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Цель настоящей работы троякая: во-первых, в контексте квазибанаховых пространств адаптировать и развить некоторые идеи и технические средства из теории банаховых решеток; во-вторых, построить чисто порядковое интерирование типа Канторовича — Райта скалярных функций относительно векторной меры, определенной на δ-кольце и принимающей значения из порядково σ-полной векторной решетки; в третьих, получить теоремы о представлении порядково полных векторных решеток и квазибанаховых решеток в виде решеток функций, интегрируемых или «слабо» интегрируемых относительно подходящей векторной меры.

Ключевые слова: квазибанахова решетка, максимальное квазинормированное расширение, векторная мера, интеграл Канторовича — Райта, оператор интегрирования, наименьшее продолжение, пространство интегрируемых функций, порядково непрерывная часть.

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The aim of this work is three-fold. First, to adopt and develope in the context of quasi-Banach spaces some ideas and technical tools coming from the theory of Banach lattices. Second, to introduce a purely order-based Kantorovich–Wright type integration of scalar function with respect to a vector measures defined on a δ-ring and taking values in a Dedekind σ-complete vector lattice. Third, to use the Kantorovich–Wright type integration for obtaining general representation theorems for Dedekind complete vector lattices and quasi-Banach lattices as spaces of integrable or “weakly” integrable functions with respect to an appropriate vector measure.

Key words: quasi-Banach lattice, maximal quasi-Banach extension, vector measure, Kantorovich–Wright integration, integration operator, smallest extension, space of integrable function, order continuous part.

Mathematics Subject Classification (2010): 46A16, 46B42, 46E30, 46G10, 47B38, 47G10.

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KANTOROVICH–WRIGHT INTEGRATION AND REPRESENTATION OF QUASI-BANACH LATTICES

A. G. Kusraev and B. B. Tasoev

1. Introduction

Integration with respect to a measure taking values in a vector lattice has its roots in spectral theory, in representation of linear operators by means of integration with respect to spectral measures. Purely order based integration theory of real-valued measurable functions with respect to countably additive vector measures with values in a Dedekind complete vector lattice was developed by Kantorovich [38, 39]. A decisive contribution was made by Wright [73, 74, 75]. The existing literature is rather sparse; aspects of the theory are reflected in the book by Kusraev [42, Ch. 6]. Some interesting application can be found in Coquand [8], Fuchssteiner and Wright [26], Khurana [40], Wright [76] (generalizations of the Riesz representation theorem); Haydon [27] (representation of injective Banach lattices); Takeuti [65], Kusraev and Kutateladze [43], Kusraev and Malyugin [44] (abstract harmonic analysis); Malyugin [51] (the moment problem in vector lattices).

Bartle, Dunford and Schwartz [5] introduced the theory of integration with respect to a σ-additive vector measure defined on a σ-algebra of subsets of some set and with values in a Banach space, see [22, Ch. IV, § 19]. Later, Lewis [45] gave an alternative duality based approach. The theory was extended to vector measures defined on δ-rings in Lewis [46] and Masani and Niemi [52, 53]. The integration of scalar measurable functions with respect to a vector measure with values in an F-space was developed in Rolewicz [59, § III.6], [60], Turpin [67], [68, Chap. 7], and Thomas [66].

For over recent 25 years the spaces of integrable functions with respect to a measure taking values in a Banach (quasi-Banach) lattice have been a field of increased interest. The spaces of integrable and weakly integrable functions with respect to a vector measure possess interesting order and metric properties and have been studied intensively by many authors. They find applications in important problems such as the representation of abstract quasi-Banach lattices as spaces of integrable functions, the study of the optimal domain of linear operators, domination and factorization of operators, spectral integration etc.

It is important to know under which condition a quasi-Banach lattice is order isometric to some quasi-Banach function space. Different aspects of this problem have been studied by variety of authors. Recent achievements are related with the vector measure integration. Curbera [10, Theorem 8] proved that every order continuous Banach lattice with a weak unit is order isometric to $L^1(\mu)$ for a vector measure μ defined on a σ-algebra. This result was extended to quasi-Banach lattices by Sánchez Pérez and Tradacete [62, Theorem 4.3]. Curbera and Ricker [13, Theorem 2.5] succeeded to prove that any Banach lattice with the σ-Fatou property with a weak order unit belonging to its order σ-continuous part as a lattice of weakly integrable function $L^1_w(\mu)$. Similar results remain valid even if the Banach space under consideration do not contains a weak order unit as it was stated in Curbera [9, Theorema 5] and proved in Delgado and Juan [18, Theorem 10]. The price one pays for this is that one should extend the integration with respect to a vector measure defined on a σ-algebra to a vector measure defined on a δ-ring, see Delgado [16], Degado and Juan [18], Calabuig, Delgado, Juan, and Sánchez Pérez [6]. It should be also noted that the standard convexification–concavification machinery (see Lindenstrauss and Tzafriri [47]) enables one to obtain similar representation results for Banach lattices with the convexity properties using the spaces of power-integrable functions, see Calabuig, Jaun, and Sánchez Pérez [7], Curbera and Ricker [15]. A basic tool for the whole study is the Bartle–Dunford–Schwartz type integration theory.

One more area of increased interest recently has been the local theory of quasi-Banach spaces [37]. It turns out that many of the ideas and achievements of the local theory of Banach spaces may be naturally transplanted to the environment of quasi-Banach spaces so that convexity is not really relevant, see, for example, Kalton [32–35].The above mentioned result by Sánchez Pérez and Tradacete [62, Theorem 4.3] is the first attempt to extend the representation of Banach lattices by using the spaces of integrable functions with respect to a vector measure to quasi-Banach setting.

The aim of this work is three-fold. First, to adopt and developed in the context of quasi-Banach spaces some ideas and technical tools coming from the theory of Banach lattices. In Section 2 we briefly sketch the needed information concerning quasi-Banach lattices. Next, we prove some Riesz–Fischer type completeness theorems for quasi-normed lattices and gave a characterization of order continuous quasi-Banach lattices. In Section 3 we examine the construction of the maximal quasi-normed extension introduced by Abramovich [1] for Banach lattices.

Second, we introduce a purely order-based Kantorovich–Wright type integration with respect to a vector measures defined on a δ-ring with values in a Dedekind σ-complete vector lattice. This is done in Sections 4 and 6. Section 5 deals with the smallest extension of the space of integrable function. The parallel Burtle–Dunford–Schwartz type integration defined on a δ-ring with values in a quasi-Banach lattice is outlined in Section 8. The definitions of countable additivity and integration are understood in the sense of order or metric convergence according to whether the Kantorovich–Wright or Bartle–Dunford–Schwartz theory is considered. To differ between them it is sometimes convenient to speak of an *order measure* (*o-measure*) and the order integration (o-integration) *operator* I_μ^o or a *topological measure* (*τ-measure*) and the *topological integration* (*τ-integration*) operator I_μ^τ, respectively.

Third, the Kantorovich–Wright type integration can be used to obtain general representation theorems for Dedekind complete vector lattices and quasi-Banach lattices. In section 7 we demonstrate that, given an arbitrary σ-Dedekind complete vector lattice, there exists an order dense ideal which is lattice isomorphic to the vector lattice $L_o^1(\mu)$ of equivalence classes of o-integrable functions with respect to a vector measure μ, while the whole vector lattice is lattice isomorphic to the domain of the smallest extension $\hat{I}_\mu^o$ of I_μ^o. Similar result is stated in Section 9 for Dedekind complete quasi-Banach lattices. In Section 10 it is shown that a Dedekind complete quasi-Banach lattice with the order dense order continuous part is representable as the domain $L^1_{\tau w}(\mu)$ of the smallest extension $\hat{I}^\tau_\mu$ of the τ-integration operator I^τ_μ. Of course, the domain of I^τ_μ is the space of τ-integrable operators $L^1_\tau(\mu)$.

In the context of Banach lattices a crucial role is played by the spaces $L^1(\mu)$ and $L^1_w(\mu)$ of integrable and weakly integrable functions with respect to a vector measure, see the survey paper by Curbera and Ricker [14] and the book by Okada, Ricker and Sánches Pérez [56]. Dealing with the functional representation of quasi-Banach lattices a duality based definition of $L^1_w(\mu)$ no longer works but there are four natural candidates for a space of weakly integrable function: maximal quasi-Banach extension $L^1_{\tau\varkappa}$ and $L^1_{o\varkappa}$ of $L^1_\tau(\mu)$ and $L^1_o(\mu)$, respectively, and the domains $L^1_{ow}(\mu)$ and $L^1_{\tau w}(\mu)$ of smallest extensions $\hat{I}^o_\mu$ and $\hat{I}^\tau_\mu$ of the integration operators I^o_μ and I^τ_μ, respectively. Several simple relationships between these quasi-Banach lattice are also presented in Sections 9 and 10. Some consequences of the obtained results for Banach lattices are discussed in Section 11.

We use the standard notation and terminology of Aliprantis and Burkinshaw [4] and Meyer-Niberg [54] for the theory of vector and Banach lattices (see also Abramovich and Aliprantis [3], Luxemburg and Zaanen [48], Schwarz [63], Vulikh [72]). Throughout the text we assume that all vector spaces are defined over the field of reals and all vector lattices are Archimedean. We let := denote the assignment by definition, while $\mathbb{N}$ and $\mathbb{R}$ symbolize the naturals and the reals.

2. Quasi-Banach Lattices

In this section, we briefly sketch the needed information concerning quasi-Banach lattices.

Definition 2.1. A *quasi-normed space* is a pair $(X, \|\cdot\|)$ where X is a real vector space and $\|\cdot\|$ is a *quasi-norm*, a function from X to $\mathbb{R}$ such that the following conditions hold:

(1) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.

(3) There exists a constant $C \geqslant 1$ with $\|x + y\| \leqslant C(\|x\| + \|y\|)$ for all $x, y \in X$.

If, in addition, for some $0 < p \leqslant 1$ the inequality

(4) $\|x + y\|^p \leqslant \|x\|^p + \|y\|^p$ holds for all $x, y \in X$,

then $\|\cdot\|$ is called a *p-norm* and $(X, \|\cdot\|)$ is called a *p-normed space.*

The best constant C in 2.1 (3) is called the *quasi-triangle constant*, or *quasi-norm multiplier*, or *modulus of concavity* of the quasi norm.

Two quasi-norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if there is a constant $A \geqslant 1$ such that $A^{-1}\|x\| \leqslant \|x\|' \leqslant A\|x\|$ for all $x \in X$. By the Aoki–Rolewicz theorem (see [34]), each quasi-norm is equivalent to some p-norm for some $0 < p \leqslant 1$.

Theorem 2.2 (Aoki–Rolewicz). *Let $(X, \|\cdot\|)$ be a quasi-normed space with the quasi-triangle constant $C \geqslant 1$ and $p = (1 + \log_2 C)^{-1}$. Define $\|\cdot\|_p : X \to \mathbb{R}$ as*

$$\|x\|_p := \inf\left\{\left(\sum_{k=1}^n \|x_k\|^p\right) :\ x = \sum_{k=1}^n x_k,\ n \in \mathbb{N}\right\} \quad (x \in X).$$

Then $0 < p \leqslant 1$, $\|\cdot\|_p$ *is a* p*-norm, and* $\|x\|_p \leqslant \|x\| \leqslant 2^{1/p}\|x\|_p$ *for all* $x \in X$.

$\lhd$ See Maligranda [49, Theorem 1.2], Pietsch [57, 6.2.5]. $\rhd$

Thus, we may assume unless otherwise is mentioned that a quasi-Banach space is equipped with a p-norm for some $0 < p \leqslant 1$.

A topological vector space X is said to be *locally bounded* if it has a bounded neighborhood of zero. A quasi-normed space is a locally bounded topological vector space if we take the sets $\{x \in X : \|x\| \leqslant \varepsilon\}$ $(0 < \varepsilon \in \mathbb{R})$ for a base of neighborhoods of zero. Moreover, this topology may be induced by metric $d(x, y) := |||x - y|||^p$ $(x, y \in X)$ where $|||\cdot|||$ is an equivalent p-norm. Conversely, Hyers [28] proved that the topology of a locally bounded topological vector space X can be deduced from a quasi-norm, which may be obtained as the Minkowski functional of a bounded balanced neighborhood B of zero:

$$\|x\| := \|x\|_B := \inf\big\{0 < \lambda \in \mathbb{R} : \ x \in \lambda B\big\} \quad (x \in X).$$

A quasi-norm may be discontinuous in its own topology [57, 6.1.9]. However, every quasi-norm is equivalent to a continuous one, since a p-norm is continuous.

Definition 2.3. A *quasi-Banach space* (*p-normed space*) is a quasi-normed space which is complete in its metric uniformity.

Theorem 2.4. *A quasi-normed space* $X := (X, \|\cdot\|)$ *with a triangle constant* $C \geqslant 1$ *is complete (and hence a quasi-Banach space) if and only if for every series* (x_k) *in* X *such that* $\sum_{k=1}^{\infty} C^k \|x_k\| < \infty$ *there exists* $\sum_{k=1}^{\infty} x_k \in X$ *and*

$$\Big\|\sum_{k=1}^{\infty} x_k\Big\| \leqslant \sum_{k=1}^{\infty} C^{k+1}\|x_k\|.$$

$\lhd$ See Maligranda [49, Theorem 1.1]. $\rhd$

The basic results of the Banach space theory such as open mapping theorem and the closed graph theorem (for linear operators) are valid also in the context of quasi-Banach spaces, see [37].

Definition 2.5. A quasi-Banach (quasi-normed, p-Banach) space $(X, \|\cdot\|)$ is called a *quasi-Banach lattice* (respectively, *quasi-normed lattice*, *p-Banach lattice*) if, in addition, X is a vector lattice and $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in X$.

Lemma 2.6. *In any quasi-normed lattice* X *lattice operations are continuous and the positive cone is closed. Moreover, if an increasing (decreasing) net* $(x_\alpha)_{\alpha\in A}$ *is quasi-norm convergent to* $x \in X$, *then* $x = \sup_{\alpha\in A} x_\alpha$ $(x = \inf_{\alpha\in A} x_\alpha)$.

