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INTERACTION BETWEEN ANALYSIS, ALGEBRA, AND LOGIC: THE WICKSTEAD PROBLEM

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Работа посвящена взаимодействию анализа, алгебры и математической логики на примере хорошо известной проблемы Викстеда об операторах, сохраняющих полосы, в универсально полных векторных решетках. Основной инструмент исследования — фундаментальная теорема Гордона, утверждающая, что интерпретация вещественных (комплексных) чисел в булевозначной модели теории множеств над полной булевой алгеброй $\mathbb{B}$ представляет собой вещественную (комплексную) универсально полную векторную решетку, у которой булева алгебра порядковых проекторов изоморфна $\mathbb{B}$. Приводятся также новые результаты о строении колец однородности аддитивных операторов, сохраняющих полосы линейных изоморфизмах, не являющихся порядковыми изоморфизмами, и классификации инъективных модулей.

Ключевые слова: векторная решетка, проблема Викстеда, теорема Гордона, функциональное уравнение, сохраняющий полосы оператор, булевозначная модель, булевозначный принцип переноса, булевозначные числа, дифференцирование, автоморфизм, рационально полное кольцо, инъективный модуль.

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The paper deals with an interaction between analysis, algebra, and mathematical logic by examining the well-known Wickstead problem for band preserving linear operators in universally complete vector lattices. The main tool is a fundamental Gordon's theorem stating that the interpretation of reals (complexes) in a Boolean valued model over a complete Boolean algebra $\mathbb{B}$ is a universally complete real (complex) vector lattice whose Boolean algebra of band projections is isomorphic to $\mathbb{B}$. Some new results on the structure of homogeneity rings of additive operators, band preserving linear isomorphisms, and a classification of injective modules are also presented.

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Keywords: vector lattice, Wickstead problem, functional equation, Gordon's theorem, band preserving operator, Boolean valued model, Boolean valued transfer principle, Boolean valued numbers, derivation, automorphism, rationally complete ring, injective module.

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INTERACTION BETWEEN ANALYSIS, ALGEBRA, AND LOGIC: THE WICKSTEAD PROBLEM

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1. Introduction

The *Boolean valued approach* is a machinery of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras. As these models, one usually takes the classical sets in the shape of the *von Neumann universe* $\mathbb{V}$ and a specially-trimmed *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ over a complete Boolean algebra $\mathbb{B}$ in which the conventional set-theoretic concepts and propositions acquire nonstandard interpretations.

A general scheme of applying the Boolean valued approach is as follows, see [24, 25]. Assume that $\mathbf{X} \subset \mathbb{V}$ and $\mathbb{X} \subset \mathbb{V}^{(\mathbb{B})}$ are two classes of mathematical objects, respectively *external* and *internal* with respect to a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ over a complete Boolean algebra $\mathbb{B}$. Suppose we are able to prove the following

Boolean Valued Representation Result: Every external $X \in \mathbf{X}$ embeds into a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ becoming an internal object $\mathscr{X} \in \mathbb{X}$.

Boolean Valued Transfer Principle then tells us that every theorem about $\mathscr{X}$ within Zermelo–Fraenkel set theory ZFC has its counterpart for the original object X interpreted as a Boolean valued object $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$.

Boolean Valued Machinery enables us to perform some translation of theorems from $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ making use of appropriate general operations and the principles of Boolean valued models.

The aim of this work is to demonstrate the power of the Boolean valued approach by treating the well-known Wickstead problem from the operator theory in vector lattices as well as to present some new results. The reader can find the necessary information on the theory of vector lattices in [1, 7]; Boolean valued analysis, in [8, 24]; field theory, in [11, 35]; functional equations in [6, 20]. We let $:=$ denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ symbolize the naturals, the rationals, the reals, and the complexes.

2. Analysis: The Wickstead Problem

Consider an arbitrary vector lattice E whose *positive cone* is denoted by E_+. Two elements $x, y \in E$ are said to be *disjoint* (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$. The *disjoint complement* of a nonempty set $M \subset E$ is defined as $M^\perp := \{x \in E : (\forall\, y \in M)\ x \perp y\}$ and the notation $M^{\perp\perp} := (M^\perp)^\perp$ is also in use. A set of the form $M^\perp$ is called a *band.* The inclusion ordered set $\mathbb{B}(E)$ of all bands in E is a *complete Boolean algebra.* A band B of E satisfying $E = B \oplus B^\perp$ is referred to as a *projection band,* while the associated projection (onto B parallel to $B^\perp$) is called a *band projection.* Let $\mathbb{P}(E)$ stand for the Boolean algebra of all band projections in E. Say that E has the *projection property* if every band of E admits a band projection. In this event the Boolean algebras $\mathbb{B}(E)$ and $\mathbb{P}(E)$ are isomorphic. The definitions and properties of complex vector lattice and complex f-algebra used below may be found in Abramovich and Aliprantis [1, Section 3.2], Meyer-Nieberg [28, Section 2.2], Schaefer [31, Chap. II, Section 11], and Zaanen [34, Sections 91 and 92].

Proposition 1. *For a linear operator $T : E \to E$ the following conditions are equivalent:*

(1) $Tx \in \{x\}^{\perp\perp}$ *for all* $x \in E$.

(2) $x \perp y$ *implies* $Tx \perp y$ *for all* $x, y \in E$.

(3) $T(K) \subset K$ *for each band* K *of* E.

If E *has the projection property then* (1)–(3) *are equivalen to*

(4) $\pi \circ T = T \circ \pi$ *for all band projections* π *in* E.

Definition 1. An operator T is called *band preserving* if one (and then any) of the conditions (1)–(4) is fulfilled and *order bounded* if T sends every order bounded set into an order bounded set.

Definition 2. A vector lattice E is said to be a *Wickstead lattice* if each band preserving linear operator in E is automatically order bounded.

Problem. *Describe the class of Wickstead lattices.*

In its full generality, the Wicksted problem is still unsettled, while it is very well understood for universally complete vector lattices. Therefore, we restrict our discussion to this class of vector lattices.

Definition 3. A vector lattice E is said to be *Dedekind complete* (respectively, *laterally complete*) whenever every non-empty order bounded set (respectively, every set of pairwise disjoint positive vectors) has a supremum. A vector lattice that is at the same time laterally complete and Dedekind complete is referred to as *universally complete.*

Definition 4. A vector lattice X is *locally one-dimensional*[1] if for every two nondisjoint $x_1, x_2 \in X$ there exist nonzero components u_1 and u_2 of x_1 and x_2 respectively such that u_1 and u_2 are proportional (see also Definition 11 below).

