

Associative function spaces: sharp description and applications *

Воркшоп по теории операторов в функциональных пространствах
и ее приложениях,
посвященный юбилею д.ф.-м.н., профессора Буренкова В.И.
28-30 сентября 2026 г.

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*Выполнена при поддержке РФФИ (Грант № 24-11-00170,
<https://rscf.ru/project/24-11-00170/>)

1. Associative spaces

Put $I := (0, \infty)$ and let $\mathfrak{M}(I)$ be the collection of all Lebesgue measurable functions on I , $\mathfrak{M}^+(I) := \{f \in \mathfrak{M}(I) : f \geq 0\}$.

Suppose $X \subset \mathfrak{M}(I)$ is a function space with a norm $\|\cdot\|_X$, X^* – the dual (conjugate) space of all linear bounded functionals on X .

The classical problem of functional analysis is the representation of elements of X^* in a concrete (e.g. *integral*) form. If X^{**} – the second dual space, then $X \subset X^{**}$. If $X = X^{**}$ (isometric isomorphism), the space X is called *reflexive*. For a Banach space criterion of reflexivity is sequential compactness of a unit ball of X .

Let $X \subset \mathfrak{M}(I)$ be a function space with a norm $\|\cdot\|_X$. Put

$$\mathfrak{D}_X := \left\{ g \in \mathfrak{M}(I) : \int_I |fg| < \infty \text{ for all } f \in X \right\}. \quad (1)$$

For any $g \in \mathfrak{D}_X$ we define functionals

$$\mathbf{J}_X(g) := \sup_{0 \neq f \in X} \frac{\int_I |fg|}{\|f\|_X} \quad \text{and} \quad J_X(g) := \sup_{0 \neq f \in X} \frac{\left| \int_I fg \right|}{\|f\|_X}$$

and associative spaces

$$X'_s := \{g \in \mathfrak{M}(I) : \|g\|_{X'_s} := \mathbf{J}_X(g) < \infty\},$$

$$X'_w := \{g \in \mathfrak{M}(I) : \|g\|_{X'_w} := J_X(g) < \infty\},$$

which will be called *strong* and *weak*, respectively.

A space X is said to be *ideal* if $|f| \leq |g|$ a.e. on I and $g \in X$ implies $f \in X$ and $\|f\| \leq \|g\|$. It is known for ideal spaces with Fatou's property

$$0 \leq f_n \uparrow f \Rightarrow \|f_n\|_X \uparrow \|f\|_X$$

that X and X'_s are complete and there is associated reflexivity

$$(X'_s)'_s = X,$$

as well $X'_s = X^*$, if the norm of X is absolutely continuous (AC-norm). Thus, $X^{**} = X$ only if X and X'_s have AC-norms.

[BS] *C. Bennett and R. Sharpley. Interpolation of operators, AP, 1988.*

A standard problem for an ideal space $(X, \|\cdot\|_X)$ is characterization of its strong associative (or the Köthe dual) space (see [BS], Ch. 1, § 2).

Observe that $J_X(g) = \mathbf{J}_X(g)$ for any ideal space X (see [BS], Ch. 1, Lemma 2.8), in such a case $X'_s = X'_w$. For a non-ideal space X , the functionals $J_X(g)$ and $\mathbf{J}_X(g)$ might be different.

Thus, there is a natural problem of characterization of double associative spaces:

$$[X'_s]'_s, [X'_s]'_w, [X'_w]'_s, [X'_w]'_w$$

and when X is equal to any of them, i.e. when X is correspondingly reflexive.

Since X'_s is ideal, therefore, $[X'_s]'_s = [X'_s]'_w$, and there are only three remaining.

If X is an ideal space with Fatou's property, then all double associative spaces coincide with X . Therefore, a particular interesting problem is for non-ideal spaces.

2. Sobolev spaces

The symbol $\mathcal{V}_p(I)$ stands for the corresponding set of weight functions (weights):

$$\mathcal{V}_p(I) := \{v \in L^p_{\text{loc}}(I) : v \geq 0, \|v\|_{L^1(I)} \neq 0\}.$$

Let $v_0, v_1 \in \mathfrak{M}^+(I)$, $v_1 < \infty$ a.e. on I , $\frac{1}{v_1} \in L^{p'}_{\text{loc}}(I)$, $v_0, v_1 \in \mathcal{V}_p(I)$. Then

$$W_p^1(I) := \left\{ u \in W_{1,\text{loc}}^1(I) : \|u\|_{W_p^1(I)} < \infty \right\}, \quad \text{where}$$

$$\|u\|_{W_p^1(I)} := \|v_0 u\|_{L^p(I)} + \|v_1 Du\|_{L^p(I)}.$$

Also we need $\overset{\circ}{\overset{\circ}{W}}_p^1(I) \subset \overset{\circ}{W}_p^1(I) \subset W_p^1(I)$, where

$$\overset{\circ}{\overset{\circ}{W}}_p^1(I) := \left\{ f \in AC(I) : \text{supp } f \text{ is compact in } [0, \infty), \right. \\ \left. \|v_0 f\|_{L^p(I)} + \|v_1 f'\|_{L^p(I)} < \infty \right\}$$

and $\overset{\circ}{W}_p^1(I)$ is the closure in $W_p^1(I)$ of the subspace $\overset{\circ}{\overset{\circ}{W}}_p^1(I)$.