$\lhd$ This can be ensured just as in the case of Banach lattice using monotonicity of the quasi-norm and quasi-triangle inequality. $\rhd$

It follows from Lemma 2.6 that the completion of a quasi-normed lattice X is a quasi-Banach lattice including X as a vector sublattice. Along similar lines, it can also be proved that Amemiya's result on completeness of normed lattices is true in the context of quasi-normed spaces: *a quasi-normed lattice* X *is complete if and only if every increasing Cauchy sequence in* X *is convergent.* This fact in combination with Theorem 2.4 leads to the following result.

Theorem 2.7. *For a quasi-normed space $X := (X, \|\cdot\|)$ with a triangle constant $C \geqslant 1$ the following assertions are equivalent:*

(1) *X is a quasi-Banach lattice.*

(2) *For every series (x_k) in X_+ such that $\sum_{k=1}^{\infty} C^k\|x_k\| < \infty$ there exists $x \in X$ with $x = \sum_{k=1}^{\infty} x_k$.*

(3) *For every series (x_k) in X_+ such that $\sum_{k=1}^{\infty} C^k\|x_k\| < \infty$ there exists $x \in X$ with $x = o\text{-}\sum_{k=1}^{\infty} x_k := \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} x_k$.*

$\lhd$ Obviously, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) in view of Lemma 2.6. Ensure that (3) $\Longrightarrow$ (1).

Assume that for a sequence (x_k) in X we have $\sum_{k=1}^{\infty} C^k\|x_k\| < \infty$. Put $u_n := x_n^+$ and $v_n := x_n^-$. By (3) there exists $u = o\text{-}\sum_{k=1}^{\infty} u_k$, since $\|u_n\| \leqslant \|x_n\|$. Choose a strictly increasing sequence of naturals (n_k) such that $\sum_{l=n_k}^{n_{k+1}} C^l\|u_l\| \leqslant k^{-3}$ for all $k \in \mathbb{N}$ and denote $y_k = \sum_{l=n_k+1}^{n_{k+1}} ku_l$. Then for the sequence (y_k) we deduce

$$\sum_{k=1}^{\infty} C^k\|y_k\| \leqslant \sum_{k=1}^{\infty} C^k \sum_{l=n_k+1}^{n_{k+1}} C^{l-n_k}\|ku_l\| \leqslant \sum_{k=1}^{\infty} \sum_{l=n_k+1}^{n_{k+1}} C^l k\|u_l\| \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

According to (3) there exists $y = o\text{-}\sum_{k=1}^{\infty} y_k$ and

$$k\Bigg(u - \sum_{l=1}^{n_k} u_l\Bigg) = \sup_{m\in\mathbb{N}} \sum_{l=n_k+1}^{m} ku_l \leqslant o\text{-}\sum_{j=k}^{\infty} y_j \leqslant y.$$

It follows that $(u - \sum_{l=1}^{n_k} u_l) \leqslant y/k$ for all $k \in \mathbb{N}$ and hence $u = \sum_{l=1}^{\infty} u_l$ in view of monotonicity of the quasi-norm. By similar arguments there exists $v = \sum_{l=1}^{\infty} v_l$. Using Lemma 2.6 once again we see that the series $\sum_{k=1}^{\infty} x_k$ converges to $x := u - v$. $\rhd$

Definition 2.8. A quasi-Banach lattice $(X, \|\cdot\|)$ (as well as the quasi-norm $\|\cdot\|$) is said to be *order continuous*, if $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$ for any net $(x_\alpha)_{\alpha\in A}$ in X. If arbitrary nets are replaced by sequences, one speak of *order σ-continuity.*

Lemma 2.9. *An order continuous quasi-Banach lattice is super Dedekind complete and every increasing order bounded net in it is convergent.*

$\lhd$ Take an increasing net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in a quasi-Banach lattice X such that $0 \leqslant x_\alpha \leqslant x$ for some $x \in X_+$. The set $\Gamma := \{\gamma \in X : x_\alpha \leqslant \gamma \text{ for all } \alpha \in \mathrm{A}\}$ is upward directed with the ordering $\gamma_1 \preccurlyeq \gamma_2$ if $\gamma_1 \geqslant \gamma_2$ holds in X. Moreover, the net $(y_\gamma - x_\alpha)_{(\alpha,\gamma)\in\mathrm{A}\times\Gamma}$ with $y_\gamma := \gamma$ is decreasing and order convergent to zero, see [4, Lemma 4.8]. By hypothesis, for an arbitrary $\varepsilon > 0$ there exist $\alpha_0 \in \mathrm{A}$ and $\gamma_0 \in \Gamma$ such that $\|y_\gamma - x_\alpha\| \leqslant C^{-1}\varepsilon$ for all $\gamma \geqslant \gamma_0$ and $\alpha \geqslant \alpha_0$. It follows that $\|x_\beta - x_\alpha\| \leqslant C(\|x_\beta - y_{\lambda_0}\| + \|y_{\lambda_0} - x_\beta\|) \leqslant 2\varepsilon$ for all $\beta, \alpha \geqslant \alpha_0$, that is, the net (x_α) is Cauchy and converges to same $x \in X$. By Lemma 2.6 $x = \sup_\alpha x_\alpha$ and X is order complete.

Let a net $(x_\alpha)_{\alpha\in\mathrm{A}}$ be such that $0 \leqslant x_\alpha \uparrow x$ for some $x \in X_+$. Then $x - x_\alpha \downarrow 0$ and in view of order continuity of the quasi-norm we have $\|x - x_\alpha\| \downarrow 0$. Choose increasing sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathrm{A}$ with $\|x - x_{\alpha_n}\| \leqslant n^{-1}$ for all $n \in \mathbb{N}$. Then, as we see, x is a quasi-norm limit of positive increasing sequence $(x_{\alpha_n})_{n=1}^{\infty}$ and by Lemma 2.6 $x = \sup_n x_{\alpha_n}$. Therefore, X is super Dedekind complete.

The second part is obvious. $\rhd$

Theorem 2.10. *For a quasi-Banach lattice X the following are equivalent:*

(1) *X is order continuous.*

(2) *Every increasing order bounded sequence in X_+ is convergent.*

(3) *X is Dedekind σ-complete and order σ-continuous.*

$\lhd$ (1) $\Longrightarrow$ (2) Follows from Lemma 2.9.

(2) $\Longrightarrow$ (3) Order σ-completeness is immediate from Lemma 2.6. To ensure order σ-continuity, take a decreasing sequence (x_n) in X order convergent to zero. Then $x_1 - x_n \uparrow x_1$ and by hypothesis there exists $\lim_n(x_1 - x_n) = y$. By Lemma 2.6 $y = x_1$ and $\lim_n x_n = 0$.

(3) $\Longrightarrow$ (1) Let (x_α) be a decreasing net order converging to zero. If (x_α) is not Cauchy then there exist $\varepsilon > 0$ and an increasing sequence (α_n) such that $\|x_{\alpha_n} - x_{\alpha_{n+1}}\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. In view of order σ-completeness of X there exists $x = \inf x_{\alpha_n}$. By hypotheses, (x_{α_n}) is convergent which is a contradiction. It follows that (x_α) is Cauchy and converges to some $y \in X$. By Lemma 2.6 $y = \inf_\alpha x_\alpha = 0$. $\rhd$

Definition 2.11. A quasi-Banach lattice $(X, \|\cdot\|)$ is said to have the *weak Fatou property* (respectively *weak σ-Fatou property*) if there exists $K > 0$ (called the *weak Fatou constant*) such that for every increasing net (x_α) (respectively sequence (x_n)) with the supremum $x \in X$ we have $\|x\| \leqslant K \sup_\alpha \|x_\alpha\|$ (respectively $\|x\| \leqslant K \sup_n \|x_n\|$). If $K = 1$ then $\|x\| = \sup_\alpha \|x_\alpha\|$ and in this situation X is said to have the *Fatou property* (respectively *σ-Fatou property*).

Definition 2.12. Say that a quasi-normed lattice $(X, \|\cdot\|)$ has the *Levi property* (respectively *σ-Levi property*) if $\sup_\alpha x_\alpha$ (respectively $\sup_n x_n$) exists for every increasing net (x_α) (respectively sequence (x_n)) in X_+ provided that $\sup_\alpha \|x_\alpha\| < \infty$ (respectively $\sup_n \|x_n\| < \infty$). A *quasi-KB-space* is an order continuous quasi-normed lattice with the Levi property.

Proposition 2.13. *Suppose that X is a quasi-normed lattice with the Levi property. Then X is a Dedekind complete quasi-Banach lattice with the weak Fatou property.*

$\lhd$ The fact that a quasi-normed lattice with the Levi property has also the weak Fatou property is the only thing that needs verification. The proof is similar to that of Proposition 2.4.19 in Meyer-Nieberg [54].

Assume that X has the Levi property but lacks the weak Fatou property. Then for every $n \in \mathbb{N}$ there exists an increasing net $(y_{n,\alpha})_{\alpha \in \mathrm{A}(n)}$ in X_+ such that $y_n = \sup_{\alpha \in \mathrm{A}(n)} y_{n,\alpha}$ exists and

$$\|y_n\| \geqslant n\tau, \quad \tau = C^n n^2 \sup_{\alpha \in \mathrm{A}(n)} \|y_{n,\alpha}\| \quad (n \in \mathbb{N}),$$

where $C \geqslant 1$ is the triangle constant of X. Putting $\bar{y}_n := y_n/\tau$, $\bar{y}_{n,\alpha} := y_{n,\alpha}/\tau$ we arrive at the following relations:

$$\bar{y}_n = \sup_{\alpha \in \mathrm{A}(n)} \bar{y}_{n,\alpha}, \quad \|\bar{y}_n\| \geqslant n, \quad \|\bar{y}_{n,\alpha}\| \leqslant C^{-n} n^{-2} \quad (n \in \mathbb{N}).$$

Let (x_γ) stands for the net of finite suprema of elements in $\{\bar{y}_{n,\alpha} :\ n \in \mathbb{N},\ \alpha \in \mathrm{A}(n)\}$.

If $x_\gamma = \bar{y}_{n_1,\alpha_1} \vee \dots \vee \bar{y}_{n_k,\alpha_k}$ with $\alpha_j \in \mathrm{A}(n_j)$, then

$$\|x_\gamma\| \leqslant \|\bar{y}_{n_1,\alpha_1} + \dots + \bar{y}_{n_k,\alpha_k}\| \leqslant \sum_{j=1}^{k} C^j \|\bar{y}_{n_j,\alpha_j}\| \leqslant \sum_{j=1}^{k} C^j C^{-n_j} n_j^{-2} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By hypothesis, $x = \sup_\gamma x_\gamma$ exists and satisfies $x \geqslant \bar{y}_n$ for all $n \in \mathbb{N}$. Consequently, $\|x\| \geqslant n$ for all $n \in \mathbb{N}$, a contradiction. ▷

3. Maximal Quasi-Normed Extension

Consider a quasi-normed lattice $(X, \|\cdot\|)$ with the quasi-triangle constant C. Let X^δ stand for the Dedekind completion of X, so that X is identified with a majorizing order dense sublattice of X^δ, while X^δ itself is a Dedekind complete vector lattice. Define a function $\|\cdot\|_\delta : X^\delta \to \mathbb{R}$ as

$$\|\bar{x}\|_\delta := \inf\big\{\|x\| : \ x \in X_+, \ |\bar{x}| \leqslant x\big\} \quad (\bar{x} \in X^\delta).$$

Clearly, $\|x\| = \|x\|_\delta$ for all $x \in X$ and $\|\bar{x}\|_\delta < \infty$ for each $\bar{x} \in X^\delta$, since X is majorizing sublattice. Positive homogeneity and monotonicity of $\|\cdot\|_\delta$ are obvious. Moreover, if $|\bar{x}| \leqslant x$ and $|\bar{y}| \leqslant y$ for some $x, y \in X$ and $\bar{x}, \bar{y} \in X^\delta$, then $|\bar{x}+\bar{y}| \leqslant x+y$ and $\|\bar{x}+\bar{y}\|_\delta \leqslant \|x+y\| \leqslant C(\|x\|+\|y\|)$ and hence $\|\bar{x}+\bar{y}\|_\delta \leqslant C(\|\bar{x}\|_\delta + \|\bar{y}\|_\delta)$. It follows that $(X_\delta, \|\cdot\|_\delta)$ is a quasi-normed lattice with the same quasi-triangle constant.