Definition 5. An element $x \in E_+$ is called *locally constant* with respect to $e \in E_+$ if $x = \sup_{\xi\in\Xi} \lambda_\xi \pi_\xi e$ for some numeric family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections in $\mathbb{P}(E)$. Recall also that $e \in E_+$ is a (*weak*) *order unit* if $\{e\}^\perp = \{0\}$ or, equivalently, $\{e\}^{\perp\perp} = E$.

Proposition 2. *For each universally complete vector lattice* E *the following conditions are equivalent:*

(1) E *is locally one-dimensional.*

(2) *All elements of* E_+ *are locally constant with respect to a fixed order unit.*

(3) *All elements of* E_+ *are locally constant with respect to every order unit.*

[1]The term *locally one-dimensional* is in use in Gutman [17], Kusraev and Kutateladze [21, 25], McPolin and Wickstead [27], while the term in Abramovich and Kitover [2] used for such spaces is *essentially one-dimensional.*

3. Historical Comments

The Wickstead problem was raised in [33] in 1977.[2] Soon afterward (in 1978), Abramovich, Veksler, and Koldunov [3, Theorem 1] announced the first example of a non order bounded band preserving linear operator. Later, it was clarified that the situation described in the paper is typical in a sense. The following result was established by Abramovich, Veksler, and Koldunov in [4, Theorem 2.1] and by McPolin and Wickstead in [27, Theorem 3.2].

Theorem 1. *All band preserving operators in a universally complete vector lattice are automatically order bounded if and only if this vector lattice is locally one-dimensional.*

This claim can be considered as a solution to the Wickstead problem. But the new notion of locally one-dimensional vector lattice crept into the answer. The novelty of this notion led to the conjecture that it coincides with that of a discrete (= atomic) vector lattice. In 1981 Abramovich, Veksler, and Koldunov [4, Theorem 2.1] gave a proof for existence of an order unbounded band preserving operator in every nondiscrete universally complete vector lattice, thus seemingly corroborating the conjecture that a locally one-dimensional vector lattice is discrete. But the proof was erroneous. Later, in 1985, McPolin and Wickstead [27, Section 3] gave an example of a nondiscrete locally one-dimensional vector lattice, confuting the conjecture. But again there was an error in the example. Finally, Wickstead [5] stated the conjecture as an open problem in 1993.

There was a similar misunderstanding in Boolean valued analysis. In a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over a complete Boolean algebra $\mathbb{B}$ there exist the internal real numbers object $\mathscr{R}$ and the standard real numbers object $\mathbb{R}^{\wedge}$. It seemed plausible that the equation $\mathbb{R}^{\wedge} = \mathscr{R}$ holds only for atomic Boolean algebras $\mathbb{B}$.

The problems were solved in 1995 by Gutman [17]: He constructed an atomless Dedekind complete locally one-dimensional vector lattice. Moreover, Gutman gave a purely algebraic description of locally one dimensional universally complete vector lattices. In more details, a universally complete vector lattice E is a Wickstead lattice if and only if $\mathbb{B} := \mathbb{P}(E)$ is a σ-distributive Boolean algebra if and only if $\mathbb{R}^{\wedge} = \mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$.

In 2004 Kusraev in [22] developed a Boolean valued approach to band preserving operators and Wickstead problem which revealed some new interconnections and possibilities. For example, the construction of an order unbounded band preserving operator can be carried out inside an appropriate Boolean valued universe by using a Hamel basis of the reals $\mathscr{R}$ considered as a vector space over its subfield $\mathbb{R}^{\wedge}$ (cp. [24, 25]). In particular, using a Hamel basis, one can construct an internal discontinuous $\mathbb{R}^{\wedge}$-linear function in $\mathscr{R}$ which gives an external order unbounded band preserving linear operator in any universally complete vector lattice E with $\mathbb{P}(E) = \mathbb{B}$. Similar constructions can be carried out on using a *transcendence basis* instead of a Hamel basis. This approach yielded new characterizations of universally

[2] In the early 2000s A. I. Veksler informed me that this problem has been formulated earlier by G. Y. Lozanovskiĭ in Leningrad mathematical seminars.

complete vector lattices with σ-distributive base in terms of narrower classes of band preserving linear operators, namely, of derivations and automorphisms [23].

It should be also mentioned that the Wickstead problem admits different answers depending on the spaces in which the operators in question are considered. There are several results that guarantee automatic boundedness for a band preserving operator in the particular classes of vector lattices. According to Abramovich, Veksler, and Koldunov [4, Theorem 2.1] (see also [3, 4]) every band preserving operator from a Banach lattice to a normed vector lattice is bounded. This claim remains valid if the Banach lattice of departure is replaced by a relatively uniformly complete vector lattice [4]. In McPolin and Wickstead [27] a similar result is obtained for the band preserving operators in a relatively uniformly complete vector lattice endowed with a locally convex locally solid topology.

Versions of the Wickstead problem can be seen in [25, § 4.14].

4. Analysis: Functional Equations

The classical *Cauchy functional equation* (CFE, for short) is the equation

$$f(x+y) = f(x) + f(y) \quad (x, y \in \mathbb{R}), \tag{1}$$

where f is an unknown function $f : \mathbb{R} \to \mathbb{R}$ which is assumed to satisfy (1) for all $x, y \in \mathbb{R}$. The solutions to CFE are called *additive functions*.

Proposition 3. *The solution set of the* CFE *coincides with the set of all endomorphisms of* $\mathbb{R}$ *considered as a vector space over* $\mathbb{Q}$.

In particular, each additive function $f : \mathbb{R} \to \mathbb{R}$ satisfies obviously also

$$f(rx) = rf(x) \quad (r \in \mathbb{Q},\ x \in \mathbb{R}).$$

Therefore, it is interesting to examine a more general situation. Let $\mathbb{F}$ stand for either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathbb{P}$ be a subfield of $\mathbb{F}$. In case $\mathbb{F} = \mathbb{C}$ we assume $i \in \mathbb{P}$ so that $\mathbb{Q} + i\mathbb{Q}$ is a subfield of $\mathbb{P}$. Denote by $\mathbb{F}_{\mathbb{P}}$ the field $\mathbb{F}$ considered as a vector space over $\mathbb{P}$. Consider now the simultaneous functional equations

$$\begin{cases} f(x+y) = f(x) + f(y), \\ f(px) = pf(x) \end{cases} \tag{L}$$

with $p \in \mathbb{P}$ and $x, y \in \mathbb{F}$. Clearly, the solutions to the simultaneous equations (L) are precisely $\mathbb{P}$-linear functions from $\mathbb{F}_{\mathbb{P}}$ to $\mathbb{F}_{\mathbb{P}}$. The following two results were obtaind by Hamel in 1905 for $\mathbb{P} = \mathbb{Q}$.