For $X = \overset{\circ}{W}_p^1(I)$ characterization of X'_s is given in:

[Oin1] R. Oinarov, *On weighted norm inequalities with three weights*, *J. London Math. Soc.* (2), 48:1 (1993), 103–116.

Let $X = \{\overset{\circ}{W}_p^1(I), W_p^1(I)\}$

$$\|gf\|_{L^q(I)} \leq C_{p,q} (\|v_0 f\|_{L^p(I)} + \|v_1 f'\|_{L^p(I)}), \quad f \in X.$$

$$C_{p,1} = \mathbf{J}_X(g)$$

[Oin2] R. Oinarov, *Boundedness of integral operators from weighted Sobolev space to weighted Lebesgue space*, *Complex Var. Elliptic Equ.*, 56:10-11 (2011), 1021–1038.

[Oin3] Oinarov, R. *Boundedness of integral operators in weighted Sobolev spaces*. *Izv. Math.* 78 (2014), no. 4, 836–853.

In **[PSU1]** Prokhorov D.V., Stepanov V.D., Ushakova E.P., *On associate spaces of weighted Sobolev space on the real line*. *Math. Nachr.* 290 (2017), no. 5-6, 890–912.

and

[PSU2] Prokhorov D.V., Stepanov V.D., Ushakova E.P., *Characterization of the function spaces associated with weighted Sobolev spaces of the first order on the real line*, *Russian Math. Survey*, 2019, 74 № 6, 1075–1115.

all associated spaces X'_s & X'_w for $X = \{\overset{\circ\circ}{W}_p^1(I), \overset{\circ}{W}_p^1(I), W_p^1(I)\}$ are characterized.

We suppose that there is $c \in I$ such that

$$\|v_1^{-1}\|_{L^{p'}(0,c)}\|v_0\|_{L^p(0,c)} = \|v_1^{-1}\|_{L^{p'}(c,\infty)}\|v_0\|_{L^p(c,\infty)} = \infty. \quad (2)$$

Then $\overset{\circ}{W}_p^1(I) = W_p^1(I)$ and by Oinarov-Otelbaev construction there are strictly increasing AC-functions $a(t)$ and $b(t)$ such that

$$\lim_{t \rightarrow 0} a(t) = \lim_{t \rightarrow 0} b(t) = 0, \quad \lim_{t \rightarrow \infty} a(t) = \lim_{t \rightarrow \infty} b(t) = \infty,$$

$$a(t) < t < b(t), \quad t > 0,$$

$$\int_{a(t)}^t v_1^{-p'} = \int_t^{b(t)} v_1^{-p'}, \quad t > 0, \quad (3)$$

and

$$\left(\int_{a(t)}^{b(t)} v_1^{-p'} \right)^{1/p'} \left(\int_{a(t)}^{b(t)} v_0^p \right)^{1/p} = 1, \quad t > 0. \quad (4)$$

Let

$$V_1(t) := \int_{\Delta(t)} v_1^{-p'}, \quad \Delta(t) := (a(t), b(t)).$$

and $a^{-1}(t)$ is the reverse function to $a(t)$. Put

$$\mathbb{G}_1(g) := \left(\int_0^\infty v_1^{-p'}(t) \left| \int_t^{a^{-1}(t)} \frac{g(x)}{V_1(x)} \left(\int_{a(x)}^t v_1^{-p'} \right) dx \right|^{p'} dt \right)^{1/p'},$$

$$\mathbb{G}_2(g) := \left(\int_0^\infty v_1^{-p'}(t) V_1^{p'}(t) \left| \int_t^{a^{-1}(t)} \frac{g(x)}{V_1(x)} dx \right|^{p'} dt \right)^{1/p'},$$

$$\mathbb{G}_0(g) := \left(\int_0^\infty \left(\int_t^{a^{-1}(t)} |g(x)| dx \right)^{p'} v_1^{-p'}(t) dt \right)^{1/p'}.$$

Theorem 2.1. *Let $1 < p < \infty$, $v_0, v_1 \in \mathcal{V}_p(I)$ and (2) holds.*

(1) Then for $g \in L^1_{\text{loc}}(I)$, if $X = W_p^1(I)$ or $X = \overset{\circ\circ}{W}_p^1(I)$,

$$\mathbf{J}_X(g) = \|g\|_{X'_s} \approx \mathbf{G}_0(g).$$

(2) If $X = \overset{\circ\circ}{W}_p^1(I)$, then

$$J_X(g) = \|g\|_{X'_w} \approx \mathbb{G}_1(g) + \mathbb{G}_2(g), \tag{5}$$

(3) Also, $J_{W_p^1}(g) < \infty$, iff $\mathbf{G}_0(g) < \infty$ and $J_{W_p^1}(g) \approx \mathbb{G}_1(g) + \mathbb{G}_2(g)$.