Lemma 3.1. *If $(X, \|\cdot\|)$ is a quasi-Banach lattice with a triangle constant C or a p-Banach lattice, then so is $(X^\delta, \|\cdot\|_\delta)$.*

◁ Assume that $\sum_{k=1}^\infty C^k \|\bar{x}_k\|_\delta < \infty$ for a sequence $(\bar{x}_k)$ in X^δ_+. Pick $x_k \in X_+$ such that $\bar{x}_k \leqslant x_k$ and $\|x_k\| \leqslant \|\bar{x}_k\|_\delta + 1/(2C)^k$. Then

$$\sum_{k=1}^{n} C^k \|x_k\| \leqslant \sum_{k=1}^{n} C^k \|\bar{x}_k\|_\delta + \sum_{k=1}^{n} \frac{1}{2^k}$$

and hence $\sum_{k=1}^\infty C^k \|x_k\| < \infty$. By Theorem 2.6 $x := o\text{-}\sum_{k=1}^\infty x_k$ exists in X. Consequently, $o\text{-}\sum_{k=1}^\infty \bar{x}_k$ exists in X^δ, since $\sum_{k=1}^n \bar{x}_k \leqslant x$ for all $n \in \mathbb{N}$. ▷

Assume now that $(X, \|\cdot\|)$ is a Dedekind complete quasi-normed lattice with a quasi-triangle constant C. Identify X with an order dense ideal in its universal completion X^u. Define a function $\|\cdot\|_\varkappa : X^u \to \mathbb{R} \cup \{+\infty\}$ by putting

$$\|\hat{x}\|_\varkappa := \sup\big\{x \in X : \ 0 \leqslant x \leqslant |\hat{x}|\big\} \quad (\hat{x} \in X^u).$$

Observe that $\|x\| = \|x\|_\varkappa$ for all $x \in X$. Denote $X^\varkappa := \{\hat{x} \in X^u : \ \|\hat{x}\|_\varkappa < \infty\}$. If $0 \leqslant u \leqslant |\hat{x}+\hat{y}| \leqslant |\hat{x}| + |\hat{y}|$ for some $\hat{x}, \hat{y} \in X^\varkappa$ and $u \in X$, then there exist $x, y \in X$ with $0 \leqslant x \leqslant |\hat{x}|$, $0 \leqslant y \leqslant |\hat{y}|$, and $u = x + y$. It follows that $\|u\| \leqslant C(\|x\| + \|y\|) \leqslant C(\|\hat{x}\|_\varkappa + \|\hat{y}\|_\varkappa)$ and thus $\|\hat{x}+\hat{y}\|_\varkappa \leqslant C(\|\hat{x}\|_\varkappa + \|\hat{y}\|_\varkappa)$. Similarly, $\|\cdot\|_\varkappa$ is a p-norm, whenever $\|\cdot\|$ is. Taking into account obvious monotonicity and positive homogeneity of $\|\cdot\|_\varkappa$, we see that $(X^\varkappa, \|\cdot\|_\varkappa)$ is a quasi-normed lattice with the quasi-triangle constant C and, if $\|\cdot\|$ is a p-norm, so is $\|\cdot\|_\varkappa$.

Definition 3.2. A *maximal quasi-normed extension* of a qusi-normed lattice $(X,\|\cdot\|)$ is the pair $(X^{\delta\varkappa},\|\cdot\|_{\delta\varkappa})$ with $X^{\delta\varkappa}:=(X^{\delta})^{\varkappa}$ and

$$\|\hat{x}\|_{\delta\varkappa}:=\sup\{\inf\{\|x\|:\ x\in X,\ |\bar{x}|\leqslant x\}:\ \bar{x}\in X^{\delta},\ 0\leqslant\bar{x}\leqslant|\hat{x}|\}\quad(\hat{x}\in X^{\delta\varkappa}).$$

Observe that if X is Dedekind complete then $X^{\delta\varkappa}=X^{\varkappa}$ and $\|\cdot\|_{\delta\varkappa}=\|\cdot\|_{\varkappa}$.

Lemma 3.3. *If $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ are quasi-normed lattices, Y is an order dense ideal in X^u containing X, and $\|x\|_X=\|x\|_Y$ for all $x\in X$, then $Y\subset X^{\varkappa}$.*

$\lhd$ This is an immediate consequence of the definition. $\rhd$

Definition 3.4. A quasi-normed lattice X is called *intervally complete* if every order interval of X is complete or, in other words, every order bounded Cauhcy sequence of X is convergent to an element of X.

It can be easily seen that each intervally complete quasi-normed lattice is an order dense ideal of its own metric completion and every order ideal of any quasi-Banach lattice is an intervally complete quasi-normed lattice. Thus, the class of intervally complete quasi-normed lattices coincides with the class of order dense ideals of quasi-Banach lattices.

Lemma 3.5. *Intervally complete quasi-normed lattice is uniformly complete.*

$\lhd$ Let (x_n) be a uniformly Cauchy sequence, that is, there exist $e\in X_+$ and a sequence of reals (ε_n) such that $\lim_n\varepsilon_n=0$ and $|x_{n+k}-x_n|\leqslant\varepsilon_n e$ for all $n,k\in\mathbb{N}$. Then $\|x_{n+k}-x_n\|\leqslant\varepsilon_n\|e\|$ and (x_n) is Cauchy in $(X,\|\cdot\|)$. Moreover, $|x_{k+1}|\leqslant|x_1|+\varepsilon_1 e$ for all $k\in\mathbb{N}$. By hypothesis, there exists $x=\lim_n x_n$ in $(X,\|\cdot\|)$. Passage to the limit in $|x_{n+k}-x_n|\leqslant\varepsilon_n e$ with $k\to\infty$ yields $|x-x_n|\leqslant\varepsilon_n e$ for all $n\in\mathbb{N}$, whence X is uniformly complete. $\rhd$

Lemma 3.6. *Let $\tilde{X}$ be the metric completion of an intervally complete quasi-normed lattice X. Then $\tilde{X}$ is Dedekind complete if and only if so is X.*

$\lhd$ If $\tilde{X}$ is Dedekind complete then so is X, since X is an order dense ideal of $\tilde{X}$. Assume that a quasi-normed lattice X is intervally complete and Dedekind complete and prove $\tilde{X}$ is Dedekind complete. It was proved by Veksler [69, 70] that an Archimedean vector lattice is Dedekind complete if and only if it is uniformly complete and has the projection property. By Lemma 3.5 it suffices to show that $\tilde{X}$ has the projection property. Consider an element $x\in\tilde{X}$ and a band $\tilde{B}$ in $\tilde{X}$ and pick a sequence (x_n) in X converging to x. Observe, that $B:=\tilde{B}\cap X$ is a band of X and $B^{\perp}=\tilde{B}^{\perp}\cap X$, since X is an order dense ideal in $\tilde{X}$. If π stands for the band projection in X onto B, then $\pi':=I_X-\pi$ is the band projection onto $B^{\perp}$. The sequences (πx_n) and $(\pi' x_n)$ are Cauchy, as so is (x_n), hence they converge to some $u\in\tilde{X}$ and $u'\in\tilde{X}$, respectively. Clearly, $u\in\tilde{B}$, $u'\in\tilde{B}^{\perp}$, and $x=u+u'$. $\rhd$

Lemma 3.7. *A quasi-normed lattice X is intervally complete if and only if every increasing order bounded Cauchy sequence in X_+ is quasi-norm convergent.*

$\lhd$ The proof given in [71, Theorem 1.1] for normed lattices works in the quasi-normed setting. $\rhd$

Lemma 3.8. *Let X be a universally complete vector lattice and $(x_\alpha)_{\alpha\in\mathrm{A}}$ an increasing net in X_+. Then there exists a band projection π on X such that $\sup_\alpha\pi x_\alpha$*

exists in X, while for the complementary band projection $\pi' := I_X - \pi$ we have $N\pi'e = \sup_\alpha \pi'(x_\alpha \wedge Ne)$ for all $N \in \mathbb{N}$ and $e \in X_+$.

$\lhd$ There is no loss of generality in assuming that $X = C_\infty(Q)$ with extremally compact space Q. (Recall that the symbol $C_\infty(Q)$ denotes the universally complete vector lattice of all continuous functions $f : Q \to [-\infty, \infty]$ for which the open set $\{q \in Q : -\infty < f(q) < \infty\}$ is dense in Q.) Let (x_α) be an increasing net in $C_\infty(Q)$ and define two functions $\bar{x}, x : Q \to [0, \infty]$ by

$$\bar{x}(q) = \sup\{x_\alpha(q) : \alpha \in \mathrm{A}\} \quad (q \in Q),$$
$$x(q) := \inf_{U \in \mathscr{N}(q)} \sup_{q' \in U} \bar{x}(q') \quad (q \in Q),$$

where $\mathscr{N}(q)$ is a basis of neighborhoods of q. Then $\bar{x}$ is lower semicontinuous and x is continuous, see [72, Lemma V.1.2 and Theorem V.1.1]. Consider an open set $Q_0 := \{q \in Q : x(q) < \infty\}$ and observe that its closure $\bar{Q}_0$ is clopen. Now, let π stands for the band projection of $C_\infty(Q)$ corresponding to $\bar{Q}_0$ and πx stands for the function coinciding with x on $\bar{Q}_0$ and vanishing on $Q_1 := Q \setminus \bar{Q}_0$. Evidently, $\pi x \in C_\infty(Q)$ and $\pi x = \sup_\alpha \pi x_\alpha$, see [72, Theorem V.2.1]. At the same time $x(q) = \infty$ for all $q \in Q_1$, so that $\bar{x}(q) = \infty$ for all $q \in Q_1 \setminus A$ where A is a meager subset of Q_1. The latter implies that $Ne(q) = \sup_\alpha x_\alpha(q) \wedge Ne(q)$ for all $q \in Q_1 \setminus A$, whence the desired equation $N\pi'e = \sup_\alpha \pi'(x_\alpha \wedge Ne)$ follows. $\rhd$

Lemma 3.9. *Let X be a quasi-normed lattice X with the weak σ-Fatou property. If X is intervally complete and Dedekind complete, then its maximal quasi-normed extension $X^\varkappa$ is intervally complete.*

$\lhd$ Take an increasing order bounded Cauchy sequence $(\hat{x}_n)$ in $X^\varkappa_+$. Since $X^\varkappa$ is Dedekind complete, there exists $\hat{x} = \sup_n \hat{x}_n$. Prove that $(\hat{x}_n)$ converges to $\hat{x}$.

We may assume without loss of generality that $A := \sum_{n=1}^\infty C^n n \|\hat{x}_{n+1} - \hat{x}_n\|_\varkappa < \infty$. Applying Lemma 3.8 to the increasing sequence $(\hat{z}_n)$ with $\hat{z}_n := \sum_{k=1}^n k(\hat{x}_{k+1} - \hat{x}_k)$ yields a band projection π on X^u such that $\hat{z} := \sup_n \pi\hat{z}_n$ exists in X^u and for $\pi' := I_X - \pi$ we have $N\pi'e = \sup_n \pi'(\hat{z}_n \wedge Ne)$ for all $N \in \mathbb{N}$ and $e \in X$, $0 \leqslant e \leqslant \hat{z}$. Making use of the weak σ-Fatou property and monotonicity of the quasi-norm we deduce

$$N\|\pi'e\| \leqslant K \sup\nolimits_m \|\pi'(\hat{z}_m \wedge Ne)\| \leqslant K \sup\nolimits_m \|\hat{z}_m\|_\varkappa \leqslant KA < \infty.$$

It follows that $\pi'e$ for all $e \in X$ with $0 \leqslant e \leqslant \hat{z}$ and hence $\pi'\hat{z} = 0$. Note that $\hat{z}_n \in \hat{x}^{\perp\perp}$ for all $n \in \mathbb{N}$ and hence $\hat{z} = \sup_n \hat{z}_n$. To ensure that $\hat{z} \in X^\varkappa$ it suffices to check that $\|x\| \leqslant A$ for an arbitrary element $x \in X$ with $0 \leqslant x \leqslant \hat{z}$. For any such x put $y_n := \hat{z}_n \wedge x$ and observe that (y_n) is an increasing sequence in X_+ with $x = \sup_n y_n$. Moreover, (y_n) is Cauchy, since for arbitrary $n, l \in \mathbb{N}$ we can estimate:

$$\begin{aligned}\|y_{n+l} - y_n\| &= \|\hat{z}_{n+l} \wedge x - \hat{z}_n \wedge x\|_\varkappa \leqslant \|\hat{z}_{n+l} - \hat{z}_n\|_\varkappa \\ &= \sum_{k=n+1}^{n+l} C^k k \|\hat{x}_{k+1} - \hat{x}_k\|_\varkappa \leqslant \sum_{k=n+1}^{\infty} C^k k \|\hat{x}_{k+1} - \hat{x}_k\|_\varkappa \to 0\end{aligned}$$

as $n \to \infty$. The interval comleteness of X implies that the sequence (y_n) is convergent in X, so that $\lim_n y_n = \sup_n y_n = x$ by Lemma 2.6. Observe now that $\|x\| \leqslant A$, since $\|y_n\| \leqslant \|\hat{z}_n\|_\varkappa \leqslant A$ and $\|x\| = \lim_n \|y_n\| \leqslant A$, whence $\hat{z} \in X^\varkappa$.

Now we are able to show that $(\hat{x}_n)$ converges to $\hat{x}$. First note that $\hat{x} - \hat{x}_n = o\text{-}\sum_{k=n}^{\infty}(\hat{x}_{k+1} - \hat{x}_k)$, and consequently

$$n(\hat{x} - \hat{x}_n) \leqslant o\text{-}\sum_{k=n}^{\infty} k(\hat{x}_{k+1} - \hat{x}_k) \leqslant \hat{z}.$$

It follows that $0 \leqslant \hat{x} - \hat{x}_n \leqslant (1/n)\hat{z}$ and $\|\hat{x} - \hat{x}_n\|_\varkappa \leqslant (1/n)\|\hat{z}\|_\varkappa \to 0$. Appealing to Lemma 3.7 completes the proof. $\triangleright$

Theorem 3.10. *Let $(X, \|\cdot\|_X)$ be a Dedekind complete quasi-normed lattice with the weak σ-Fatou property. The maximal quasi-normed extension $(X^\varkappa, \|\cdot\|_\varkappa)$ is a quasi-Banach lattice if and only if X is intervally complete.*[1]

$\triangleleft$ The necessity is immediate from the fact that X is an order dense ideal of $X^\varkappa$. To prove the sufficiency observe that the metric completion $(Y, \|\cdot\|_Y)$ of $(X^\varkappa, \|\cdot\|_\varkappa)$ is Dedekind complete by Lemma 3.6. At the same time $X^\varkappa$ is order dense ideal of Y, since $X^\varkappa$ is intervally complete by Lemma 3.9 and an intervally complete quasi-normed lattice is an order dense ideal of its metric completion. Thus, $X \subset Y \subset (X^\varkappa)^u = X^u$ and $\|x\| = \|x\|_Y$ for all $x \in X$ so that $Y \subset X^\varkappa$ by Lemma 3.3. It follows that $Y = X^\varkappa$ and $X^\varkappa$ is complete. $\triangleright$

It is evident that if X has the Levi property then $X = X^\varkappa$ but the converse is false, see [1, Examples 2 and 5]. The next result asserts that the maximal quasi-normed extension with the weak Fatou property has the Levi property.

