$ and w be as in (1.1). Then, for any $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$, $Tu(x)$ exists for $\mathcal{H}^d$ -almost every $x \in \Gamma$.*

Definition 2. Let $1 \leq q \leq p < \infty$. The space $Q_q^p(\Gamma)$ is defined to be the set of all $\mathcal{H}^d$ -measurable functions f on Γ such that

$$\begin{aligned} & \|f\|_{Q_q^p(\Gamma)} \\ &:= \sup_{v \in \Gamma, R \in (0, \infty)} |B(v, R)|^{\frac{1}{p} - \frac{1}{q}} \\ & \quad \times \left[\iint_{\Gamma(v, R)^2} \frac{|f(x) - f(y)|^q}{|x - y|^{q+d-1}} d\mathcal{H}^d(x) d\mathcal{H}^d(y) \right]^{\frac{1}{q}} \end{aligned}$$

is finite

Theorem 5.14. *Let $1 < q \leq p < \infty$ and w be as in (1.1). Then the trace operator T can be extended to a linear bounded mapping*
$$T : \dot{W}^1 \mathcal{M}_q^p(\Omega, w) \longrightarrow Q_q^p(\Gamma).$$

Theorem 5.15. *Let $1 < q \leq p < \infty$ and w be as in (1.1). Then the extension operator E is bounded from $Q_q^p(\Gamma)$ to $\dot{W}^1 \mathcal{M}_q^p(\Omega, w)$. Moreover, for any $g \in Q_q^p(\Gamma)$ and $\mathcal{H}^d$ -almost every $x \in \Gamma$, $(T \circ E)g(x) = g(x)$.*

Theorem 5.16. *Let $1 < q_0 \leq p_0 < \infty$, $1 < q_1 \leq p_1 < \infty$ satisfy $\frac{n}{p_i} - \frac{n-d-1}{q_i} > 0$ for any $i \in \{0, 1\}$, $1 \leq q \leq p < \infty$, $\theta \in (0, 1)$ be such that (5.1) and (5.2) hold, and w be as in (1.1). Then*

$$\left[\dot{W}^1 \mathcal{M}_{q_0}^{p_0}(\Omega, w), \dot{W}^1 \mathcal{M}_{q_1}^{p_1}(\Omega, w) \right]^\theta = \dot{W}^1 \mathcal{M}_q^p(\Omega, w).$$

Corollary 5.17. *Let $1 < q_0 \leq p_0 < \infty$, $1 < q_1 \leq p_1 < \infty$ satisfy $\frac{n}{p_i} - \frac{n-d-1}{q_i} > 0$ for any $i \in \{0, 1\}$, $1 \leq q \leq p < \infty$, $\theta \in (0, 1)$ be such that (5.1) and (5.2) hold, and w be as in (1.1). Then*

$$\left[Q_{q_0}^{p_0}(\Gamma), Q_{q_1}^{p_1}(\Gamma) \right]^\theta = Q_q^p(\Gamma).$$

6 Applications

In this subsection, we recall the concept of solutions and establish the reverse Hölder estimates for local solutions. Let w be as in (1.1) and $L := -\operatorname{div}(A\nabla\cdot)$ be a divergence-form degenerate second-order elliptic operator on Ω , where A is a real-valued $n \times n$ matrix of measurable functions on Ω satisfying the *degenerate elliptic condition*; i.e., there exists a constant $C_1 \in [1, \infty)$ such that, for any $x \in \Omega$

and $\xi, \eta \in \mathbb{R}^n$,

$$|A(x)\xi \cdot \eta| \leq C_1 w(x) |\xi| |\eta| \quad (6.1)$$

and

$$A(x)\xi \cdot \xi \geq C_1^{-1} w(x) |\xi|^2.$$

Denote the matrix $w^{-1}A$ by $\mathcal{A}$. Then the matrix $\mathcal{A}$ satisfies the *uniformly elliptic condition*; i.e., for any $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$,

$$|\mathcal{A}(x)\xi \cdot \eta| \leq C_1 |\xi| |\eta| \quad \text{and} \quad \mathcal{A}(x)\xi \cdot \xi \geq C_1^{-1} |\xi|^2.$$

Let $E \subset \Omega$ be an open set. Recall that a function $u \in W_r(E, w)$ is called a *weak solution* to the divergence-form degenerate second-order elliptic equation

$$Lu = -\operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } E \quad (6.2)$$

if, for any $\varphi \in C_c^\infty(E)$,

$$\int_{\Omega} A(x)\nabla u(x) \cdot \nabla \varphi(x) \, dx = 0.$$

Corollary 6.1. *Let ε be small,
 $q \in (\max\{2 - \varepsilon, 1\}, 2 + \varepsilon)$, $\theta \in (0, \frac{2+\varepsilon-q}{2+\varepsilon})$,
 $p \in (q, \frac{qn}{n-(d+1)\theta}]$, and w be as in (1.1). Then,
for any given $f := (f^{(1)}, \dots, f^{(n)}) \in \mathcal{M}_q^p(\Omega, w)$,
there exists a unique weak solution
 $u \in \dot{W}^1 \mathcal{M}_q^p(\Omega, w)$ to the Dirichlet problem*

$$\begin{cases} Lu = -\operatorname{div}(A\nabla u) = \operatorname{div}_w(f) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma \end{cases} \quad (6.3)$$

such that

$$\|\nabla u\|_{\mathcal{M}_q^p(\Omega, w)} \lesssim \|f\|_{\mathcal{M}_q^p(\Omega, w)}. \quad (6.4)$$

Congratulations on your 85th birthday, Professor Burenkov.

ブレンコフ先生 85 歳のお誕生日おめでとうございます。