3. Sobolev, Cesàro and Copson spaces

(1) Let $1 < p < \infty$, $\beta = \alpha + 1 > \frac{1}{p'}$ and $W_{p,\beta}^1(I)$ — Sobolev space with weights

$$v_0(x) = \eta_p x^\alpha, \quad v_1(x) = x^\beta. \quad (6)$$

(2) If $1 < p < \infty$, $\gamma = \sigma + 1 < \frac{1}{p'}$, then $W_{p,\gamma}^1(I)$ Sobolev space with weights

$$v_0(x) = \theta_p x^\sigma, \quad v_1(x) = x^\gamma. \quad (7)$$

Observe that condition (2) for these spaces is satisfied. Therefore,

$$\overset{\circ}{W}_{p,\beta}^1(I) = W_{p,\beta}^1(I), \quad \overset{\circ}{W}_{p,\gamma}^1(I) = W_{p,\gamma}^1(I).$$

Let $p \in (1, \infty)$, $\beta > \frac{1}{p}$. We define Cesàro type spaces as spaces

$$Ces_{p,\beta}(I) := \left\{ f \in \mathfrak{M}(I) : \|f\|_{Ces_{p,\beta}(I)} < \infty \right\}$$

of functions with finite norm

$$\|f\|_{Ces_{p,\beta}(I)} := \left(\int_0^\infty \left(\frac{1}{x^\beta} \int_0^x |f| \right)^p dx \right)^{\frac{1}{p}}.$$

Along with $\mathcal{Ces}_{p,\beta}(I)$, we consider non-ideal spaces of Cesáro type

$$\mathcal{Ces}_{p,\beta}(I) := \left\{ f \in L^1_{\text{loc}}([0, \infty)) : \|f\|_{\mathcal{Ces}_{p,\beta}(I)} < \infty \right\}$$

with the norm

$$\|f\|_{\mathcal{Ces}_{p,\beta}(I)} := \left(\int_0^\infty \left| \frac{1}{x^\beta} \int_0^x f \right|^p dx \right)^{\frac{1}{p}}.$$

Furthermore, for $\gamma < 1/p$ we set a Copson type function space

$$\mathcal{Cop}_{p,\gamma}(I) := \{g \in \mathfrak{M}(I) : \|g\|_{\mathcal{Cop}_{p,\gamma}(I)} < \infty\},$$

where

$$\|g\|_{\mathcal{Cop}_{p,\gamma}(I)} := \left(\int_0^\infty \left(\frac{1}{x^\gamma} \int_x^\infty |g| \right)^p dx \right)^{\frac{1}{p}}.$$

Put

$$L^1_{\text{loc}}((0, \infty]) := \left\{ f \in \mathfrak{M}(I) : \int_t^\infty |f| < \infty \text{ for all } t \in I \right\}$$

and, by analogy with Cesáro spaces, we define a non-ideal space of Copson type of the form

$$\mathcal{Cop}_{p,\gamma}(I) := \left\{ g \in L^1_{\text{loc}}((0, \infty]) : \|g\|_{\mathcal{Cop}_{p,\gamma}(I)} < \infty \right\},$$

where

$$\|g\|_{\mathcal{Cop}_{p,\gamma}(I)} := \left(\int_0^\infty \left| \frac{1}{x^\gamma} \int_x^\infty g \right|^p dx \right)^{\frac{1}{p}}.$$

Observe that $\mathcal{Ces}_{p,\beta}(I)$ and $\mathcal{Cop}_{p,\beta}(I)$ are incomplete normed spaces.

Example. For $\varepsilon \in (0, 1/p)$ we put

$$f_\varepsilon(x) = \frac{\sin x}{x^{1+\varepsilon}}.$$

Then $\|f_\varepsilon\|_{\mathcal{Cop}_{p,\gamma}(I)} < \infty$, but $\|f_\varepsilon\|_{\mathcal{Cop}_{p,\gamma}(I)} = \infty$, i.e. $f_\varepsilon \in \mathcal{Cop}_{p,\gamma}(I) \setminus \mathcal{Ces}_{p,\gamma}(I)$.

Theorem 3.1.

$$\|g\|_{[\mathcal{Ces}_{p,\beta}(I)]'_s} \approx \left(\int_0^\infty \left(x^{\beta-1} \|g\|_{L^\infty([x,\infty))} \right)^{p'} dx \right)^{\frac{1}{p'}} =: \|g\|_{\mathcal{D}_{p',\beta}(I)},$$

$$\|g\|_{[\mathcal{Cop}_{p,\gamma}(I)]'_s} \approx \left(\int_I \left(x^{\gamma-1} \|g\|_{L^\infty((0,x])} \right)^{p'} dx \right)^{\frac{1}{p'}} =: \|g\|_{\mathcal{D}_{p',\gamma}(I)},$$

and $X = [X'_s]'_s$ for $X \in \{\mathcal{Ces}_{p,\beta}(I), \mathcal{Cop}_{p,\gamma}(I)\}$.