Theorem 3.11. *Let X be a Dedekind complete quasi-normed lattice. Then the maximal quasi-normed extension $X^\varkappa$ has the Levi property if and only if X has the weak Fatou property.*

$\triangleleft$ Let X be a Dedekind complete quasi-normed lattice with the weak Fatou constant K. Take an increasing net $(\hat{x}_\alpha)$ in $X^\varkappa$ with $B := \sup_\alpha \|\hat{x}_\alpha\|_\varkappa < \infty$. By Lemma 3.8 there exists a band projection π on $X^\varkappa$ such that $\hat{x} = \sup_\alpha \pi\hat{x}_\alpha$ exists in X^u and for every $N \in \mathbb{N}$ and $e \in X_+$ we have $N\pi^\perp e = \sup_\alpha \pi^\perp(\hat{x}_\alpha \wedge Ne)$. Making use of the weak Fatou property we deduce $N\|\pi^\perp e\| \leqslant K\sup_\alpha \|\pi^\perp(\hat{x}_\alpha \wedge Ne)\| \leqslant K\sup_\alpha \|\hat{x}_\alpha\|_\varkappa = BK$ and $\pi^\perp = 0$, since N and e are arbitrary. It follows that π is the identity operator and $\hat{x} = \sup_\alpha \hat{x}_\alpha$. Show that $\hat{x} \in X^\varkappa$. If $x \in X$ and $0 \leqslant x \leqslant \hat{x}$ then $x \wedge \hat{x}_\alpha \in X$ and $(x \wedge \hat{x}_\alpha)$ is an increasing net with the supremum x. By the weak Fatou property we have $\|x\| \leqslant K\sup_\alpha \|x \wedge \hat{x}_\alpha\| \leqslant K\sup_\alpha \|\hat{x}_\alpha\|_\varkappa = KB$. It follows that $\sup\{\|x\| : x \in X,\, 0 \leqslant x \leqslant \hat{x}\} \leqslant KB$ and $\hat{x} \in X^\varkappa$.

To prove the converse, it suffices to observe that if $X^\varkappa$ has the Levi property, then X has the weak Fatou property by Proposition 2.13. $\triangleright$

Proposition 3.12. *Let X be a Dedekind complete quasi-normed lattice. Then the maximal quasi-normed extension $X^\varkappa$ has the Fatou property if only if X has the Fatou property.*

[1] In the case of normed lattices Theorem 3.10 is true without the weak σ-Fatou property, see Abramovich [1]. We do not know whether or not the assumption about the weak σ-Fatou property is superfluous in Theorem 3.10.

$\lhd$ The necessity is obvious. To prove the sufficiency take an increasing net $(\hat{x}_\alpha)$ in $X_+^\varkappa$ such that $\hat{x} = \sup_\alpha \hat{x}_\alpha$ for some $\hat{x} \in X_+^\varkappa$. Pick an arbitrary $x \in X_+$ with $0 \leqslant x \leqslant \hat{x}$ and note that $\hat{x}_\alpha \wedge x$ is an increasing set in X_+ and $\sup_\alpha \hat{x}_\alpha \wedge x = x$. In virtue of the Fatou property we have $\|x\| = \sup_\alpha \|\hat{x}_\alpha \wedge x\| \leqslant \sup_\alpha \|\hat{x}_\alpha\|_\varkappa$. Hence, $\|x\| \leqslant \sup_\alpha \|\hat{x}_\alpha\|_\varkappa \leqslant \|\hat{x}\|_\varkappa$ for all $x \in X_+$ with $0 \leqslant x \leqslant \hat{x}$. The latter implies that $\|\hat{x}\|_\varkappa = \sup_\alpha \|\hat{x}_\alpha\|_\varkappa$. $\rhd$

Corollary 3.13. *Let X be a Dedekind complete quasi-normed lattice. If X has the Fatou property then the maximal quasi-normed extension $X^\varkappa$ has the Fatou and the Levi property.*

$\lhd$ The proof follows immediately from Theorem 3.11 and Proposition 3.12. $\rhd$

Remark 3.14. The maximal normed extension of a Dedekind complete normed lattice was introduced and the Theorem 3.10 was proved in Abramovich [1, Definition on p. 8 and Theorem 3]. Lemmas 3.6 and 3.7 for normed lattices can be seen in Veksler [70, Lemma 2] and [71, Theorem 1.1], respectively.

4. Kantorovich–Wright Integration

In this section X is a Dedekind σ-complete vector lattice and Ω is a nonempty set. Let $\mathscr{P}(\Omega)$ stands for the powerset of Ω. A *ring* (of subsets of Ω) is a subset $\mathscr{R} \subset \mathscr{P}(\Omega)$ such that $A \setminus B \in \mathscr{R}$ and $A \cup B \in \mathscr{R}$ for all $A, B \in \mathscr{R}$. A δ-*ring* is a ring, which is closed under the countable intersections. Let $\mathscr{R}^{\mathrm{loc}}$ stand for the collection of set $A \subset \Omega$ such that $A \cap B \in \mathscr{R}$ for all $B \in \mathscr{R}$; in symbols,

$$\mathscr{R}^{\mathrm{loc}} := \big\{A \in \mathscr{P}(\Omega) : A \cap B \in \mathscr{R} \text{ for all } B \in \mathscr{R}\big\}. \tag{1}$$

If $\mathscr{R}$ is a δ-ring then the collection $\mathscr{R}^{\mathrm{loc}}$ is a σ-algebra containing $\mathscr{R}$. Indeed given $A \in \mathscr{R}^{\mathrm{loc}}$ and a sequence $(A_n)_{n=1}^\infty$ in $\mathscr{R}^{\mathrm{loc}}$, we have $(\Omega \setminus A) \cap B = B \setminus (B \cap A) \in \mathscr{R}$ and $(\bigcap_{n=1}^\infty A_n) \cap B = \bigcap_{n=1}^\infty (A_n \cap B) \in \mathscr{R}$ for all $B \in \mathscr{R}$. Thus, $\Omega \setminus A \in \mathscr{R}^{\mathrm{loc}}$ and $\bigcap_{n=1}^\infty A_n \in \mathscr{R}^{\mathrm{loc}}$. Moreover $\mathscr{R} \subset \mathscr{R}^{\mathrm{loc}}$ trivially.

Definition 4.1. A function $\mu : \mathscr{R} \to X_+$ is said to be a *measure*, if $\mu(\varnothing) = 0$ and for every sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint sets $A_n \in \mathscr{R}$ with $\bigcup_{n=1}^\infty A_n \in \mathscr{R}$ the series $\sum_{n=1}^\infty \mu(A_n)$ is order convergent to $\mu(\bigcup_{n=1}^\infty A_n)$; in symbols,

$$\mu\bigg(\bigcup_{n=1}^\infty A_n\bigg) = o\text{-}\sum_{n=1}^\infty \mu(A_k) := \bigvee_{n=1}^\infty \bigg(\sum_{k=1}^n \mu(A_k)\bigg).$$

A triple $(\Omega, \mathscr{R}, \mu)$ is said to be a *vector measure space* if Ω is a nonempty set, $\mathscr{R}$ is a δ-ring of subsets of Ω, and $\mu : \mathscr{R} \to X_+$ is a measure.

Everywhere below $A_n \uparrow$ means that $A_n \subset A_{n+1}$ for all $n \in \mathbb{N}$, while $A_n \uparrow A$ means that $A_n \uparrow$ and $\bigcup_{n=1}^\infty A_n = A$. The meanings of $A_n \downarrow$ and $A_n \downarrow A$ are similar.

Lemma 4.2. *Let $A, B \in \mathscr{R}$ and $(A_n)_{n=1}^\infty$ be a sequence in $\mathscr{R}$. Then for a measure $\mu : \mathscr{R} \to X_+$ the following hold:*

(1) *If $A \subset B$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$ and $\mu(A) \leqslant \mu(B)$.*

(2) *If $A_n \uparrow A$ then $\mu(A_n) \uparrow$ and $\mu(A) = o\text{-}\lim_n \mu(A_n)$.*

(3) *If* $A_n \downarrow \varnothing$ *then* $\mu(A_n) \downarrow$ *and* $o\text{-}\lim_n \mu(A_n) = 0$.

(4) *If* $o\text{-}\sum_{n=1}^{\infty} \mu(A_n)$ *exists in* X *and* $A = \bigcup_{n=1}^{\infty} A_n$, *then* $\mu(A) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$.

$\lhd$ This can be proved by standard arguments from measure theory. $\rhd$

Definition 4.3. A set $A \in \mathscr{R}^{\mathrm{loc}}$ is called *negligible* (or, more precisely, *μ-negligible*) if $\mu(B \cap A) = 0$ for all $B \in \mathscr{R}$. Say that a property $P(\cdot)$ is true for *almost all* $\omega \in \Omega$ or *almost everywhere* (μ-a.e., for short) on Ω whenever $\{\omega \in \Omega : \ P(\omega) \text{ is false}\}$ is μ-negligible. Define a family $\mathscr{R}^*$ of subsets of Ω and a function $\mu^* : \mathscr{R}^* \to X_+$ by putting, as

$$\mathscr{R}^* := \{B \cup A : \ B \in \mathscr{R} \text{ and } A \text{ is negligible}\},$$
$$\mu^*(B \cup A) := \mu(B) \quad (B \in \mathscr{R}, \ A \in \mathscr{N}).$$

It follows from Lemma 4.2 that the collection $\mathscr{N}(\mu)$ of all μ-negligible sets is a *σ-ideal* in the sense that (1) $\varnothing \in \mathscr{N}(\mu)$, (2) if $A \in \mathscr{R}^{\mathrm{loc}}$, $B \in \mathscr{N}(\mu)$, and $A \subset B$, then $A \in \mathscr{N}(\mu)$, (3) countable union of negligible sets is negligible.

Lemma 4.4. *The family* $\mathscr{R}^*$ *is a* δ*-ring containing* $\mathscr{R}$ *and the function* μ^* *from* $\mathscr{R}^*$ *to* X_+ *is a measure extending* μ *from* $\mathscr{R}$ *to the* δ*-ring* $\mathscr{R}^*$.

$\lhd$ Clearly, $\mathscr{R}^*$ is closed under finite unions and intersections. Show that $\mathscr{R}^*$ contains also differences. Assume first that $B \in \mathscr{R}$ and A is negligible. Since $B \cap A \in \mathscr{R}$ and $\mathscr{R}$ is a ring, we have $B \setminus A = B \setminus (B \cap A) \in \mathscr{R}$. Now, for $B_1 \cup A_1, \ B_2 \cup A_2 \in \mathscr{R}^*$ with arbitrary $B_1, B_2 \in \mathscr{R}$ and negligible A_1, A_2 we deduce

$$(B_1 \cup A_1) \setminus (B_2 \cup A_2) = (B_1 \cup A_1) \cap (B_2^c \cap A_2^c) =$$
$$(B_1 \cap B_2^c \cap A_2^c) \cup (A_1 \cap B_2^c \cap A_2^c) = ((B_1 \setminus B_2) \setminus A_2) \cup C,$$

where $C = A_1 \cap B_2^c \cap A_2^c$ is negligible, since $\mathscr{N}(\mu)$ is an ideal of sets, while $(B_1 \setminus B_2) \setminus A_2 \in \mathscr{R}$ by virtue of the above observation. Consequently, $(B_1 \cup A_1) \setminus (B_2 \cup A_2) \in \mathscr{R}^*$.

It remains to ensure that the family $\mathscr{R}^*$ is closed under countable intersections. For a sequence $(B_n \cup A_n)_{n=1}^{\infty}$ in $\mathscr{R}^*$ with $B_n \in \mathscr{R}$ and negligible A_n put $B := \bigcap_{n=1}^{\infty} B_n \in \mathscr{R}$ and $A := \bigcup_{n=1}^{\infty} A_n$. Then $B \subset \bigcap_{n=1}^{\infty} (B_n \cup A_n) \subset B \cup A$. Taking into account the fact that $\mathscr{N}(\mu)$ is a σ-ideal we conclude that A is negligible and the desire relation $\bigcap_{n=1}^{\infty} (B_n \cup A_n) \in \mathscr{R}^*$ holds. $\rhd$

By virtue of Lemma 4.4 we can assume without loss of generality that the δ-ring $\mathscr{R}$ contains all negligible sets and $\mu(A) = 0$ for each negligible set $A \in \mathscr{R}^{\mathrm{loc}}$. Now we present briefly the construction of Kantorovich–Wright integration for positive vector measure takin values in a Dedekind σ-complete vector lattice coming back to Kantorovich [39, 38] and Wright [73, 74].

Definition 4.5. A function $f : \Omega \to \mathbb{R}$ is called an *$\mathscr{R}$-simple* if it is a finite linear combination of characteristic functions of sets in $\mathscr{R}$, that is, f admits a representation $f = \sum_{k=1}^{n} a_k \chi_{A_k}$ with $A_1, \dots, A_n \in \mathscr{R}$ and $a_1, \dots, a_n \in \mathbb{R}$. In this representation neither $a_1, \dots, a_n$ are distinct nor $A_1, \dots, A_n$ are nonempty. (By definition $\chi_{\varnothing} = 0$.) But $A_1, \dots, A_n$ always may be chosen pairwise disjoint. Denote by $S(\mathscr{R})$ the set of all $\mathscr{R}$-simple functions. Given an $\mathscr{R}$-simple function $f = \sum_{k=1}^{n} a_k \chi_{A_k}$, the integral $\int f \, d\mu$ is defined by

$$I_{\mu}^{o}(f) := \int f \, d\mu := \sum_{i=1}^{n} a_i \mu(A_i).$$

Integral of a simple function is well-defined, i.e., if a simple function f is representable as $f = \sum_{i=1}^{n} a_i\chi_{A_i}$ and $f = \sum_{j=1}^{m} b_j\chi_{B_j}$, then $\sum_{i=1}^{n} a_i\mu(A_i) = \sum_{j=1}^{m} b_j\mu(B_j)$. We omit the verification of this fact, as well as the proofs of the following two lemmas, as routine exercises.