Theorem 2. *Let* $\mathscr{E}$ *be a Hamel basis for a vector space* $\mathbb{F}_{\mathbb{P}}$*, and let* $\mathscr{F}(\mathscr{E}, \mathbb{F})$ *be the space of all functions from* $\mathscr{E}$ *to* $\mathbb{F}$*. The solution set of* (L) *is a vector space over* $\mathbb{F}$ *isomorphic to* $\mathscr{F}(\mathscr{E}, \mathbb{F})$*. Such an isomorphism can be implemented by sending a solution* f *to the restriction* $f|_{\mathscr{E}}$ *of* f *to* $\mathscr{E}$*. The solution* f *is continuous if and only if* $f(x)/x = \mathrm{const}$ $(x \in \mathscr{E})$.

Theorem 3. *If* $\mathbb{F} \neq \mathbb{P}$*, then there exist discontinuous solutions of* (L)*. In particular, there exist discontinuous solutions to* CFE.

$\lhd$ The assumption $\mathbb{F} \neq \mathbb{P}$ implies that each Hamel basis $\mathscr{E}$ for the vector space $\mathbb{F}_{\mathbb{P}}$ contains at least two nonzero distinct elements $e_1, e_2 \in \mathscr{E}$. Define the function $\tau : \mathscr{E} \to \mathbb{F}$ so that $\tau(e_1)/e_1 \neq \tau(e_2)/e_2$ (for example, $\tau(\mathscr{E}) \subset \mathbb{Q}$). Then the $\mathbb{P}$-linear function $f = f_\tau : \mathbb{F} \to \mathbb{F}$, coinciding with τ on $\mathscr{E}$, exists and is discontinuous by Theorem 2. $\rhd$

Corollary 1. *If $\mathbb{F}$ and $\mathbb{P}$ are as above, then $\mathbb{F} = \mathbb{P}$ if and only if each linear function from $\mathbb{F}_{\mathbb{P}}$ to $\mathbb{F}_{\mathbb{P}}$ sends bounded sets into bounded sets.*

5. Algebra: Derivations and Endomorphisms of Fields

Consider two more systems of functional equations (with $x, y \in \mathbb{F}$ and $p \in \mathbb{P}$):

$$\begin{cases} f(x+y) = f(x) + f(y), \\ f(px) = pf(x), \\ f(xy) = f(x)f(y), \end{cases} \tag{A}$$

$$\begin{cases} f(x+y) = f(x) + f(y), \\ f(px) = pf(x), \\ f(xy) = f(x)y + xf(y). \end{cases} \tag{D}$$

Definition 6. The solutions of (A) are called $\mathbb{P}$-*endomorphisms* of $\mathbb{F}$, while the solutions of (D) are named $\mathbb{P}$-*derivations* of $\mathbb{F}$. A bijective $\mathbb{P}$-endomorphisms are called $\mathbb{P}$-*automorphisms* of $\mathbb{F}$. The identity automorphism and the zero derivation are conventionally called *trivial.*

For examining the systems (A) and (D) more subtle tool than Hamel basis, namely a *transcendence basis,* is needed (see the proof of Theorem 5 below). By Steinitz Theorem each extension $\mathbb{F}$ of a field $\mathbb{P}$ has a transcendence basis $\mathscr{E}$ over K. In this event $\mathbb{F}$ is an algebraic extension of the pure extension $\mathbb{P}(\mathscr{E})$ of K, see Bourbaki [11, Chapter 5, Section 5, Theorem 1].

Theorem 4. *Let $\mathbb{F}$ be a field of characteristic zero, $\mathbb{P}$ a subfield of $\mathbb{F}$, $\mathscr{E}$ a transcendental base of $\mathbb{F}$ over $\mathbb{P}$, if it exists, and $\mathscr{E} = \varnothing$ otherwise. If $d : \mathbb{P} \to \mathbb{F}$ is a derivation and $\phi : \mathscr{E} \to \mathbb{F}$ is an arbitrary function, then there exists a unique derivation $D : \mathbb{F} \to \mathbb{F}$ with $D|_{\mathbb{P}} = d$ and $D|_{\mathscr{E}} = \phi$.*

$\lhd$ See [11, Chap. V, § 9, Propositions 4 and 5], [20, Theorem 14.2.1]. $\rhd$

Theorem 5. *Let $\mathbb{C}$ be an extension of an algebraically closed subfield $\mathbb{P} = \mathbb{P}_0 + i\mathbb{P}_0$ with $\mathbb{P}_0$ a subfield of $\mathbb{R}$. Then the following are equivalent:*

(1) $\mathbb{P} = \mathbb{C}$.

(2) *There is no nontrivial $\mathbb{P}$-derivation of $\mathbb{C}$.*

(3) *Every $\mathbb{P}$-endomorphism of $\mathbb{C}$ is the zero or the identity function.*

(4) *There is no nontrivial $\mathbb{P}$-automorphism of $\mathbb{C}$.*

$\lhd$ If $\mathbb{P} = \mathbb{C}$ then every $\mathbb{P}$-linear function $f : \mathbb{C} \to \mathbb{C}$ is of the form $f(z) = cz$ $(z \in \mathbb{C})$ for some $c \in \mathbb{C}$; therefore (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) trivially. The implication (3) $\Longrightarrow$ (4) is evident. If f is multiplicative then $c^2 = c$ and hence $c = 0$

or $c = 1$, whence (1) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (5). Prove (3) $\Longrightarrow$ (1). Given a mapping $d : \mathscr{E} \to \mathbb{C}$, Theorem 4 yields a unique derivation $D_0 : \mathbb{P}(\mathscr{E}) \to \mathbb{C}$ with $D_0(e) = d(e)$ ($e \in \mathscr{E}$). Since $\mathbb{C}$ is an algebraic extension of $\mathbb{P}(\mathscr{E})$, we may apply Theorem 4 again and find a unique extension of D_0 to some derivation $D : \mathbb{C} \to \mathbb{C}$. The freedom in the choice of d guarantees that D is nontrivial. The implication (4) $\Longrightarrow$ (1) can be proved in a similar way making use of the isomorphism extension theorem (see [11, Chap. V, § 6, Proposition 1]) instead of Theorem 4. $\rhd$

6. Algebra: σ-Distributive Boolean Algebras

Denote $\Phi := \mathbb{N}^{\mathbb{N}}$, i. e., Φ stands for the set of all mappings $\varphi : \mathbb{N} \to \mathbb{N}$.