[S1] V.D. Stepanov, *On Spaces Associated with Weighted Cesaro and Copson Spaces*, *Math. Notes*, **111:3** (2022), 470–477.

Theorem 3.2. *Let $1 < p < \infty$, $\beta > \frac{1}{p'}$ and $W_{p,\beta}^1(I)$ be Sobolev space defined above. Then*

$$\mathbf{J}_{W_{p,\beta}^1(I)}(g) \approx \|g\|_{\mathcal{Ces}_{p',\beta}(I)}.$$

Corollary 3.1. *If $X \in \{\mathring{W}_{p,\beta}^1(I), W_{p,\beta}^1(I)\}$, then $X'_s = \mathcal{Ces}_{p',\beta}(I)$ and*

$$\|g\|_{X'_s} = \mathbf{J}_{W_{p,\beta}^1(I)}(g) \approx \|g\|_{\mathcal{Ces}_{p',\beta}(I)}$$

for $g \in \mathcal{Ces}_{p',\beta}(I)$.

Theorem 3.3. *Let $1 < p < \infty$, $\beta = \alpha + 1 > \frac{1}{p'}$, $X = \mathring{W}_p^1(I)$. Then*

$$X'_w = \left\{ g \in L^1_{\text{loc}}([0, \infty)) : J_{\mathring{W}_p^1(I)}(g) < \infty \right\} = \mathcal{Ces}_{p',\beta}(I) \quad (8)$$

and $\|g\|_{X'_w} = J_{\mathring{W}_p^1(I)}(g) \approx \|g\|_{\mathcal{Ces}_{p',\beta}(I)}$ for $g \in \mathcal{Ces}_{p',\beta}(I)$.

4. $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ spaces

Let $X \subset L^1(\mathbb{R}^n)$ be the Hardy space with the norm

$$\|u\|_{H^1(\mathbb{R}^n)} := \|Mu\|_{L^1(\mathbb{R}^n)},$$

where Mu maximal function of u . Also, let BMO be the space of all function with bounded mean oscillation with the norm

$$\|f\|_{BMO} := \sup_{I \subset \mathbb{R}^n} \frac{1}{|I|} \int_I |f - f_I|.$$

Definition. $a \in \mathfrak{M}(\mathbb{R}^n)$ is an *atom* if there is a cube $Q \subset \mathbb{R}^n$ such that $\text{supp } a \subset Q$, $|Q| \|a\|_\infty \leq 1$, $\int_{\mathbb{R}^n} a = 0$. Then by Ch. Fefferman's theorem

(i) For any $f \in BMO(\mathbb{R}^n)$ $\exists!$ $T \in (H^1(\mathbb{R}^n))^*$ such that

$$T(a) = \int_{\mathbb{R}^n} af \quad \forall \text{ atoms } a. \quad (9)$$

Moreover,

$$\|T\|_{(H^1(\mathbb{R}^n))^*} \lesssim \|f\|_{BMO}.$$

(ii) For any $T \in (H^1(\mathbb{R}^n))^*$ $\exists f \in BMO(\mathbb{R}^n)$ such that (9) holds for all atoms and

$$\|f\|_{BMO} \lesssim \|T\|_{(H^1(\mathbb{R}^n))^*}.$$

(iii) There exist $g \in H^1(\mathbb{R}^n)$ and $f \in BMO(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |gf| = \infty.$$

Relations for associative norms of $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ we take from the recent paper [†]

Theorem 4.1. 1. $(H^1(\mathbb{R}^n))'_s = L^\infty(\mathbb{R}^n)$.

2. $(H^1(\mathbb{R}^n))'_w = (L^\infty(\mathbb{R}^n), \|\cdot\|_{BMO})$.

3. $((H^1(\mathbb{R}^n))'_w)'_s = \{0\}$.

4. $((H^1(\mathbb{R}^n))'_w)'_w = H^1(\mathbb{R}^n)$.

5. $(BMO(\mathbb{R}^n))'_s = \{0\}$,

6. $((BMO(\mathbb{R}^n))'_w)'_w = BMO(\mathbb{R}^n)$,

7. $((BMO(\mathbb{R}^n))'_w)'_s = L^\infty(\mathbb{R}^n)$.

Theorem 4.2. $(BMO(\mathbb{R}^n))'_w \subset H^1(\mathbb{R}^n)$ strictly and $\|u\|_{H^1(\mathbb{R}^n)} \lesssim \|u\|_{(BMO(\mathbb{R}^n))'_w}$.

Moreover, $(BMO(\mathbb{R}^n))'_w = \{u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |uf| < \infty \ \forall \ f \in BMO(\mathbb{R}^n)\}$.