Lemma 4.6. *The set $S(\mathscr{R})$ with the pointwise operations and ordering is a vector lattice and the mapping $I_\mu^o : f \mapsto I_\mu^o(f)$ from $S(\mathscr{R})$ to X is a positive linear operator. Moreover, $I_\mu^o(|f|) = 0$ if and only if $f = 0$ μ-a.e. for all $f \in S(\mathscr{R})$.*

Lemma 4.7. *If $0 \leqslant g = \sum_{i=1}^{n} b_i\chi_{B_i}$ is a $\mathscr{R}$-simple function and $0 \leqslant f = \sum_{j=1}^{m} a_j\chi_{A_j}$ is a $\mathscr{R}^{\mathrm{loc}}$-simple function, then the function $g \wedge f$ is $\mathscr{R}$-simple.*

Lemma 4.8. *Let (f_n) be a monotone decreasing sequence of positive functions in $S(\mathscr{R})$ such that $\lim_n f_n = 0$ μ-almost everywhere. Then $\bigwedge_{n=1}^{\infty} I_\mu^o(f_n) = 0$.*

◁ The proof is similar to that of [73, Proposition 3.1]. ▷

Definition 4.9. Say that a $\mathscr{R}^{\mathrm{loc}}$-measurable real-valued function f defined on a conegligible subset of Ω is *integrable*, if there exists a sequence $(f_n)_{n=1}^{\infty}$ of $\mathscr{R}$-simple functions such that $0 \leqslant f_n \uparrow f$ μ-a.e. and $\bigvee_{n=1}^{\infty} \int f_n\,d\mu$ exists in X_+. In this occasion we put

$$I_\mu^o(f) := \int f\,d\mu := \bigvee_{n=1}^{\infty} \int f_n\,d\mu.$$

(A subset $A \subset \Omega$ is *conegligible* if $\Omega \setminus A$ is negligible.) An arbitrary $\mathscr{R}^{\mathrm{loc}}$-measurable function f is *integrable*, whenever so are f^+ and f^-. The integral of f is defined as $I_\mu^o(f) = I_\mu^o(f^+) - I_\mu^o(f^-)$.

The following lemma says that the integral I_μ^o is well defined.

Lemma 4.10. *Let $f : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ be an $\mathscr{R}^{\mathrm{loc}}$-measurable function and let (f_n) and (g_n) be increasing sequences of positive $\mathscr{R}$-simple functions. If $\lim_n f_n = f = \lim_n g_n$ μ-almost everywhere then*

$$x := \bigvee_{n=1}^{\infty} I_\mu^o(f_n) = \bigvee_{n=1}^{\infty} I_\mu^o(g_n) =: y,$$

provided that at least one of the least upper bounds exist in X.

◁ Observe that the sequence $(f_n \wedge g_m)_{n\in\mathbb{N}}$ increases and converges μ-a.e. g_m for each fixed $m \in \mathbb{N}$. Moreover, $(f_n \wedge g_m)_{n\in\mathbb{N}}$ lies in $S(\mathscr{R})$ by Lemma 4.7. Applying Lemma 4.8 yields

$$I_\mu^o(g_m) = \bigvee_{n=1}^{\infty} I_\mu^o(f_n \wedge g_m) \leqslant \bigvee_{n=1}^{\infty} I_\mu^o(f_n) = x.$$

It follows that $y \leqslant x$ and similarly $x \leqslant y$. ▷

Denote by $\mathscr{L}^0(\mu) := \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ the set of all real-valued $\mathscr{R}^{\mathrm{loc}}$-measurable functions defined on conegligible subsets of Ω. Say that two functions $f, g \in \mathscr{L}^0(\mu)$ are equivalent and write $f \sim g$ if $f(\omega) = g(\omega)$ for μ-a.e. $\omega \in \Omega$. Let $L^0(\mu) := L^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ stand for the set of equivalence classes in $\mathscr{L}^0(\mu)$ under $\sim$. For $f \in \mathscr{L}^0(\mu)$, write $\tilde{f}$ for its equivalence class in $L^0(\mu)$. The linear structure and the ordering of $L^0(\mu)$ are conventionally defined using pointwise operations and order relation, see Fremlin [24, § 241].

Let $\mathscr{L}^1_o(\mu) := \mathscr{L}^1_o(\Omega, \mathscr{R}^{\rm loc}, \mu)$ be the set of real-valued μ-integrable functions defined on conegligible subsets of Ω. Thus, for $f \in \mathscr{L}^0(\mu)$, we have $f \in \mathscr{L}^1_o(\mu)$ if there is a $g \in \mathscr{L}^1_o(\mu)$ such that $|f| \leqslant |g|$ μ-a.e.; in particular, $\mathscr{L}^1_o(\mu) \subset \mathscr{L}^0(\mu)$. Let $L^1_o(\mu) := L^1_o(\Omega, \mathscr{R}, \mu)$ be the set of equivalence classes of members of $\mathscr{L}^1_o(\mu)$. Define an operator $I^o_\mu : L^1_o(\mu) \to X$ by writing $I^o_\mu(\tilde{f}) := I^o_\mu(f)$ for every $f \in \mathscr{L}^1_o(\mu)$.

Lemma 4.11. *The following assertions hold:*

(1) $L^0(\mu)$ *is a Dedekind σ-complete vector lattice.*

(2) $L^1_o(\mu)$ *is an order dense ideal in* $L_0(\mu)$*.*

(3) $L^0(\mu)$ *is a Dedekind complete if and only if μ is localizable.*

$\lhd$ The proof of (1) is standard and the proof of (3) is similar to [24, Theorem 241G]. The fact that $L^1_o(\mu)$ is an order ideal in $L^0(\mu)$ follows from Lemma 4.7. Show that $L^1_o(\mu)$ is order dense in $L^0(\mu)$.

Take $0 < \tilde{f} \in L_0(\mu)$ and put $\{f \geqslant n^{-1}\} := \{\omega \in \Omega : f(\omega) \geqslant n^{-1}\} \in \mathscr{R}^{\rm loc}$ with $f \in \mathscr{L}_0(\mu)$ and $n \in \mathbb{N}$. If $\mu(B \cap \{f \geqslant n^{-1}\}) = 0$ for all $B \in \mathscr{R}$ and $n \in \mathbb{N}$ then $\{f \geqslant n^{-1}\}$ is μ-negligible for all $n \in \mathbb{N}$. Moreover, $\{f \geqslant n^{-1}\} \in \mathscr{R}$ and $\mu(\{f \geqslant n^{-1}\}) = 0$ for all $n \in \mathbb{N}$ by the remark after the proof of Lemma 4.4. It follows that $f = 0$ μ-a.e. which contradicts the choice of $\tilde{f} > 0$. Thus, $\mu(B \cap \{f \geqslant n_0^{-1}\}) > 0$ for some $n_0 \in \mathbb{N}$ and $B_0 \in \mathscr{R}$. Evidently, $0 < \chi_C \in L^1_o(\mu)$ and $n_0^{-1}\chi_C \leqslant f$ with $C := B_0 \cap \{f \geqslant n_0^{-1}\}$. $\rhd$

The variants of the convergence theorems of Lebesgue integration theory are true for Kantorovich–Wright integral. The proofs of the following three results can be given along the lines of [73, Propositions 3.3–3.5] and [42, Theorems 6.1.4 and 6.1.5].

Theorem 4.12 (Monotone convergence)**.** *Let (f_n) be a sequence in $\mathscr{L}^1_o(\mu)$ such that $f_n \leqslant f_{n+1}$ μ-a.e. for each $n \in \mathbb{N}$ and $\{I^o_\mu(f_n) : n \in \mathbb{N}\}$ is order bounded in X. Then there exists $f \in \mathscr{L}^1_o(\mu)$ such that (f_n) converges to f μ-a.e. and*

$$I^o_\mu(f) = o\text{-}\lim_n I^o_\mu(f_n).$$

Theorem 4.13 (Fatou's Lemma)**.** *Let $(f_n)_{\in\mathbb{N}}$ be a sequence in $\mathscr{L}^1_o(\mu)_+$ such that $0 \leqslant f_n$ μ-a.e. for each $n \in \mathbb{N}$ and $\liminf_n I^o_\mu(f_n) := \bigvee_{n=1}^\infty \bigwedge_{k=n} I^o_\mu(f_k)$ exists in X_+. Then the function $\liminf_n f_n \in L^1_o(\Omega, \mathscr{R}^{\rm loc}, \mu)_+$ and*

$$I^o_\mu(\liminf_n f_n) \leqslant \liminf_n I^o_\mu(f_n).$$

Theorem 4.14 (Dominated convergence)**.** *Let (f_n) be a sequence of functions in $\mathscr{L}^1_o(\mu)$ converging to $f \in \mathscr{L}^0(\mu)$ μ-a.e. If there exists $g \in \mathscr{L}^1_o(\mu)$ such that $|f_n| \leqslant g$ μ-a.e. for each $n \in \mathbb{N}$, then f is integrable, $(I^o_\mu(f_n))$ is order convergent, and*

$$I^o_\mu(f) = o\text{-}\lim_n I^o_\mu(f_n).$$

Theorem 4.15. *Let X be a Dedekind σ-complete vector lattice and $\mu : \mathscr{R} \to X_+$ a measure. Then $L^1_o(\mu)$ is a Dedekind σ-complete vector lattice and the mapping $I^o_\mu : L^1_o(\mu) \to X$ is a strictly positive order σ-continuous linear operator.*

$\lhd$ This is immediate from Lemmas 4.6 and 4.11 and Theorem 4.14. $\rhd$

5. The Smallest Extension of the Kantorovich–Wright Integration

Definition 5.1. Let E and F be vector lattices with F Dedekind complete and let G be an order ideal of E. Consider a positive operator $S : G \to F$ and denote by $\hat{G}$ the collection of all $x \in X$ such that the set $\{S(g) : g \in G,\, 0 \leqslant g \leqslant |x|\}$ is order bounded in F. Then $\hat{G}$ is an order ideal in E and we may define

$$\hat{S}x := \sup\{Sg : g \in G,\ 0 \leqslant g \leqslant x\} := \sup\{S(g \wedge x) : g \in G\} \quad (x \in E_+).$$

The operator $\hat{S} : \hat{G}_+ \to F$ is additive and positively homogeneous, so it can be extended to $\hat{G}$ by differences. The resulting operator, which we denote by $\hat{S}$ again, extends S and is less or equal to every other positive extension of S to all of $\hat{G}$. The operator $\hat{S}$ is naturally called the *smallest extension*[2] *of S with respect to E*, see [4, Theorem 1.30] and [42, 3.1.3].

Specify some properties of the smallest extension. Below, in Lemmas 5.2–5.5, E, F, G, $\hat{G}$, S, and $\hat{S}$ are the same as in Definition 5.1.

Lemma 5.2. *The smallest extension $\hat{S}$ is order continuous or order σ-continuous if and only if so is S.*

$\lhd$ See [4, Theorem 1.64]. $\rhd$

Lemma 5.3. *The following assertions hold:*

(1) *$\hat{S}$ is a lattice homomorphism if and only if so is S.*

(2) *If G is order dense in E then $\hat{S}$ is strictly positive if and only if so is S.*

$\lhd$ In both cases the necessity is trivial.

(1) Assume that S is a lattice homomorphism and ensure that $\hat{S}(x_1) \wedge \hat{S}(x_2) = 0$ for all $x_1, x_2 \in \hat{G}$ with $x_1 \wedge x_2 = 0$. Indeed, making use of Definition 5.1 we deduce

$$\hat{S}(x_1) \wedge \hat{S}(x_2) = \Big(\bigvee\nolimits_{g_1 \in G_+} S(g_1 \wedge x_1)\Big) \wedge \Big(\bigvee\nolimits_{g_2 \in G_+} S(g_2 \wedge x_2)\Big)$$
$$\bigvee\nolimits_{g_1, g_2 \in G_+} S(g_1 \wedge x_1) \wedge S(g_2 \wedge x_2) = 0$$

which implies that $\hat{S}$ is a lattice homomorphism by [4, Theorem 2.14].

(2) Given $0 < x \in \hat{G}$, we can pick $g \in G$ with $0 < g \leqslant x$, since G is order dense in E and hence in $\hat{G}$. Now, if S is strictly positive then $\hat{S}(x) \geqslant \hat{S}(g) = S(g) > 0$. $\rhd$

Lemma 5.4. *Assume that $(x_\alpha)_{\alpha \in \mathrm{A}}$ is an increasing net in $\hat{G}_+$ and there exists $x = \sup_\alpha x_\alpha$ in E. If $\hat{S}$ is order continuous and $\sup_\alpha \hat{S}(x_\alpha)$ exists in F, then $x \in \hat{G}$ and $\hat{S}(x) = \sup_\alpha \hat{S}(x_\alpha)$. A similar statement holds also for an order σ-continuous $\hat{S}$ and an increasing sequence (x_n) in $\hat{G}_+$.*

$\lhd$ We restrict ourselves to an order continuous $\hat{S}$. By Lemma 5.2 S is also order continuous, so that it suffices to show $x \in \hat{G}$. Take an arbitrary $g \in G$ with $0 \leqslant g \leqslant x$ and note that $(x_\alpha \wedge g)$ is an increasing net in G_+ and $g = \sup_\alpha x_\alpha \wedge g$. Order continuity of S yields then $0 \leqslant S(g) = \sup_\alpha S(x_\alpha \wedge g) \leqslant \sup_\alpha \hat{S}(x_\alpha)$. It follows that $S([0, x] \cap G)$ is order bounded in F and hence $x \in \hat{G}$. $\rhd$

[2]Evidently, $\hat{S} \leqslant T$ for any other positive extension $T : \hat{G} \to F$ of S to all of E, see [4, p. 27]. At the same time $\hat{G}$ is the largest order ideal of E to which S extends positively, [42, 3.6.1 (4)].