Definition 7. A σ-complete Boolean algebra $\mathbb{B}$ is said to be *σ-distributive* if for every double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{B}$ we have [32]:

$$\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b_{n,m} = \bigwedge_{\varphi\in\Phi} \bigvee_{n\in\mathbb{N}} b_{n,\varphi(n)}.$$

Examples. (1) The powerset algebra $\mathscr{P}(A)$ with nonempty A is σ-distributive. Moreover, $\mathscr{P}(A)$ is completely distributive, i. e., the above equation holds for $\Phi := A^B$ for arbitrary nonempty A and B.

(2) Let (Ω, Σ, μ) be a measure space with the direct sum property (see [21, 1.1.7 and 1.1.8]). The Boolean algebra $\mathbb{B} := \mathbb{B}(\Omega, \Sigma, \mu)$ of *measurable sets modulo μ-negligible sets* is σ-distributive if and only if $\mathbb{B}$ is atomic, or, equivalently, μ is atomic. In this event $\mathbb{B}$ is isomorphic to $\mathscr{P}(A)$ for some nonempty A [19, 5.3.3].

(3) Let $\mathbb{B}$ be the powerset $\mathscr{P}(\mathbb{N})$ modulo finite sets, that is $\mathbb{B} := \mathscr{P}(\mathbb{N})/\mathscr{I}$ where $\mathscr{I}$ is the ideal of finite subsets of $\mathbb{N}$. Then the *Dedekind completion* of $\mathbb{B}$ is an atomless σ-distributive Boolean algebra [17].

The following result is due to Gutman [17].

Theorem 6. *The following assertions hold:*

(1) *The vector lattice $L^0 := L^0(\Omega, \Sigma, \mu)$ is atomic (or equivalently, the measure μ is atomic) if and only if $\mathbb{P}(L^0)$ is σ-distributive.*

(2) *There exists an atomless σ-distributive complete Boolean algebra.*

$\lhd$ Follows from the examples (2) and (3). $\rhd$

7. Logic: Boolean Valued Models

Let $\mathbb{B}$ be a complete Boolean algebra. The Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ is defined by recursion on $\alpha \in \mathrm{On}$

$$\mathbb{V}_\alpha^{(\mathbb{B})} = \left\{ x : \ (\exists\, \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B},\ \mathrm{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \right\},$$

$$\mathbb{V}^{(\mathbb{B})} := \bigcup\nolimits_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})} \quad (\mathrm{On}\ \text{ is the class of all ordinals}).$$

For making statements about $\mathbb{V}^{(\mathbb{B})}$ take a formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of set theory ($\equiv$ ZFC) and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in$

$\mathbb{V}^{(\mathbb{B})}$. Then $\varphi(x_1,\dots,x_n)$ is a statement about $x_1,\dots,x_n$. To verify whether or not $\varphi(x_1,\dots,x_n)$ is true within $\mathbb{V}^{(\mathbb{B})}$, there is a natural way of assigning to each such statement an element $[\![\varphi(x_1,\dots,x_n)]\!]\in\mathbb{B}$ which acts as the *'Boolean truth-value'* of $\varphi(x_1,\dots,x_n)$ in the universe $\mathbb{V}^{(\mathbb{B})}$. We say that the statement $\varphi(x_1,\dots,x_n)$ *is valid within* $\mathbb{V}^{(\mathbb{B})}$ and write $\mathbb{V}^{(\mathbb{B})}\models\varphi(x_1,\dots,x_n)$ if $[\![\varphi(x_1,\dots,x_n)]\!]=\mathbb{1}$.

Theorem 7 (Transfer Principle)**.** *All theorems of Zermelo–Fraenkel set theory with choice (ZFC, for short) are true within $\mathbb{V}^{(\mathbb{B})}$. More precisely, if $\varphi(u_1,\dots,u_n)$ is a theorem of ZFC then "$[\![\varphi(x_1,\dots,x_n)]\!]=\mathbb{1}$ for all $x_1,\dots,x_n\in\mathbb{V}^{(\mathbb{B})}$" is also a theorem of ZFC.*

Given an arbitrary $X\in\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}$ as the set $X{\downarrow}:=\{x\in\mathbb{V}^{(\mathbb{B})}:[\![x\in X]\!]=\mathbb{1}\}$. Assume that $X,Y,f,P\in\mathbb{V}^{(\mathbb{B})}$ are such that $[\![f:X\to Y]\!]=\mathbb{1}$ and $[\![P\subset X^2]\!]=\mathbb{1}$, i.e., f is a mapping from X to Y and P is a binary relation on X within $\mathbb{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ for which $[\![f{\downarrow}(x)=f(x)]\!]=\mathbb{1}$ $(x\in X{\downarrow})$ and $P{\downarrow}$ is a unique binary relation on $X{\downarrow}$ with $(x_1,x_2)\in P{\downarrow}\iff[\![(x_1,x_2)\in P]\!]=\mathbb{1}$. The *descent of an algebraic structure* is the descent of an underlying set endowed with the descended operations and relations.

The *ascent* is a transformation acting in the reverse direction, i.e., sending any subset $X\subset\mathbb{V}^{(\mathbb{B})}$ into an element $X{\uparrow}\in\mathbb{V}^{(\mathbb{B})}$. One more important these transformation is the *canonical embedding* $X\mapsto X^{\wedge}$ of the class of standard sets ($\equiv$ *von Neumann universe*) $\mathbb{V}$ into a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Ascent, descent, and canonical embedding enables one to perform an interaction between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$, see [8, 24].

8. Logic: Boolean Valued Numbers

Let $\mathscr{R}$ and $\mathscr{C}$ stand respectively for the fields of reals and complexes within $\mathbb{V}^{(\mathbb{B})}$ i.e., $\mathscr{R}:=(\mathrm{R},\oplus,\odot,0,1,\leqq)$ and $[\![\varphi(\mathscr{R})]\!]=\mathbb{1}$, where $\varphi(\mathscr{R})$ is the conjunction of axioms of the reals, while $\mathscr{C}=\mathscr{R}+i\mathscr{R}$. Consider the descent $\mathbf{R}:=\mathscr{R}{\downarrow}$ of the algebraic structure $\mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$. In other words, $\mathbf{R}:=(\mathrm{R}{\downarrow},\oplus{\downarrow},\odot{\downarrow},\leqq{\downarrow},0,1)$ is considered as the descent $\mathrm{R}{\downarrow}$ of the underlying set R together with the descended operations $\oplus{\downarrow}$ and $\odot{\downarrow}$ and the descended order relation $\leqq{\downarrow}$ of the structure $\mathscr{R}$. By definition, $\mathscr{C}{\downarrow}:=\mathscr{R}{\downarrow}+i\mathscr{R}{\downarrow}$.

The following fundamental result due to Gordon [14] tells us that the interpretation of reals (complexes) in a Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ is a universally complete real (complex) vector lattice with the Boolean algebra of band projections isomorphic to $\mathbb{B}$.