[†]Prokhorov, D. V. On the associated spaces of the Hardy space. Dalnevost. Mat. Zh. 24 (2024), no. 1, 96–106.

5. Boundedness criterion for the Hilbert transform from a weighted Sobolev space to a weighted Lebesgue space

An important problem in harmonic analysis is characterization of weighted norm inequalities for the Hilbert operator.

$$Hf(x) = p.v. \int_0^\infty \frac{f(t)}{x-t} dt, \quad x > 0.$$

The first work in this direction was an article K.I. Babenko (1948), where it was shown the inequality

$$\left(\int_0^\infty |Hf(x)|^p x^\alpha dx \right)^{\frac{1}{p}} \leq C \left(\int_0^\infty |f(x)|^p x^\alpha dx \right)^{\frac{1}{p}}, \quad 1 < p < \infty, -1 < \alpha < p-1.$$

Further, this problem has been considered since the seventies of the last century, in [HMW] R. Hunt, B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, *Trans. Amer. Math. Soc.*, **176** (1973), 227–251.

was found that the Hilbert transform is bounded in weighted Lebesgue spaces L_w^p , $1 < p < \infty$, if and only if the weight w belongs to the Muckenhoupt class A_p .

M.T. Lacey, E.T. Sawyer, Shen, Chun-Yen, I. Uriarte-Tuero, Two-weight inequality for the Hilbert transform: a real variable characterization, I. *Duke Math. J.* 163 (2014), no. 15, 2795–2820.

M.T. Lacey, E.T. Sawyer, Shen, Chun-Yen, I. Uriarte-Tuero, —||—, II. *Duke Math. J.* 163 (2014), no. 15, 2821–2840. The case $p = q = 2$.

In this part we study operation of the Hilbert transform from weighted Sobolev space of the first order to weighted Lebesgue space on the semi-axis. Basing on the results of the first part we obtain an analog of Babenko's inequality.

Theorem 5.1. Let $1 < p \leq q < \infty$, $\beta > 1/p'$, $\gamma < 1/q'$,

$$W_{p,\beta}^1(I) := \{f \in AC(I) : \|f\|_{W_{p,\beta}^1} := \|f\|_{L_{x^\beta-1}^p} + \|f'\|_{L_{x^\beta}^p} < \infty\}.$$

Then

$$\|Hf\|_{L_{x^\gamma}^q} \lesssim \|f\|_{W_{p,\beta}^1}, \quad (10)$$

iff

$$\gamma + \frac{1}{q} + \frac{1}{p'} - \beta = 0. \quad (11)$$

Proof. (10) is equivalent to

$$\|Hf\|_{L_{x^\gamma}^q} \lesssim \|f'\|_{L_{x^\beta}^p}. \quad (12)$$

Put $f_\tau(s) := f(\tau s)$, $\tau > 0$, $s > 0$. Then

$$\begin{aligned} \|f'_\tau\|_{L_{x^\beta}^p} &= \tau^{\frac{1}{p'}-\beta} \|f'\|_{L_{x^\beta}^p}, \\ \|Hf_\tau\|_{L_{x^\gamma}^q} &= \tau^{-\frac{1}{q}-\gamma} \|Hf\|_{L_{x^\gamma}^q}. \end{aligned}$$

Thus,

$$\tau^{-\frac{1}{q}-\gamma} \|Hf\|_{L_{x^\gamma}^q} \lesssim \tau^{\frac{1}{p'}-\beta} \|f'\|_{L_{x^\beta}^p}$$

and (11) follows.

Sufficiency follows by applying **Theorem 3.3**.

Now we study a more general situation. In particular, we find sufficient conditions of the boundedness of $H : \overset{\circ\circ}{W}_p^1(I) \rightarrow L_w^q(I)$ for $1 < p, q < \infty$ and some classes of weights $v_0, v_1, w \in \mathfrak{M}^+(I)$.

We assume that weight functions v_0 and v_1 are such that

$$\|f\|_{p,v_0} \lesssim \|f'\|_{p,v_1}, \quad f \in \overset{\circ\circ}{W}_p^1(I). \quad (13)$$

The requirement (13) is satisfied if and only if

$$\mathbf{A}_0 := \sup_{(s,t) \subset I} \left(\int_s^t v_0^p \right)^{\frac{1}{p}} \left(\min \left\{ \int_0^s v_1^{-p'}, \int_t^\infty v_1^{-p'} \right\} \right)^{\frac{1}{p'}} < \infty.$$

In such a case

$$\|f\|_{W_p^1(I)} \approx \|f'\|_{p,v_1}, \quad (14)$$

i.e. the norm in the two-weighted Sobolev space $W_p^1(I)$ is equivalent to the norm $\|f'\|_{p,v_1}$ of "one-weighted Sobolev space". In the general case, obviously, the first space is strictly less than the second one.