Lemma 5.5. *Assume that E is universally complete, $\hat{G}$ is order dense in E, and $\hat{S}$ is strictly positive and order continuous. Assume further that (x_α) is increasing net in $\hat{G}_+$ such that $y := \sup_\alpha \hat{S}(x_\alpha)$ exists in F. Then there exists $x \in \hat{G}$ with $x = \sup_\alpha x_\alpha$ and $y = \hat{S}(x)$.*

$\lhd$ By Lemma 3.8 there exists a band projection π in E and an element $x \in E_+$ such that $x = \sup_\alpha \pi x_\alpha$, while for the complementary projection $\pi' := I_E - \pi$ we have $k\pi' e = \sup_\alpha \pi'(x_\alpha \wedge ke)$ for all $e \in \hat{G}_+$ and $k \in \mathbb{N}$. If $\pi' = 0$ then $x = \sup_\alpha \pi x_\alpha$ and the claim follows from Lemma 5.4. Using order continuity of $\hat{S}$ yields $0 \leqslant k\hat{S}(\pi' e) = \sup_\alpha \hat{S}(\pi'(x_\alpha \wedge ke)) \leqslant \sup_\alpha \hat{S}(x_\alpha) = y$. Since $k \in \mathbb{N}$ is arbitrary, $\hat{S}(\pi' e) \leqslant y/k$ implies $\hat{S}(\pi' e) = 0$ and hence $\pi' e = 0$ for all $e \in \hat{G}_+$ because S is strictly positive. It follows that $\pi' = 0$ and the proof is complete. $\rhd$

Given a vector measure $\mu : \mathscr{R} \to X_+$, apply Definition 5.1 to $E := L^0(\mu)$, $G := L^1_o(\mu)$ and $S := I^o_\mu$. Denote by $\hat{I}^o_\mu$ the smallest extension of I^o_μ and by $L^1_{ow}(\mu)$ the domain $\hat{G}$ of $\hat{I}^o_\mu$. Then $L^1_o(\mu) \subset L^1_{ow}(\mu) \subset L^0(\mu)$.

Definition 5.6. The vector lattice $L^1_{ow}(\mu) \subset L^0(\mu)$ is called the *space of weakly integrable* function with respect to μ, while a measurable function $f \in \mathscr{L}^0(\mu)$ is called *weakly integrable* with respect to μ if $\tilde{f} \in L^1_{ow}(\mu)$.

In view of Lemma 5.5 it is important to know when the vector lattice $L^0(\mu)$ is Dedekind complete. As in the case of scalar measures the answer is given in terms of localizability.

Definition 5.7. A measure $\mu : \mathscr{R} \to X_+$ is called *semi-finite* if for every not μ-negligible set $A \in \mathscr{R}^{\rm loc}$ there exists $B \in \mathscr{R}$ such that $B \subset A$ and $\mu(B) > 0$.

Definition 5.8. A measure $\mu : \mathscr{R} \to X$ is said to be *localizable* if for every collection $\mathscr{A} \subset \mathscr{R}^{\rm loc}$, there exists $B \in \mathscr{R}^{\rm loc}$ such that (i) $A \setminus B$ is μ-negligible for all $A \in \mathscr{A}$ and (ii) if $C \in \mathscr{R}^{\rm loc}$ and $A \setminus C$ is μ-negligible for all $A \in \mathscr{A}$, then $B \setminus C$ is also μ-negligible.

Consider the measure space $(\Omega, \mathscr{R}^{\rm loc}, \mu)$ and denote by $\mathscr{N}(\mu)$ the ideal of μ-negligible sets in $\mathscr{R}^{\rm loc}$. Define an equivalence relation $\sim$ on $\mathscr{R}^{\rm loc}$ by putting $A_1 \sim A_2$ whenever $A_1 \triangle A_2 \in \mathscr{N}(\mu)$. Let $\mathbb{B}(\mu)$ stands for the Boolean algebra quotient $\mathscr{R}^{\rm loc}/\mathscr{N}(\mu)$. The coset of a set $A \in \mathscr{R}^{\rm loc}$ we will denote by $\tilde{A} \in \mathbb{B}(\mu)$. Then $\mathbb{B}(\mu)$ is a Dedekind σ-complete Boolean algebra and the canonical map $A \mapsto \tilde{A}$ from $\mathscr{R}^{\rm loc}$ onto $\mathbb{B}(\mu)$ is an order σ-continuous Boolean homomorphism. Note that for $\tilde{A}, \tilde{B} \in \mathbb{B}(\mu)$ we have $\tilde{A} \leqslant \tilde{B}$ if and only if there exist $A_1 \in \widetilde{A}$ and $B_1 \in \widetilde{B}$ such that $A_1 \subset B_1$.

Lemma 5.9. *A measure $\mu : \mathscr{R} \to X_+$ is localizable if and only if the Boolean algebra $\mathbb{B}(\mu)$ is Dedekind complete.*

$\lhd$ The proof given in [25, 322B (d, e)] for scalar measures carries over verbatim. $\rhd$

Theorem 5.10. *For a vector measure $\mu : \mathscr{R} \to X_+$ the following are equivalent:*

(1) *$\mu : \mathscr{R} \to X_+$ is localizable.*

(2) *$\mathbb{B}(\mu)$ is complete Boolean algebra.*

(3) *$L^0(\mu)$ is universally complete vector lattice.*

$\lhd$ The equivalence (1) $\iff$ (2) follows from Lemma 5.9. To ensure that (2) $\iff$ (3) it suffices to observe that $L^0(\mu)$ is a universally σ-complete vector lattice with the constant function one on Ω taken as an order unit $\mathbb{1}$, the Boolean algebras $\mathbb{B}(\mu)$ and $\mathscr{C}(\mathbb{1})$ are isomorphic, and $L^0(\mu)$ is Dedekind complete if and only if so is $\mathscr{C}(\mathbb{1})$, see [72, Theorem V.4.3]. ▷

Corollary 5.11. *If the measure $\mu : \mathscr{R} \to X_+$ is semi-finite and localizable then $L^1_o(\mu)$ and $L^1_{ow}(\mu)$ are Dedekind complete and order dense ideals of $L_0(\mu)$.*

◁ This is immediate from Theorem 5.10, since $L^1_o(\mu)$ and $L^1_{ow}(\mu)$ are order dense ideals of $L^0(\mu)$. ▷

Lemma 5.12. *Let X be a Dedekind complete vector lattice and $\mu : \mathscr{R} \to X_+$ a semi-finite localizable measure. Assume that an operator $\hat{I}^o_\mu : L^1_{ow}(\mu) \to X$ is order continuous. If $(f_\alpha)_{\alpha\in\mathrm{A}}$ is an increasing net in $L^1_{ow}(\mu)$ and there exists $y := \sup_{\alpha\in\mathrm{A}} \hat{I}^o_\mu(f_\alpha)$ in X_+ then there is $f \in L^1_{ow}(\mu)$ such that $\sup_\alpha f_\alpha = f$ and $\hat{I}^o_\mu(f) = y$.*

◁ According to Corollary 5.11 and Theorem 5.10 $L^1_{ow}(\mu)$ is an order dense ideal of universally complete vector lattice $L^0(\mu)$. By Theorem 4.15 and Lemma 5.3(2) the operator $\hat{I}^o_\mu$ is order continuous and strictly positive. Hence, we can apply Lemma 5.5 to $\hat{I}^o_\mu$. ▷

6. Direct Sums of Vector Measures

Now we introduce a basic construction of the direct sums of a family of vector measure spaces. Everywhere in this section X is a Dedekind σ-complete vector lattice.

Consider an indexed family $\big(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\big)_{\alpha\in\mathrm{A}}$ of vector measure spaces. We will assume that Ω_α are pairwise disjoint. (Otherwise we replace Ω_α by $\Omega_\alpha \times \{\alpha\}$.) Moreover, for simplicity, we assume that Σ_α is a σ-algebra for all $\alpha \in \mathrm{A}$.

Definition 6.1. Say that a triple $(\Omega, \mathscr{R}, \mu)$ is the *direct sum* of the family of measure spaces $\big((\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\big)_{\alpha\in\mathrm{A}}$ or μ is the *direct sum* of the family of vector measures $(\mu_\alpha)_{\alpha\in\mathrm{A}}$, whenever it satisfies the following conditions:

(1) $\Omega = \bigcup_{\alpha\in\mathrm{A}} \Omega_\alpha$;

(2) the collection $\mathscr{R} \subset 2^\Omega$ comprises the sets of the form $\bigcup_{\alpha\in\mathrm{A}} A_\alpha \subset \Omega$, where $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$ and $\mu_\alpha(A_\alpha) = 0$ except for at most a finite set of $\alpha \in \mathrm{A}$;

(3) $\mu(A) = \sum_{\alpha\in\mathrm{A}} \mu_\alpha(A_\alpha)$ for any $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha \in \mathscr{R}$ with $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$.

Remark 6.2. Every set $A \in \mathscr{R}$ has a unique representation $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha$ with $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$, since $(\Omega_\alpha)_{\alpha\in\mathrm{A}}$ is a family of mutually disjoint subsets of Ω. In particular, the mapping $\mu : \mathscr{R} \to X_+$ in 6.1 (3) is well defined. The next lemma asserts that the direct sum $(\Omega, \mathscr{R}, \mu)$ in Definition 6.1 is a vector measure space.

Lemma 6.3. *The collection $\mathscr{R}$ is a δ-ring and the mapping μ is a measure.*

◁ Consider two sets $A_1 = \bigcup_{\alpha\in\mathrm{A}} A_{1,\alpha}$, $A_2 = \bigcup_{\alpha\in\mathrm{A}} A_{2,\alpha} \in \mathscr{R}$. There exist finite sets $\theta_1, \theta_2 \subset \mathrm{A}$ such that $\mu_\alpha(A_{i,\alpha}) = 0$ for all $\alpha \in \mathrm{A} \setminus \theta_i$, $i = 1, 2$. Consequently,

$$A_1 \cup A_2 = \Big(\bigcup\nolimits_{\alpha\in\mathrm{A}} A_{1,\alpha}\Big) \cup \Big(\bigcup\nolimits_{\alpha\in\mathrm{A}} A_{2,\alpha}\Big) = \bigcup\nolimits_{\alpha\in\mathrm{A}} (A_{1,\alpha} \cup A_{2,\alpha}) \in \mathscr{R},$$

since $\{\alpha \in \mathrm{A} : \mu_\alpha(A_{1,\alpha} \cup A_{2,\alpha}) \neq 0\} = \theta_1 \cup \theta_2$. At the same time

$$A_1 \setminus A_2 = \bigcup\nolimits_{\alpha\in\mathrm{A}} (A_{1,\alpha} \setminus A_{2,\alpha}) \in \mathscr{R},$$

since $A_{1,\alpha} \setminus A_{2,\alpha} \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$, $\{\alpha \in \mathrm{A} : \mu_\alpha(A_{1,\alpha} \setminus A_{2,\alpha}) \neq 0\} \subset \theta_1$, and $\Omega_\alpha \cap \Omega_\beta = \varnothing$ for all $\alpha \neq \beta$. Moreover, $A_1 \cap A_2 = A_1 \setminus (A_1 \setminus A_2)$ implies $A_1 \cap A_2 \in \mathscr{R}$.

Assume now that for $n \in \mathbb{N}$ a set $A_n \in \mathscr{R}$ has the representation $A_n := \bigcup_{\alpha\in\mathrm{A}} A_{n,\alpha}$ with $A_{n,\alpha} \in \Sigma_\alpha$ for all $n \in \mathbb{N}$ and $\alpha \in \mathrm{A}$. Then the equality

$$\bigcap_{n=1}^{\infty} A_n = \bigcup_{\alpha\in\mathrm{A}} \bigcap_{n=1}^{\infty} A_{n,\alpha} \in \mathscr{R}$$

holds with the right-hand side in $\mathscr{R}$. Indeed, if $x \in \bigcap_{n=1}^{\infty} A_n$, then there exists a function $\nu : \mathbb{N} \to \mathrm{A}$ such that $x \in \bigcap_{n=1}^{\infty} A_{n,\nu(n)}$. Since $A_{n,\nu(n)} \cap A_{m,\nu(m)} = \varnothing$ for all $\nu(n) \neq \nu(m)$, there is $\bar{n} \in \mathbb{N}$ such that $x \in \bigcap_{n=1}^{\infty} A_{n,\nu(\bar{n})}$. This proves the inclusion $\subset$ and the converse inclusion is trivial. It follows that $\mathscr{R}$ is a δ-ring.

It remains to prove that $\mu : \mathscr{R} \to X_+$ is a measure. First note that $\mu(\varnothing) = 0$. Let (A_n) be the above sequence of $\mathscr{R}$ with the additional assumptions that A_n are disjoint sets and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$.

Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{\alpha\in\mathrm{A}} \bigcup_{n=1}^{\infty} A_{n,\alpha}$ and, as $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$, there exists a finite subset $\theta \subset \mathrm{A}$ such that $\mu_\alpha(\bigcup_{n=1}^{\infty} A_{n,\alpha}) = 0$ for all $\alpha \in \mathrm{A} \setminus \theta$. Consequently,

$$\mu\biggl(\bigcup_{n=1}^{\infty} A_n\biggr) = \mu\biggl(\bigcup_{\alpha\in\mathrm{A}} \bigcup_{n=1}^{\infty} A_{n,\alpha}\biggr) = \sum_{\alpha\in\theta} \mu\biggl(\bigcup_{n=1}^{\infty} A_{n,\alpha}\biggr) =$$

$$\sum_{\alpha\in\theta} o\text{-}\sum_{n=1}^{\infty} \mu_\alpha(A_{n,\alpha}) = o\text{-}\sum_{n=1}^{\infty} \sum_{\alpha\in\theta} \mu_\alpha(A_{n,\alpha}) = o\text{-}\sum_{n=1}^{\infty} \mu(A_n).$$

Thus, μ is countable additive, as claimed. $\rhd$

Recall that the σ-algebra $\mathscr{R}^{\mathrm{loc}}$ comprises all sets $A \subset \Omega$ such that $A \cap B \in \mathscr{R}$ for all $B \in \mathscr{R}$.