Theorem 8. *The algebraic structure $\mathscr{R}{\downarrow}$ (respectively, $\mathscr{C}{\downarrow}$) with the descended operations and order relation is a universally complete real (respectively, complex) vector lattice and a semiprime f-algebra with a ring and order unit $\mathbb{1}:=1^{\wedge}$. Moreover, within $\mathbb{V}^{(\mathbb{B})}$, $\mathbb{R}^{\wedge}$ and $\mathbb{C}^{\wedge}$ are dense subfields of $\mathscr{R}$ and $\mathscr{C}$, respectively.*

The following result is due to Gutman [17].

Theorem 9. *Let $\mathbb{B}$ be a complete Boolean algebra and $\mathscr{C}$ the field of complexes within $\mathbb{V}^{(\mathbb{B})}$. The following assertions are equivalent:*

(1) $\mathbb{B}$ *is* σ*-distributive.*

(2) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^{\wedge}$.

(3) $\mathscr{R}{\downarrow}$ *is locally one-dimensional.*

Remark 1. Since the relations $\mathscr{R} = \mathbb{R}^{\wedge}$ and $\mathscr{C} = \mathbb{C}^{\wedge}$ are equivalent within $\mathbb{V}^{(\mathbb{B})}$, Theorem 9 implies that $\mathbb{B}$ is σ-distributive if and only if $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathbb{C}^{\wedge}$ if and only if $\mathscr{C}{\downarrow}$ is a locally one-dimensional universally complete vector lattice.

9. An Interaction

Definition 8. Let A be an algebra over a field $\mathbb{P}$. A $\mathbb{P}$-linear operator D from a subalgebra A_0 to A is called a $\mathbb{P}$-*derivation* (*real derivation* or *complex derivation* if $\mathbb{P} = \mathbb{R}$ or $\mathbb{P} = \mathbb{C}$, respectively) provided that $D(uv) = D(u)v + uD(v)$ for all $u, v \in A_0$. A nonzero derivation is referred as *nontrivial.*

Definition 9. A $\mathbb{P}$-*endomorphism of an algebra* A is a $\mathbb{P}$-linear multiplicative operator $M : A \to A$, i.e., A is $\mathbb{P}$-linear and satisfies the equation $M(uv) = M(u)M(v)$ for all $u, v \in A$. A bijective $\mathbb{P}$-endomorphism is a $\mathbb{P}$-*automorphism.* The identical automorphism is commonly referred to as the *trivial automorphism.* If $\mathbb{P} = \mathbb{R}$ or $\mathbb{P} = \mathbb{C}$ then we speak of *real* or *complex automorphisms.*

Let G be a real universally complete vector lattice with a fixed f-algebra multiplication and X an f-subalgebra of G.

Proposition 4. *Let* D *be a linear operator from* $X_{\mathbb{C}} := X \oplus iX$ *to* $G_{\mathbb{C}} := G \oplus iG$ *and* $D = D_1 + iD_2$*. Then* D *is a complex derivation if and only if* D_1 *and* D_2 *are real derivations from* X *into* G*. If* $X^{\perp\perp} = G$ *then each derivation from* $X_{\mathbb{C}}$ *into* $G_{\mathbb{C}}$ *is a band preserving operator.*

Proposition 5. *An order bounded derivation and an order bounded band preserving automorphism of a universally complete* f*-ring* $G_{\mathbb{C}}$ *are trivial.*

Consider two internal objects, the elements of $\mathbb{V}^{(\mathbb{B})}$: the sets of all $\mathbb{C}^{\wedge}$-derivations $\mathscr{D}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ and all $\mathbb{C}^{\wedge}$-endomorphisms $\mathscr{M}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ of $\mathscr{C}$. Define also two external objects: Let $\mathscr{D}(\mathscr{C}{\downarrow})$ be the set of all complex derivations on the f-algebra $\mathscr{C}{\downarrow}$ and let $\mathscr{M}_{bp}(\mathscr{C}{\downarrow})$ be the set of all complex band preserving endomorphisms of $\mathscr{C}{\downarrow}$.

Theorem 10. *Assume* $\tau \in \mathbb{V}^{(\mathbb{B})}$ *is an internal derivation (automorphism), that is,* $\tau \in \mathscr{D}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ *(respectively,* $\tau \in \mathscr{M}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$*). Then there exists a unique complex derivation* $T \in \mathscr{D}(\mathscr{C}{\downarrow})$ *(respectively, band preserving automorphism* $T \in \mathscr{M}_{bp}(\mathscr{C}{\downarrow})$*) such that* $[\![\tau(x) = T(x)]\!] = \mathbb{1}$ *for all* $x \in \mathscr{C}{\downarrow}$*. The correspondence* $\tau \leftrightarrow T$ *is a bijection between* $\mathscr{D}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ *and* $\mathscr{D}(\mathscr{C}{\downarrow})$ *as well as between* $\mathscr{M}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ *and* $\mathscr{M}_{bp}(\mathscr{C}{\downarrow})$*.*

Theorem 10 reduces the study of derivations and band preserving endomorphisms of a universally complete vector lattice respectively to that of the solution sets of simultaneous functional equations (D) and (A) with $\mathbb{F} = \mathscr{C}$ and $\mathbb{P} = \mathbb{C}^{\wedge}$.

10. The Results

Interpreting Theorem 5 and Corollary 1 in an appropriate Boolean valued model and combining Gordon's theorem with Theorems 7–10 yields the following:

Theorem 11. *Let X be a real universally complete vector lattice, $X_{\mathbb{C}} := X \oplus iX$, $\mathbb{B} := \mathbb{P}(X)$, while $\mathscr{R}$ and $\mathscr{C}$ be the fields respectively of reals and complexes within $\mathbb{V}^{(\mathbb{B})}$. The following assertions are equivalent:*

(1) *X is a Wickstead lattice.*

(2) *$\mathbb{B}$ is σ-distributive.*

(3) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{R} = \mathbb{R}^{\wedge}$.

(4) $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathbb{C}^{\wedge}$.