Lemma 5.2. *Let $1 < p < \infty$ and weights v_0 and v_1 be such that*

$$\mathbf{A}_1 := \sup_{t \in I} \left(\int_0^t v_0^p \right)^{\frac{1}{p}} \left(\int_t^\infty v_1^{-p'} \right)^{\frac{1}{p'}} < \infty. \quad (15)$$

Then

$$Hf(x) = \int_0^\infty k(x,s)f'(s)ds =: Lf'(x), \quad f \in \overset{\circ}{\overset{\circ}{W}}_p^1(I), \quad (16)$$

where

$$k(x,s) = \int_0^s \frac{dt}{x-t} = \begin{cases} \log \frac{x}{x-s}, & 0 < s < x, \\ \log \frac{x}{s-x}, & x < s < 2x, \\ \log \frac{x}{s-x}, & s \geq 2x, \end{cases} \quad (17)$$

and

$$\begin{aligned} Lg(x) &= \int_0^x \log \frac{x}{x-s} g(s) ds + \int_x^{2x} \log \frac{x}{s-x} g(s) ds \\ &+ \int_{2x}^\infty \log \frac{x}{s-x} g(s) ds =: L_1 g(x) + L_2 g(x) - L_3 g(x). \end{aligned}$$

Hence, based on (14), (16), (17) and due to density $\overset{\circ\circ}{W}_p^1(I) \subset \overset{\circ}{W}_p^1(I)$, we have

$$\|H\|_{\overset{\circ\circ}{W}_p^1(I) \rightarrow L_w^q(I)} = \sup_{0 \neq f \in \overset{\circ\circ}{W}_p^1(I)} \frac{\|Hf\|_{q,w}}{\|f\|_{W_p^1(I)}} \approx \sup_{0 \neq f \in \overset{\circ}{W}_p^1(I)} \frac{\|Lf'\|_{q,w}}{\|f'\|_{p,v_1}}$$

and

$$\|H\|_{\overset{\circ}{W}_p^1(I) \rightarrow L_w^q(I)} = \|H\|_{\overset{\circ\circ}{W}_p^1(I) \rightarrow L_w^q(I)} \lesssim \sum_{i=1}^3 \|L_i\|_{L_{v_1}^p(I) \rightarrow L_w^q(I)}, \quad (18)$$

where H is extended on $\overset{\circ}{W}_p^1(I)$ by continuity.

The three theorems below give auxiliary two-sided estimates for the norms $\|L_i\|_{L_{v_1}^p(I) \rightarrow L_w^q(I)}$, $i = 1, 2, 3$, those are of independent interest as well.

The inequality $\|L_1 g\|_{q,w} \lesssim \|g\|_{p,v_1}$ is characterized under the condition $v_1 \in \mathfrak{M}^\uparrow$.

[AP] A.M. Abylayeva, L.-E. Persson, *Hardy type inequalities and compactness of a class of integral operators with logarithmic singularities*, *Math. Ineq. & Appl.*, **21:1** (2018), 201–215.

Lemma 5.3. *Let $1 < p < \infty$, $0 < q < \infty$, $1/r = (1/q - 1/p)_+$, $v_1 \in \mathfrak{M}^\uparrow$. Then the inequality*

$$\left(\int_0^\infty w^q(x) \left(\int_0^x \log \frac{x}{x-y} g(y) dy \right)^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (g v_1)^p \right)^{\frac{1}{p}}, \quad g \in \mathfrak{M}^+,$$

holds if and only if

(i) $A < \infty$ for $1 < p \leq q < \infty$, where

$$A := \sup_{t \in I} \left(\int_t^\infty \frac{w^q(x) dx}{x^q} \right)^{\frac{1}{q}} \left(\int_0^t \frac{x^{p'} dx}{v_1^{p'}(x)} \right)^{\frac{1}{p'}}, \quad (19)$$

and $C = \|L_1\|_{L_{v_1}^{p_1}(I) \rightarrow L_w^q(I)} \approx A$;

(ii) $B < \infty$ for $0 < q < p < \infty$ and $p > 1$, where

$$B := \left(\int_0^\infty \left(\int_t^\infty \frac{w^q(x) dx}{x^q} \right)^{\frac{r}{q}} \left(\int_0^t \frac{x^{p'} dx}{v_1^{p'}(x)} \right)^{\frac{r}{q'}} \frac{t^{p'}}{v_1^{p'}(t)} dt \right)^{\frac{1}{r}}, \quad (20)$$

and $C = \|L_1\|_{L_{v_1}^{p_1}(I) \rightarrow L_w^q(I)} \approx B$.

The following Theorem gives estimates for L_2 .