Lemma 6.4. *A subset $A \subset \Omega$ belongs to $\mathscr{R}^{\mathrm{loc}}$ if and only if the representation $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha$ holds with $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$. Moreover such representation is unique.*

$\lhd$ Let $\mathscr{C}$ stand for the collection of sets representable as $\bigcup_{\alpha\in\mathrm{A}} A_\alpha$ with $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$. If $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha \in \mathscr{C}$ and $B = \bigcup_{\beta\in\mathrm{A}} B_\beta \in \mathscr{R}$, then $A \cap B = \bigcup_{\alpha\in\mathrm{A}} (A_\alpha \cap B_\alpha)$, since $A_\alpha \cap B_\beta = \varnothing$ for $\alpha \neq \beta$. By 6.1 (2) the relation $B \in \mathscr{R}$ implies the existence of a finite set $\theta \subset \mathrm{A}$ such that $\mu_\alpha(B_\alpha) = 0$ for $\alpha \in \mathrm{A} \setminus \theta$. From this we see that $\mu_\alpha(A_\alpha \cap B_\alpha) = 0$ for all $\alpha \in \mathrm{A} \setminus \theta$, whence $A \cap B \in \mathscr{R}$. As $B \in \mathscr{R}$ is arbitrary, $A \in \mathscr{R}^{\mathrm{loc}}$ and we get the inclusion $\mathscr{C} \subset \mathscr{R}^{\mathrm{loc}}$. To ensure the converse inclusion, note that $A \in \mathscr{R}^{\mathrm{loc}}$ implies $A \cap \Omega_\alpha \in \mathscr{R}$, since $\Omega_\alpha \in \mathscr{R}$. Thus, the relation $A \in \mathscr{C}$ follows from the representation $A = A \cap \Omega = \bigcup_{\alpha\in\mathrm{A}} (A \cap \Omega_\alpha)$. Consequently, $\mathscr{C} = \mathscr{R}^{\mathrm{loc}}$. The uniqueness of the representation $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha$ follows from the fact that $(\Omega_\alpha)_{\alpha\in\mathrm{A}}$ is a family of mutually disjoint sets. $\rhd$

Lemma 6.5. *Let the equality $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha$ hold with $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \mathrm{A}$. Then A is μ-negligible if and only if A_α is μ_α-negligible for all $\alpha \in \mathrm{A}$.*

$\lhd$ Let $\mu(A_\alpha) = 0$ for all $\alpha \in \mathrm{A}$ and ensure that then A is μ-negligible. Indeed, take an arbitrary $B \in \mathscr{R}$ with $B \subset A$ and note that, as $B = \bigcup_{\alpha\in\mathrm{A}} (A_\alpha \cap B)$, in view of Remark 6.2 there exists a finite subset $\theta \subset \mathrm{A}$ such that $\mu(B) = \sum_{\alpha\in\theta} \mu(B \cap A_\alpha) = 0$. By Definition 4.3 A is μ-negligible.

Conversely, if $A = \bigcup_{\alpha\in\mathrm{A}} A_\alpha \in \mathscr{R}^{\mathrm{loc}}$ is μ-negligible, then taking $B := A_\alpha$ in Definition 4.3 yields $\mu(A_\alpha) = 0$ for all $\alpha \in \mathrm{A}$. $\rhd$

Remark 6.6. For a function $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}\mu)$ denote by $f|_\alpha$ the restriction of f to Ω_α. Observe that if $g = \sum_{k=1}^n a_k\chi_{A_k}$ and $A_k = \bigcup_{\alpha\in\mathrm{A}} A_{k,\alpha} \in \mathscr{R}^{\mathrm{loc}}$, then $g|_\alpha = \sum_{k=1}^n a_k\chi_{A_{k,\alpha}}$, i. e. $g|_\alpha$ is a Σ_α-simple function for all $\alpha \in \mathrm{A}$.

Lemma 6.7. *A function $f : \Omega \to \mathbb{R}\cup\{\pm\infty\}$ belongs to $\mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}\mu)$ if and only if $f|_\alpha$ belongs to $\mathscr{L}^0(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ for all $\alpha \in \mathrm{A}$.*

$\lhd$ It is immediate from Lemma 6.4 that $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}\mu)$ implies $f|_\alpha \in \mathscr{L}^0(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ for all $\alpha \in \mathrm{A}$. Conversely, assuming $f|_\alpha \in \mathscr{L}^0(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ for all $\alpha \in \mathrm{A}$, we may apply Lemma 6.4 again, taking the representation $f^{-1}(B) = \bigcup_{\alpha\in\mathrm{A}} (f|_\alpha)^{-1}(B)$ into account, to get $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}\mu)$. $\rhd$

Lemma 6.8. *For $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ and $\alpha \in \mathrm{A}$ we have $f\chi_{\Omega_\alpha} \in \mathscr{L}^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ if and only if $f|_\alpha \in \mathscr{L}^1_o(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$. Moreover, $\int f\chi_{\Omega_\alpha}\, d\mu = \int f|_\alpha\, d\mu_\alpha$.*

$\lhd$ It suffices to verify the claim for positive $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$. Suppose that $f\chi_{\Omega_\alpha} \in \mathscr{L}^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$. Take an $\mathscr{R}$-simple function $g = \sum_{i=1}^n a_i\chi_{A_i}$ such that $0 \leqslant g \leqslant f\chi_{\Omega_\alpha}$ μ-a.e. and $A_k = \bigcup_{\alpha\in\mathrm{A}} A_{k,\alpha} \in \mathscr{R}$ for all $k = 1, \dots, n$. Because $\{g \neq 0\} \subset \Omega_\alpha$ up to a μ-negligible set, $\mu(A_k) = \mu_\alpha(A_{k,\alpha})$ for all $k = 1, \dots, n$. It follows that

$$\int g\, d\mu = \sum_{k=1}^n a_i\mu(A_k) = \sum_{k=1}^n a_i\mu_\alpha(A_{k,\alpha}) = \int g|_\alpha\, d\mu_\alpha. \tag{2}$$

Consider now a sequence of positive $\mathscr{R}$-simple functions $(g_n)_{n\in\mathbb{N}}$ converging to $f\chi_{\Omega_\alpha}$ μ-a.e. Then the sequence $(g_n|_\alpha)$ increases and converges to $f|_\alpha$ μ_α-a.e., so by (2) we have $\int f\chi_{\Omega_\alpha}\, d\mu = \bigvee_{n=1}^\infty \int g_n\, d\mu = \bigvee_{n=1}^\infty \int g_n|_\alpha\, d\mu_\alpha$. Therefore, $f|_\alpha \in \mathscr{L}^1_o(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ and $\int f|_\alpha\, d\mu_\alpha = \bigvee_{n=1}^\infty \int g_n|_\alpha\, d\mu = \int f\chi_{\Omega_\alpha}\, d\mu$.

Conversely, assume that $f|_\alpha \in \mathscr{L}^1_o(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ and pick an increasing sequence of positive Σ_α-simple function $(f_n)_{n=1}^\infty$ converging to $f|_\alpha$ μ_α-a.e. Define a function g_n by writing $g_n(t) := f_n(t)$ for $t \in \Omega_\alpha$ and $g_n(t) := 0$ for $t \in \Omega \setminus \Omega_\alpha$. Then $(g_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive $\mathscr{R}$-simple functions converging to $f\chi_{\Omega_\alpha}$ μ-a.e. Moreover, $g_n|_{\Omega_\alpha} = f_n$ for all $n \in \mathbb{N}$ and, according to (2), the equalities $\int f|_\alpha\, d\mu_\alpha = \bigvee_{n=1}^\infty \int f_n\, d\mu_\alpha = \bigvee_{n=1}^\infty \int g_n\, d\mu$ hold. Consequently, $f\chi_{\Omega_\alpha} \in \mathscr{L}^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ and $\int f\chi_{\Omega_\alpha}\, d\mu = \int f|_\alpha\, d\mu_\alpha$. $\rhd$

Theorem 6.9. *Let $(\Omega, \mathscr{R}, \mu)$ be the direct sum of the family of measure spaces $\big((\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)\big)_{\alpha\in\mathrm{A}}$. Then $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ belongs to $\mathscr{L}^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$, if and only if there is a function $\nu := \nu_f : \mathbb{N} \to \mathrm{A}$ depending on f such that $|f| = \bigvee_{n\in\mathbb{N}} |f|\chi_{\Omega_{\nu(n)}}$ μ-a.e. and o-$\sum_{n=1}^\infty \int |f|\chi_{\Omega_{\nu(n)}}\, d\mu$ exists in X. In this event, $\int |f|\, d\mu = o$-$\sum_{n=1}^\infty \int |f|\chi_{\Omega_{\nu(n)}}\, d\mu$.*

$\lhd$ Obviously, it suffices to check this statement for a positive $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$. If $0 \leqslant f \in \mathscr{L}^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ then there exists a sequence of $\mathscr{R}$-simple functions $(g_n)_{n=1}^\infty$ such that $0 \leqslant g_n \uparrow f$ μ-a.e. and $\int f\, d\mu = \bigvee_{n=1}^\infty \int g_n\, d\mu$. Since $\{g_n \neq 0\} \in \mathscr{R}$ up to a μ-negligible set, there exists a finite subset $\theta_n \subset \mathrm{A}$ such that $\{g_n \neq 0\} \subset \bigcup_{\alpha\in\theta} \Omega_\alpha$ up to a μ-negligible set for all $n \in \mathbb{N}$. Clearly, $\theta_n \uparrow$ and $\Theta := \bigcup_{n=1}^\infty \theta_n$ is a countable subset of A. Pick an appropriate mapping ν from $\mathbb{N}$ onto Θ (there exists an increasing sequence of naturals (n_k) with $n_0 = 0$ such that ν sends $\{n_{k-1}+1, \dots, n_k\}$ onto θ_k)

and denote $f_m := \bigvee_{n=1}^m f\chi_{\Omega_{\nu(n)}}$ for all $m \in \mathbb{N}$. Then $0 \leqslant f_m \uparrow_m f$ μ-a.e., because for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $g_n \leqslant f_m$ μ-a.e. Taking into account the fact that the inequality $f_m \leqslant \bigvee_{n\in\mathbb{N}} f\chi_{\Omega_\nu(n)}$ are true for all $m \in \mathbb{N}$ we get the desired representation $f = \bigvee_{n\in\mathbb{N}} f\chi_{\Omega_{\nu(n)}}$ μ-a.e. Moreover, the relations $f_m = \sum_{n=1}^m f\chi_{\Omega_{\nu(n)}}$ and $0 \leqslant f_m \uparrow_m f$ hold μ-a.e., so that making use of Theorem 4.12 we deduce

$$\int f\,d\mu = \bigvee_{m\in\mathbb{N}} \int f_m\,d\mu = \bigvee_{m=1}^{\infty} \sum_{n=1}^{m} \int f\chi_{\Omega_{\nu(n)}} d\mu = o\text{-}\sum_{n=1}^{\infty} \int f\chi_{\Omega_{\nu(n)}}\,d\mu$$

whence $\int f\,d\mu = o\text{-}\sum_{n=1}^{\infty} \int f\chi_{\Omega_\nu(n)} d\mu$.

Conversely, assume that $0 \leqslant f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ admits a representation $f = \bigvee_{n\in\mathbb{N}} f\chi_{\Omega_{\nu(n)}}$ μ-a.e., where $\nu : \mathbb{N} \to \mathrm{A}$ is an into mapping and $o\text{-}\sum_{n=1}^{\infty} \int f\chi_{\Omega_{\nu(n)}}\,d\mu$ exists in X_+. Put $f_m := \bigvee_{n=1}^m f\chi_{\Omega\nu(n)}$ μ-a.e. for every $m \in \mathbb{N}$. Then $0 \leqslant f_m \uparrow$ μ-a.e. and

$$f = \bigvee_{n\in\mathbb{N}} f\chi_{\Omega_{\nu(n)}} = \bigvee_{m=1}^{\infty} \bigvee_{n=1}^{m} f\chi_{\Omega_{\nu(n)}} = \bigvee_{m=1}^{\infty} f_m \ \mu\text{-a.e.},$$

that is $0 \leqslant f_m \uparrow f$ μ-a.e. Now, making use of Theorem 4.12 and the equality $f_m = \sum_{n=1}^m f\chi_{\Omega\nu(n)}$ μ-a.e. we see that

$$\int f\,d\mu = \bigvee_{m=1}^{\infty} \int f_m\,d\mu = \bigvee_{m=1}^{\infty} \sum_{n=1}^{m} \int f\chi_{\Omega_{\nu(n)}}\,d\mu = o\text{-}\sum_{n=1}^{\infty} \int f\chi_{\Omega_{\nu(n)}}\,d\mu.$$

Thus, $f \in \mathscr{L}_o^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ and $\int f\,d\mu = o\text{-}\sum_{n=1}^{\infty} \int f\chi_{\Omega_{\nu(n)}}\,d\mu$. ▷

Remark 6.10. Theorem 6.9 remains valid when Σ_α is a δ-ring for all $\alpha \in \mathrm{A}$ but in this generality the result is not used in the present work.

7. Representation of Dedekind Complete Vector Lattices

In this section we demonstrate that, given an arbitrary Dedekind complete vector lattice, there exists an order dense ideal in it which is lattice isomorphic to the vector lattice of equivalence classes of integrable functions with respect to a vector measure. Moreover, the latter admits a natural extension which is lattice isomorphic to the whole vector lattice. We start with the following preliminary result.