(5) *X is locally one-dimensional.*

(6) *$X_{\mathbb{C}}$ is locally one-dimensional.*

(7) *X has no nontrivial derivation.*

(8) *$X_{\mathbb{C}}$ has no nontrivial derivation.*

(9) *$X_{\mathbb{C}}$ has no nontrivial band preserving automorphism.*

(10) *Band preserving endomorphisms of $X_{\mathbb{C}}$ are precisely band projections.*

$\lhd$ The equivalence of (1)–(6) follows from Theorems 1 and 9. By Transfer Principle, Theorem 5 is true within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{C} := \mathscr{C}$ and $\mathbb{P} := \mathbb{C}^{\wedge}$, so that we have within $\mathbb{V}^{(\mathbb{B})}$ (taking into account that $\mathbb{C}^{\wedge}$ is algebraically closed in $\mathscr{C}$, see [25, Theoem 4.12.1])

$$\mathscr{C} = \mathbb{C}^{\wedge} \iff \mathscr{D}_{\mathbb{C}^{\wedge}}(\mathscr{C}) = \{0\} \iff \mathscr{M}_{\mathbb{C}^{\wedge}}(\mathscr{C}) = \{0, I_{\mathscr{C}}\}.$$

By Transfer Principle, Theorem 10 is true within $\mathbb{V}^{(\mathbb{B})}$, consequently, $\mathscr{C} = \mathbb{C}^{\wedge}$ is equivalent to both $\mathscr{D}(\mathscr{C}{\downarrow}) = \{0\}$ and $\mathscr{M}_{bp}(\mathscr{C}{\downarrow}) = \{0, I_{\mathscr{C}\downarrow}\}$. Using Gordon's theorem we can replace $\mathscr{C}{\downarrow}$ by $X_{\mathbb{C}}$. It follows that (6) $\Longrightarrow$ (8) and (6) $\Longrightarrow$ (10). Similarly, (6) $\Longrightarrow$ (8). The converse implications as well as (9) $\iff$ (10) are trivial. $\rhd$

Definition 10. A derivation D (respectively, an automorphism A) on X is called *essentially nontrivial* provided that $\pi D = 0$ (respectively, $\pi A = \pi I_X$) imply $\pi = 0$ for every band projection $\pi \in \mathbb{P}(X)$.

Corollary 2. *Let (Ω, Σ, μ) be an atomless Maharam measure space and $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$ the space of all (cosets of) measurable complex-valued functions on Ω. Then the following hold:*

(1) *There exists an essentially nontrivial $\mathbb{C}$-derivation on $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$.*

(2) *There exists an essentially nontrivial band preserving automorphism of $L^0_{\mathbb{C}}(\Omega, \Sigma, \mu)$.*

$\lhd$ This is immediate from Theorem 6 (1) and Theorem 11 (5, 6). $\rhd$

Remark 2. (1) The equivalences (4) $\iff$ (5) $\iff$ (6) $\iff$ (7) in Theorem 11 and Corollary 2 are due to Kusraev [23]. Detailed presentation of these results as well as other interesting properties of band preserving operators can be found in Kusraev and Kutatelsdze [25, Chap. 4].

(2) Ber, Chilin, and Sukochev [9] proved independently that the algebra $L^0_{\mathbb{C}}([0,\, 1])$ admits nontrivial derivations. Some extensions of this result and interesting related questions are discussed in Ber, de Pagter, and Sukochev [10].

(3) It is well known that if Q is a compact space then there are no nontrivial derivations on the algebra $C(Q,\mathbb{C})$ of complex-valued continuous functions on Q, see Aczél and Dhombres [6, Chapter 19, Theorem 21]. At the same time, we see from Theorem 11 that if Q is an extremally disconnected compact space and the Boolean algebra of the clopen sets of Q is not σ-distributive then there is a nontrivial derivation on $C_\infty(Q,\mathbb{C})$.

(4) Using the same arguments as above, we can infer that in the class of universally complete real vector lattices with a fixed structure of an f-algebra the absence of nontrivial derivations is equivalent to the σ-distributivity of the Boolean algebra of bands of the algebra under consideration. At the same time there are no nontrivial band preserving automorphisms of the f-algebra $\mathscr{R}{\downarrow}$.

11. Variations on the Theme

A. Homogeneity rings of additive operators. Let K be a ring with an identity $1 \neq 0$ and M, L two unitary K-modules. Then, for any additive mapping $f : M \to L$, the set $H_f := \{a \in K : \ f(ax) = af(x) \text{ for all } x \in M\}$ forms a subring of K, the *homogeneity ring* of f. It is proved by Rätz [30] that, for $M \neq \{0\}$, $L \neq \{0\}$ and any subring S of K for which M is a free S-module, there exists an additive mapping $f : M \to L$ such that $H_f = S$. In particular, the following is true:

Proposition 6. *If $\mathbb{F}$ is an arbitrary subfield of $\mathbb{R}$ then there exists an additive function $f : \mathbb{R} \to \mathbb{R}$ such that $H_f = \mathbb{F}$.*

Consider an f-algebra A. Given an additive operator $S : A \to A$, define the homogeneity set $H_S \subset A$ of S as $H_S := \{a \in A : \ S(ax) = aSx \text{ for all } x \in A\}$. Then H_S is evidently a subring of A and our problem is to examine what subrings of A have the form H_S for some additive operator S in A?

Definition 11. Let E be a vector lattice and $x \in E$. An element $y \in E$ is called a *component* of x if $|y| \wedge |x-y| = 0$. The collection of all components of x is denoted by $\mathfrak{C}(x)$. A subspace X_0 of X is *component-wise closed* if, for each $x \in X_0$, the set $\mathfrak{C}(x)$ is contained in X_0.

Definition 12. An annihilator ideal of K is a subset of the form $S^\perp := \{k \in K : (\forall\, s \in S)\ ks = 0\}$ with a nonempty subset $S \subset K$. A subset S of K is called *dense* provided that $S^\perp = \{0\}$; i. e., the equality $k \cdot S := \{k \cdot s : \ s \in S\} = \{0\}$ implies $k = 0$ for all $k \in K$. A ring K is said to be *rationally complete* whenever, to each dense ideal $J \subset K$ and each group homomorphism $h : J \to K$ such that $h(kx) = kh(x)$ for all $k \in K$ and $x \in J$, there is an element r in K with $h(x) = rx$ for all $x \in J$.

Observe that K is rationally complete if and only if the complete ring of quotients $Q(K)$ is isomorphic to K canonically, see Lambek [26, § 2.3]. The following result is due to Gordon [15, 16].

Theorem 12. *If $\mathscr{K}$ is a field within $\mathbb{V}^{(\mathbb{B})}$ then $\mathscr{K}{\downarrow}$ is a rationally complete semiprime ring, and there is an isomorphism χ of $\mathbb{B}$ onto the Boolean algebra $\mathbb{A}(\mathscr{K}{\downarrow})$ of the annihilator ideals of $\mathscr{K}{\downarrow}$ such that*

$$b \leqslant [\![x = 0]\!] \iff x \in \chi(b^*) \quad (x \in K,\ b \in \mathbb{B}).$$

Conversely, if K is a rationally complete semiprime ring and $\mathbb{B}$ stands for the Boolean algebra $\mathbb{A}(K)$ of all annihilator ideals of K, then there is an internal field $\mathscr{K} \in \mathbb{V}^{(\mathbb{B})}$ such that the ring K is isomorphic to $\mathscr{K}{\downarrow}$.