Lemma 5.4. *Let $1 < p < \infty$, $1 < q < \infty$, $1/r = (1/q - 1/p)_+$, $v_1 \in \mathfrak{M}^\uparrow$, and also there exists $\gamma > 0$ such that $x^{-\gamma}v_1(x) \in \mathfrak{M}^\downarrow$. Then the inequality*

$$\left(\int_0^\infty w^q(x) \left(\int_x^{2x} \log \frac{x}{y-x} g(y) dy \right)^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (gv_1)^p \right)^{\frac{1}{p}}, \quad g \in \mathfrak{M}^+,$$

is valid if and only if

$$\left(\int_0^\infty \frac{w^q(x)}{x^q} \left(\int_{\frac{3x}{2}}^{2x} (2x-y)g(y)dy \right)^q dx \right)^{\frac{1}{q}} \leq D \left(\int_0^\infty (gv_1)^p \right)^{\frac{1}{p}}, \quad g \in \mathfrak{M}^+, \quad (21)$$

and $\mathbb{A} < \infty$, where

$$\mathbb{A} := \begin{cases} \sup_{k \in \mathbb{Z}} \sup_{t \in [a^k, a^{k+1}]} \left(\int_{a^k}^t \omega^q \right)^{\frac{1}{q}} \left(\int_t^{a^{k+2}} u^{p'} \right)^{\frac{1}{p'}}, & p \leq q, \\ \left(\sum_{k \in \mathbb{Z}} \int_{a^k}^{a^{k+1}} \omega^q(x) \left(\int_{a^k}^x \omega^q \right)^{\frac{r}{p}} \left(\int_x^{a^{k+2}} u^{p'} \right)^{\frac{r}{p'}} dx \right)^{\frac{1}{r}}, & q < p, \end{cases} \quad (22)$$

$a \in (1, \sqrt{2})$, $\omega(x) := \frac{w(x)}{x^\gamma}$, $u(y) := \frac{y^\gamma}{v_1(y)}$. Moreover,

$$C = \|L_2\|_{L_{v_1^p}^p(I) \rightarrow L_w^q(I)} \approx D + \mathbb{A}.$$

Now we estimate the norm of the operator L_3 .

Lemma 5.5. *The inequality*

$$\left(\int_0^\infty w^q(x) \left(\int_{2x}^\infty \log \left(\frac{s-x}{x} \right) \frac{h(s)}{v_1(s)} ds \right)^q dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty h^p \right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}^+,$$

is equivalent to

$$\left(\int_0^\infty \frac{w^q(x)}{x^q} \left(\int_{2x}^{3x} (s-2x) \frac{h(s)}{v_1(s)} ds \right)^q dx \right)^{\frac{1}{q}} \leq D_1 \left(\int_0^\infty h^p \right)^{\frac{1}{p}} \quad (23)$$

and

$$\left(\int_0^\infty w^q(x) \left(\int_{3x}^\infty \log \left(\frac{s}{x} \right) \frac{h(s)}{v_1(s)} ds \right)^q dx \right)^{\frac{1}{q}} \leq D_2 \left(\int_0^\infty h^p \right)^{\frac{1}{p}}, \quad (24)$$

besides, $C = \|L_3\|_{L_{v_1}^p(I) \rightarrow L_w^q(I)} \approx D_1 + D_2$.

Remark 5.6. (a) Let $a(x) = 3x, k(y, x) = \log(y/x)$. Then

$$k(y, x) \approx k(y, z) + k(a(z), x), \quad 0 < x \leq z \leq a^{-1}(y) < \infty.$$

This means that the kernel $k(y, x) = \log(y/x)$ belongs to the Oinarov class $\mathbb{O}_a$ and

$$D_2 \approx \begin{cases} A_0 + A_1, & p \leq q, \\ B_0 + B_1, & q < p, \end{cases}$$

where

$$\begin{aligned}
A_0 &:= \sup_{t \in I} \left(\int_0^t \log^q \left(\frac{3t}{x} \right) w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{3t}^\infty v_1^{-p'} \right)^{\frac{1}{p'}}, \\
A_1 &:= \sup_{t \in I} \left(\int_0^t w^q \right)^{\frac{1}{q}} \left(\int_{3t}^\infty \log^{p'} \left(\frac{y}{t} \right) v_1^{-p'}(y) dy \right)^{\frac{1}{p'}}, \\
B_0 &:= \left(\left(\int_0^{t/3} \log^q \left(\frac{t}{x} \right) w^q(x) dx \right)^{\frac{r}{q}} \left(\int_t^\infty v_1^{-p'} \right)^{\frac{r}{q'}} v_1^{-p'}(t) dt \right)^{\frac{1}{r}}, \\
B_1 &:= \left(\left(\int_0^t w^q \right)^{\frac{r}{p}} \left(\int_{3t}^\infty \log^{p'} \left(\frac{y}{t} \right) v_1^{-p'}(y) dy \right)^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}.
\end{aligned}$$

(b) The inequalities (21) (constant D) and (23) (constant D_1) are characterized in

[SU] V.D. Stepanov, E.P. Ushakova, Kernel operators with variable intervals of integration in Lebesgue spaces and applications, *Math. Inequal. & Appl.*, **13**:3 (2010), 449–510.

We will formulate estimates from **[SU]** (Theorem 3.2 and Theorem 3.1) in the case when $p \leq q$. The case $q < p$ is also characterized, but in a discrete form, so we omit the details.

By applying Theorem 3.2 [SU] for the best constant D in (21), we find

$$D \approx \mathcal{A}_0 + \mathcal{A}_1, \quad p \leq q,$$

where

$$\begin{aligned} \mathcal{A}_0 &:= \sup_{t \in I} \sup_{\frac{3t}{2} < s < t} \left(\int_s^t (x-s)^q x^{-q} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\frac{3t}{2}}^{2s} v_1^{-p'} \right)^{\frac{1}{p'}}, \\ \mathcal{A}_1 &:= \sup_{t \in I} \sup_{\frac{3t}{2} < s < t} \left(\int_s^t x^{-q} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{\frac{3t}{2}}^{2s} (2s-y)^{p'} v_1^{-p'}(y) dy \right)^{\frac{1}{p'}}. \end{aligned}$$

Similarly, by applying Theorem 3.1 [SU] for the smallest constant D_1 in (23), we obtain

$$D_1 \approx \mathcal{A}_0^* + \mathcal{A}_1^*, \quad p \leq q,$$

where

$$\begin{aligned} \mathcal{A}_0^* &:= \sup_{t \in I} \sup_{s < t < \frac{3s}{2}} \left(\int_s^t (t-x)^q x^{-q} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{2t}^{3s} v_1^{-p'} \right)^{\frac{1}{p'}}, \\ \mathcal{A}_1^* &:= \sup_{t \in I} \sup_{s < t < \frac{3s}{2}} \left(\int_s^t x^{-q} w^q(x) dx \right)^{\frac{1}{q}} \left(\int_{2t}^{3s} (y-2t)^{p'} v_1^{-p'}(y) dy \right)^{\frac{1}{p'}}. \end{aligned}$$

The estimate

$$\|H\|_{\dot{W}_p^1(I) \rightarrow L_w^q(I)} = \|H\|_{\dot{W}_p^1(I) \rightarrow L_w^q(I)}^{\circ\circ} \lesssim \sum_{i=1}^3 \|L_i\|_{L_{v_1}^p(I) \rightarrow L_w^q(I)},$$

and the above Theorems imply the main result of this Section.

Theorem 5.7. *Let $1 < p, q < \infty$. Suppose that $\mathbf{A}_1 < \infty$ (see (15)), $v_1 \in \mathfrak{M}^\uparrow(I)$ and there exists $\gamma > 0$ such that $x^{-\gamma}v_1(x) \in \mathfrak{M}^\downarrow$. Then*

$$\|H\|_{\dot{W}_p^1(I) \rightarrow L_w^q(I)}^{\circ} \lesssim \mathbb{D} + \mathbb{A} + D + D_1 + D_2,$$

where $\mathbb{D} = A$ (see (19)) if $p \leq q$ and $\mathbb{D} = B$ (see (20)) if $q < p$, and constants $\mathbb{A}, D, D_1, D_2$ are determined in (22), (21), (23), (24), respectively. Explicit values of the constants D, D_1, D_2 are given in Remark 3.6.

The main results are contained in the following publications:

- [1] Prokhorov D.V., Stepanov V.D., Ushakova E.P., *On associate spaces of weighted Sobolev space on the real line*. *Math. Nachr.* 290 (2017), no. 5-6, 890–912.
- [2] Prokhorov D.V., Stepanov V.D., Ushakova E.P., *Characterization of the function spaces associated with weighted Sobolev spaces of the first order on the real line*, *Russian Math. Survey*, 2019, 74, № 6, 1075–1115.
- [3] Stepanov V.D., *On Spaces associated with weighted Cesàro and Copson spaces*, *Math. Notes*, 2022, 111, № 3, 148–155.
- [4] Prokhorov, D. V. *On the associated spaces for the altered Cesaro space*. *Anal. Math.* 48 (2022), no. 4, 1169–1183.
- [5] Prokhorov, D. V., *On the associated spaces of the Hardy space*. *Dalnevost. Mat. Zh.* 24 (2024), no. 1, 96–106.
- [6] Stepanov V.D., *On Cesàro and Copson type function spaces. Reflexivity*. *J. Math. Anal. Appl.*, 2022, № 1, 18 pp.
- [7] Prokhorov, D. V., *On the dual spaces for weighted altered Cesàro and Copson spaces*. *J. Math. Anal. Appl.*, 2022, no. 2, Paper No. 126325, 11 pp.
- [8] Stepanov V.D., *On the boundedness of the Hilbert transform from weighted Sobolev space to weighted Lebesgue space*, *J. Fourier Anal. Appl.*, 2022, 28, № 3, Paper No. 46, 17 pp.
- [9] Stepanov V.D., Ushakova E.P., *Strong and weak associativity of weighted Sobolev spaces of the first order*, *Russian Math. Survey*, 2023, 78, № 1, 167–204.
- [10] Stepanov V.D., *Weighted inequalities for the Hilbert transform on Sobolev classes*, *Dokl. Math.* 2026 (in press).

Thank you for your attention!