$\lhd$ See [24, Theorem 8.3.1] and [24, Theorem 8.3.2]. $\rhd$

Theorem 13. *Let A be a universally complete semiprime f-algebra and let K be a componentwise closed rationally complete subring of A with $A = K^{\perp\perp}$. Then there exists a band preserving additive operator S in A such that $H_S = K$.*

$\lhd$ By Gordon's theorem we can assume that $A = \mathscr{R}{\downarrow}$. Our assumptions that K is componentwise closed in A and $A = K^{\perp\perp}$ imply that K is an f-subalgebra and the mapping $B \mapsto B \cap K$ is a Boolean isomorphism of $\mathbb{B}(A)$ onto $\mathbb{B}(K)$. Recall that the annihilator ideals in a commutative semiprime ring R form a complete Boolean algebra $\mathbb{A}(R)$, see [26, § 2.4, Proposition 2]. Since A and K are semiprime f-algebras, we have $\mathbb{A}(K) = \mathbb{B}(K)$ and $\mathbb{A}(A) = \mathbb{B}(A)$. By Theorem 12, there exists a subfield $\mathscr{F} \subset \mathscr{R}$ within $\mathbb{V}^{(\mathbb{B})}$ such that $K = \mathscr{F}{\downarrow}$. The Transfer Principle (Theorem 7) guarantees that Proposition 6 is true within $\mathbb{V}^{(\mathbb{B})}$, so that there exists an additive function $\sigma : \mathscr{R} \to \mathscr{R}$ with $H_\sigma = \mathscr{F}$. Put $S := \sigma{\downarrow}$ and note that $H_S = H_\sigma{\downarrow}$. It follows that $H_S = K$. $\rhd$

B. Band preserving linear isomorphisms. Abramovich and Kitover raised the question in [2, p. 1, Problem B] as to whether the vector lattices E and F are lattice isomorphic whenever there exists a linear disjointness preserving operator $T : E \to F$ such that T^{-1} is also disjointness preserving? A negative answer was given in the same work, see [2, Theorem 13.4]. Below we demonstrate that this problem has a negative solution even in the class of band preserving operators.

Proposition 7. *Let $\mathbb{P}$ be a proper subfield of $\mathbb{R}$. There exists a $\mathbb{P}$-linear subspace $\mathscr{X}$ in $\mathbb{R}$ such that $\mathscr{X}$ and $\mathbb{R}$ are isomorphic vector spaces over $\mathbb{P}$ but they are not isomorphic as ordered vector spaces over $\mathbb{P}$.*

$\lhd$ Recall that the real field $\mathbb{R}$ has no proper subfield of which it is a finite extension; see, for example, Coppel [13, Lemma 17]. It follows that $\mathbb{R}$ is an infinite dimensional vector space over the field $\mathbb{P}$. Let $\mathscr{E}$ be a Hamel basis of a $\mathbb{P}$-vector space $\mathbb{R}$. Since $\mathscr{E}$ is infinite, we can choose a proper subset $\mathscr{E}_0 \subsetneq \mathscr{E}$ of the same cardinality: $|\mathscr{E}_0| = |\mathscr{E}|$. If $\mathscr{X}$ denotes the $\mathbb{P}$-subspace of $\mathbb{R}$ generated by $\mathscr{E}_0$, then $\mathscr{X}_0 \subsetneq \mathbb{R}$ and $\mathscr{X}$ and $\mathbb{R}$ are isomorphic as vector spaces over $\mathbb{P}$. If $\mathscr{X}$ and $\mathbb{R}$ were isomorphic as ordered vector spaces over $\mathbb{P}$, then $\mathscr{X}$ would be order complete and, as a consequence, we would have $\mathscr{X} = \mathbb{R}$; a contradiction. $\rhd$

Theorem 14. *Let X be a real universally complete vector lattice without locally one-dimensional bands. Then there exist a vector sublattice $X_0 \subset X$ and a band preserving linear bijection $T : X_0 \to X$ such that $T^{-1} : X \to X_0$ is also band preserving but X_0 and X are not lattice isomorphic.*

$\lhd$ We can assume without loss of generality that $X = \mathscr{R}{\downarrow}$ and $[\![\mathscr{R} \neq \mathbb{R}^\wedge]\!] = \mathbb{1}$. By Proposition 7 there exist an $\mathbb{R}^\wedge$-linear subspace $\mathscr{X}$ in $\mathscr{R}$ and $\mathbb{R}^\wedge$-linear isomorphism τ from $\mathscr{X}$ onto $\mathscr{R}$, while $\mathscr{X}$ and $\mathscr{R}$ are not isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$. Put $X_0 := \mathscr{X}{\downarrow}$, $T := \tau{\downarrow}$, and $S := \tau^{-1}{\downarrow}$. The maps S and T are band preserving and $\mathbb{R}$-linear by [25, Theorem 4.3.4]. Moreover, $S = (\tau{\downarrow})^{-1} = T^{-1}$.

It remains to observe that X_0 and X are lattice isomorphic if and only if $\mathscr{X}$ and $\mathscr{R}$ are isomorphic as ordered vector spaces over $\mathbb{R}^\wedge$. $\rhd$

Remark 3. By the same kind of reasoning one can prove the following. If X is a real universally complete vector lattice without locally one-dimensional bands then there exist component-wise closed vector sublattices $X_1 \subset X$ and $X_2 \subset X$ and band preserving linear bijections $T_1 : X_1 \to X$ and $T_2 : X_2 \to X$ such that: 1) $T_k^{-1} : X \to X_k$ is also band preserving ($k = 1, 2$); 2) $X = X_1 \oplus X_2$; 3) the canonical projection $\pi_k : X \to X_k$ is band preserving ($k = 1, 2$); 4) neither X_1 nor X_2 is Dedekind complete and hence lattice isomorphic to X.

C. Classification of injective modules. In what follows, K stands for a commutative semiprime ring with unit and X denotes a unitary K-module.

Definition 13. A K-module X is *separated* provided that for every dense ideal $J \subset K$ the identity $xJ = \{0\}$ implies $x = 0$. Recall that a K-module X is *injective* whenever, given a K-module Y, a K-submodule $Y_0 \subset Y$, and a K-homomorphism $h_0 : Y_0 \to X$, there exists a K-homomorphism $h : Y \to X$ extending h_0.

The *Baer criterion* says that a K-module X is injective if and only if for each ideal $J \subset K$ and each K-homomorphism $h : J \to X$ there exists $x \in X$ with $h(a) = xa$ for all $a \in J$; see Lambek [26]. All modules under consideration are assumed to be *faithful*, that is, $Xk \neq \{0\}$ for any $0 \neq k \in K$, or equivalently, the canonical representation of K by endomorphisms of the additive group X is one-to-one.

The following result is due to Gordon [15, 16].

Theorem 15. *Let $\mathscr{X}$ be a vector space over a field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$, and let $\chi : \mathbb{B} \to \mathbb{B}(\mathscr{K}{\downarrow})$ be a Boolean isomorphism in Theorem 12. Then $\mathscr{X}{\downarrow}$ is a separated unital injective module over $\mathscr{K}{\downarrow}$ such that $b \leqslant [\![x = 0]\!]$ and $\chi(b)x = \{0\}$ are equivalent for all $x \in \mathscr{X}{\downarrow}$ and $b \in \mathbb{B}$.*

Conversely, if K is a rationally complete semiprime ring, $\mathbb{B} := \mathbb{A}(K)$, and $\mathscr{K}$ is as in Theorem 12, then for every unital separated injective K-module X there exists an internal vector space $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ over $\mathscr{K}$ such that the K-module X is isomorphic to $\mathscr{X}{\downarrow}$. Moreover if $\jmath : K \to \mathscr{K}{\downarrow}$ is an isomorphism in Theorem 12, then one can choose an isomorphism $\imath : X \to \mathscr{X}{\downarrow}$ such that $\imath(ax) = \jmath(a)\imath(x)$ $(a \in K,\ x \in X)$.

$\lhd$ See [24, Theorems 8.3.12] and [24, and 8.3.13]. $\rhd$

Thus, Theorem 15 enables us to apply Boolean valued approach to unital separated injective modules over commutative semiprime rationally complete rings.

Definition 14. A family $\mathscr{E}$ in a K-module X is called *K-linearly independent* or symply *linearly independent* whenever, for all $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in K$, and $e_1, \dots, e_n \in \mathscr{E}$, the equality $\sum_{k=1}^n \alpha_k e_k = 0$ implies $\alpha_1 = \ldots = \alpha_n = 0$. An inclusion maximal K-linearly independent subset of X is called a *Hamel K-basis* for X.

Every unital separated injective K-module X has a Hamel K-basis. A K-linearly independent set $\mathscr{E}$ in X is a Hamel K-basis if and only if for every $x \in X$ there exist a partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{P}(K)$ and a family $(\lambda_{k,e})_{k\in\mathbb{N},e\in\mathscr{E}}$ in K such that

$$\pi_k x = \sum\nolimits_{e\in\mathscr{E}} \lambda_{k,e}\pi_k e \quad (k \in \mathbb{N})$$

and for every $k \in \mathbb{N}$ the set $\{e \in \mathscr{E} : \lambda_{k,e} \neq 0\}$ is finite.

Definition 15. Let γ be a cardinal. A K-module X is said to be *Hamel γ-homogeneous* whenever there exists a Hamel K-basis of cardinality γ in X. For $\pi \in \mathbb{P}(X)$ denote by $\varkappa(\pi)$ the least cardinal γ for which πX is Hamel γ-homogeneous. Say that X is *strictly Hamel γ-homogeneous* whenever X is Hamel γ-homogeneous and $\varkappa(\pi) = \gamma$ for all nonzero $\pi \in \mathbb{P}(X)$.

Theorem 16. *Let K be a semiprime rationally complete commutative ring and let X be a separated injective module over K. There exists a partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ in $\mathbb{P}(K)$ with Γ a set of cardinals such that $e_\gamma X$ is strictly Hamel γ-homogeneous for all $\gamma \in \Gamma$. Moreover, X is isomorphic to $\prod_{\gamma\in\Gamma} e_\gamma X$ and the partition of unity $(e_\gamma)_{\gamma\in\Gamma}$ is unique up to permutation.*

$\lhd$ According to Theorems 12 and 15 we may assume that $K = \mathscr{K}{\downarrow}$ and $X = \mathscr{X}{\downarrow}$, where $\mathscr{X}$ is a vector space over the field $\mathscr{K}$ within $\mathbb{V}^{(\mathbb{B})}$. Moreover, $\dim(\mathscr{X}) \in \mathbb{V}^{(\mathbb{B})}$, the algebraic dimension of $\mathscr{X}$, is an internal cardinal and, since each Boolean valued cardinal is a mixture of some set of relatively standard cardinals [25, 1.9.11], we have $\dim(\mathscr{X}) = \operatorname{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$ where Γ is a set of cardinals and $(b_\gamma)_{\gamma\in\Gamma}$ is a partition of unity. Thus, for all $\gamma \in \Gamma$ we have $e_\gamma \leqslant [\![\dim(\mathscr{X}) = \gamma^\wedge]\!]$, whence $e_\gamma X$ is strictly Hamel γ-homogeneous. The remaining details are elementary. $\rhd$

Remark 4. (1) Recently, Chilin and Karimov [12, Theorem 4], without using the Boolean valued approach, obtained that particular case of Theorem 16 when $K = L^0(\mathscr{B})$ is a real or complex universally complete f-algebra. In this event $\mathscr{X}$ is a vector spaces over the field of reals $\mathscr{K} = \mathscr{R}$ or complexes $\mathscr{K} = \mathscr{C}$ within $\mathbb{V}^{(\mathbb{B})}$. Another particular case of Theorem 16 when $\mathscr{X}$ is a vector subspace of $\mathscr{R}$ (considered as a vector space over $\mathbb{R}^\wedge$) was examined by Kusraev and Kutateladze [25, Chap. 4].

(2) The family $(e_\gamma)_{\gamma\in\Gamma}$ in Theorem 16 is called the *passport* for X. Thus, the passport $\Gamma(X)$ is the interpretation of the algebraic dimension $\dim(\mathscr{X})$ in $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{A}(X)$. Chilin and Karimov [12, Theorem 4.3] proved that separated injective modules over $K = L^0(\mathscr{B})$ are isomorphic if and only if their passports coincide. This result remains valid for any general commutative semiprime rationally complete ring.

(3) The family $(e_\gamma)_{\gamma\in\Gamma}$ in Theorem 16 is called a *decomposition series* if $e_\gamma X$ is (not necessarily strict) Hamel γ-homogeneous for all $\gamma \in \Gamma$. It can be also proved that separated injective modules over $K = L^0(\mathscr{B})$ are isomorphic if and only if their decomposition series are *congruent* in the sense of Ozawa [29].

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ВЗАИМОДЕЙСТВИЕ АНАЛИЗА, АЛГЕБРЫ И
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