Theorem 7.1. *Let X be a Dedekind σ-complete vector lattice with a weak order unit $e > 0$ and let Q be the Stone representation space of a σ-algebra $\mathscr{C}(e)$ of components of e. Then there exists a Baire measure μ on Q with values in X_+ such that the integration operator $I_\mu^o : \tilde{f} \mapsto \int f\,d\mu$ is a lattice isomorphism of $L_o^1(\mu)$ onto X.*

◁ Denote by Σ the Baire σ-algebra of Q and observe that Σ coincides with the σ-algebra of subsets of Q generated by the set $\mathrm{Clop}(Q)$ of all clopen subsets of Q. Let Δ stand for the collection of all meager subsets (= sets of first category) of Q contained in Σ. Then by Loomis–Sikorski Theorem [42, Theorem 1.2.6(1)] there exists a Boolean isomorphism h from the quotient algebra Σ/Δ onto $\mathrm{Clop}(Q)$ and the quotient mapping $\varphi : \Sigma \to \Sigma/\Delta$ is a Boolean σ-homomorphism. Define $\mu : \Sigma \to X_+$ as $\mu := \iota^{-1} \circ h \circ \varphi$ where $\iota : \mathscr{C}(e) \to \mathrm{Clop}(Q)$ is the Stone representation. Then μ is a Bair measure on Q. It is immediate from the definition of μ that Δ coincides with the collection of μ-negligible sets.

By Theorem 4.15 the integration operator $I^o_\mu : \tilde{f} \to \int f\,d\mu$ from the Dedekind σ-complete vector lattice $L^1_o(Q, \mathrm{Clop}_\sigma(Q), \mu)$ to X is strictly positive and order σ-continuous. Moreover, I^o_μ is injective lattice homomorphism, since μ is a Boolean homomorphism.

To complete the proof we have to show that I^o_μ is onto. Recall that an e-step element in X is any vector $x \in X$ representable as $x = \sum_{k=1}^n \lambda_k e_k$ where $e_1, \dots, e_n$ are pairwise disjoint components of e with $e = e_1 + \dots + e_n$ and $\lambda_1, \dots, \lambda_n$ are arbitrary reals. Note that for every e-step element $x \in X$ there exists a simple function $f \in L^1_o(\mu)$ such that $I^o_\mu(f) = x$, since the restriction $\mu|_{\mathrm{Clop}(Q)}$ is an isomorphism of $\mathrm{Clop}(Q)$ onto $\mathscr{C}(e)$. Now, take an element $x \in X_+$ and put $x_n := x \wedge (ne)$ for all $n \in \mathbb{N}$. Then the sequence $(x_n)_{n\in\mathbb{N}}$ lies in the order ideal X_e generated by e. By Freudenthal's Spectral Theorem [4, Theorem 2.8] for every $n \in \mathbb{N}$ there exists an e-step element u_n satisfying $0 \leqslant x_n - u_n \leqslant e/n$. For $n \in \mathbb{N}$ pick a simple function $g_n \in L^1_o(\mu)$ such that $u_n = I^o_\mu(g_n)$ and put $f_n := g_1 \vee \dots \vee g_n$ and $v_n := u_1 \vee \dots \vee u_n$. Clearly, (f_n) and (v_n) are increasing sequences connected by the formula $v_n = I^o_\mu(f_n)$ $(n \in \mathbb{N})$. Moreover, $0 \leqslant x_n - v_n \leqslant x_n - u_n \leqslant e/n$ and hence $x = \bigvee_{n=1}^\infty v_n$. By the monotone convergence Theorem 4.12 there exists $f \in \mathscr{L}^1_o(\mu)$ such that $f = \bigvee_{n=1}^\infty f_n$ μ-a.e. and $x = I^o_\mu(f)$. Thus I^o_μ is onto and the proof is complete. $\rhd$

Lemma 7.2. *Let X be a Dedekind σ-complete vector lattice. There exists a disjoint family of nonzero positive elements $\Gamma \subset X$ such that each element $x \in X_+$ admits a unique representation $x = \bigvee_{\gamma\in\Gamma} x_\gamma$ with $0 \leqslant x_\gamma \in X_\gamma := \{\gamma\}^{\perp\perp}$ for all $\gamma \in \Gamma$.*

$\lhd$ Each disjoint collection Γ of nonzero positive elements in X with $X = \Gamma^{\perp\perp}$ is suitable, see Vulikh [72, Lemma IV.7.1, Theorems IV.5.2 and IV.5.3]. $\rhd$

Let X, Γ, and $(X_\gamma)_{\gamma\in\Gamma}$ be as in Lemma 7.2. By Theorem 7.1 for each $\gamma \in \Gamma$ there exist a nonempty set Ω_γ, a σ-algebra Σ_γ of its subsets, and a measure $\mu_\gamma : \Sigma_\gamma \to X_\gamma$ such that the integration operator $I^o_{\mu_\gamma} : \tilde{f} \mapsto \int f\,d\mu_\gamma$ is a lattice isomorphism of $L^1_o(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)$ onto X_γ. Make the direct sum $(\Omega, \mathscr{R}, \mu)$ of the family $(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)_{\gamma\in\Gamma}$, see Definition 6.1.

Definition 7.3. Define X_Γ as the set of elements $x \in X$, whose representation in Lemma 7.2 has at most countable nonzero projections; in symbols,

$$X_\Gamma := \Big\{ x \in X : \ (\exists\, \nu : \mathbb{N} \to \Gamma)\ |x| = \bigvee\nolimits_{n=1}^{\infty} \pi_{\nu(n)}|x| \Big\}, \tag{3}$$

where π_γ is a band projection in X onto the band $X_\gamma := \{\gamma\}^{\perp\perp}$.

Clearly, the inclusion $\Gamma \subset X_\Gamma$ implies that X_Γ is an order dense ideal of X. If Γ is at most countable, then $X_\Gamma = X$. Below, μ stands for the direct sum of the family of measures $(\mu_\gamma)_{\gamma\in\Gamma}$. Note that $\mu : \mathscr{R} \to X_\Gamma$.

Theorem 7.4. *Let X be a Dedekind σ-complete vector lattice and Γ be chosen as in Lemma 7.2. Then the integration operator $I^o_\mu : f \mapsto \int f\,d\mu$ is a lattice isomorphism from $L^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X_Γ.*

$\triangleleft$ In virtue of Theorem 4.15 the integration operator $I^o_\mu : L^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu) \to X_\Gamma$ is strictly positive. Show that I^o_μ is a lattice homomorphism. To do this, it suffices to ensure that $I^o_\mu(f) \wedge I^o_\mu(g) = 0$ for all $f, g \in \mathscr{L}^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)_+$ with $f \wedge g = 0$ μ-a.e., see [4, Theorem 2.14]. By Theorem 6.9 there are functions $\nu, \xi : \mathbb{N} \to \Gamma$ such that

$$f = \bigvee_{n\in\mathbb{N}} f\chi_{\Omega_{\nu(n)}} \ \mu\text{-a.e.}, \qquad \int f\,d\mu = \bigvee_{n\in\mathbb{N}} \int f\chi_{\Omega_{\nu(n)}}\,d\mu,$$
$$g = \bigvee_{n\in\mathbb{N}} g\chi_{\Omega_{\xi(n)}} \ \mu\text{-a.e.}, \qquad \int g\,d\mu = \bigvee_{n\in\mathbb{N}} \int f\chi_{\Omega_{\xi(n)}}\,d\mu.$$

Pick $m, n \in \mathbb{N}$ and observe that if $\nu(n) \neq \xi(m)$ then by Definition 6.1 and Lemma 6.8

$$\int f\chi_{\Omega_{\nu(n)}}\,d\mu \wedge \int g\chi_{\Omega_{\xi(m)}}\,d\mu = \int f|_{\nu(n)}\,d\mu_{\nu(n)} \wedge \int g|_{\xi(m)}\,d\mu_{\xi(m)} \in X_{\nu(n)} \cap X_{\xi(m)} = \{0\}.$$

If $\nu(n) = \xi(m)$ then by Theorem 7.1 and Lemma 6.8 we have

$$\int f\chi_{\Omega_{\nu(n)}}\,d\mu \wedge \int g\chi_{\Omega_{\xi(m)}}\,d\mu = \int f|_{\nu(n)}\,d\mu_{\nu(n)} \wedge \int g|_{\nu(n)}\,d\mu_{\nu(n)} = \int f|_{\nu(n)} \wedge g|_{\nu(n)}\,d\mu_{\nu(n)} = 0.$$

It follows that $\int f\chi_{\Omega_{\nu(n)}}\,d\mu \wedge \int g\chi_{\Omega_{\xi(m)}}\,d\mu = 0$ for all $n, m \in \mathbb{N}$. From this we can deduce

$$\int f\,d\mu \wedge \int g\,d\mu = \Big(\bigvee_{n\in\mathbb{N}} \int f\chi_{\Omega_{\nu(n)}}\,d\mu\Big) \wedge \Big(\bigvee_{\xi(m)\in\mathbb{N}} \int g\chi_{\Omega_{\xi(m)}}\,d\mu\Big) =$$
$$\bigvee_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} \Big(\int f\chi_{\Omega_{\nu(n)}}\,d\mu \wedge \int g\chi_{\Omega_{\xi(m)}}\,d\mu\Big) = 0,$$

whence I^o_μ is a lattice homomorphism.

The fact that I^o_μ is one-to-one follows by observing that if $\tilde f \in L^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ and $I^o_\mu(\tilde f) = 0$, then $0 = |I^o_\mu(\tilde f)| = I^o_\mu(|\tilde f|)$ and $f = 0$ μ-a.e. or $\tilde f = 0$ by strictly positivity.

It remains to snow that I^o_μ is onto. For an arbitrary $0 \leqslant x \in X_\Gamma$ there exists a mapping $\nu : \mathbb{N} \to \Gamma$ such that $x = \bigvee_{n\in\mathbb{N}} x_{\nu(n)}$, where $0 \leqslant x_{\nu(n)} \in X_{\nu(n)}$ for all $n \in \mathbb{N}$. By Definition 6.1, for every $n \in \mathbb{N}$ one can chose $0 \leqslant f_{\nu(n)} \in \mathscr{L}_1(\Omega_{\nu(n)}, \Sigma_{\nu(n)}, \mu_{\nu(n)})$ such that $I_{\mu_{\nu(n)}}(f_{\nu(n)}) = \int f_{\nu(n)}\,d\mu_{\nu(n)} = x_{\nu(n)}$. Put $f_\gamma(t) = 0$ for all $t \in \Omega_\gamma$ and $\gamma \in \Gamma \setminus \nu(\mathbb{N})$. Define a function $f : \Omega \to \mathbb{R} \cup \{\infty\}$ by putting $f(t) := f_\gamma(t)$ for $t \in \Omega_\gamma$ and $\gamma \in \Gamma$. Then the restriction $f|_\gamma$ of f onto Ω_γ belongs to $\mathscr{L}^0(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)$ for all $\gamma \in \Gamma$ so that $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)_+$ by Lemma 6.7. Moreover, $f = \bigvee_{n\in\mathbb{N}} f\chi_{\Omega_{\nu(n)}}$ μ-a.e. and by Lemma 6.8 we have

$$\bigvee_{n\in\mathbb{N}} \int f\chi_{\Omega_{\nu(n)}}\,d\mu = \bigvee_{n\in\mathbb{N}} \int f|_{\nu(n)}\,d\mu_{\nu(n)} = \bigvee_{n\in\mathbb{N}} x_{\nu(n)} = x.$$

By Theorem 6.9 it follows that $\tilde f \in L^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)_+$ and $I^o_\mu(\tilde f) = x$. $\triangleright$

Lemma 7.5. *The direct sum of the family* $(\Omega_\gamma, \Sigma_\gamma, \mu_\gamma)_{\gamma\in\Gamma}$ *(see Definition* 6.1*) is a vector measure space with a semi-finite and localizable measure* $\mu : \mathscr{R} \to X_+$.

$\lhd$ It follows from Lemmas 6.4 and 6.5 that μ is semi-finite. By virtue of Theorem 5.10 it suffices to ensure Dedekind completeness of $B(\Omega) := \mathscr{R}^{\rm loc}/\mathscr{N}$, where $\mathscr{N} := \{A \in \mathscr{R}^{\rm loc} : \mu(A) = 0\}$. Denote by $B(\Omega_\gamma)$ the quotient algebra $\Sigma_\gamma/\mathscr{N}_\gamma$, where $\mathscr{N}_\gamma$ is the collection of μ_γ-negligible sets for all $\gamma \in \Gamma$. Then Lemmas 6.4 and 6.5 imply that the Boolean algebra $B(\Omega)$ is isomorphic to the product $\prod_{\gamma\in\Gamma} B(\Omega_\gamma)$ of a family of Dedekind complete Boolean algebras $(B(\Omega_\gamma))_{\gamma\in\Delta}$. Therefore, $B(\Omega)$ is Dedekind complete. $\rhd$

Recall that the lattice isomorphism I_μ^o of $L_o^1(\mu)$ onto X_Γ in Theorem 7.4 has the smallest extension $\hat{I}_\mu^o$ whose domain $L_{ow}^1(\mu)$ is taken as the space of weakly integrable functions (see Definition 5.6). Say that a disjoin set $\Gamma \subset X_+$ is *complete* if $X = \Gamma^{\perp\perp}$.

Theorem 7.6. *Let* X *be a Dedekind complete vector lattice and* Γ *a complete disjoint set in* X_+. *Then there exists a vector measure space* $(\Omega, \mathscr{R}, \mu)$ *with semi-finite localizable* μ *such that the integration operator* I_μ^o *is a lattice isomorphism of* $L_o^1(\mu)$ *onto* X_Γ. *Moreover, the minimal extension* $\hat{I}_\mu^o$ *of* I_μ^o *with respect to* $L^0(\mu)$ *is a lattice isomorphism of* $L_{ow}^1(\mu)$ *onto* X.

$\lhd$ The existence of a vector measure space $(\Omega, \mathscr{R}, \mu)$ as well as the fact that the integration operator I_μ^o is a lattice isomorphism of $L_o^1(\mu)$ onto X_Γ follows from Theorem 7.4. Consequently, $I_\mu^o : L_\mu^o(\mu) \to X$ is order continuous, since X_Γ is order dense ideal of X. It follows from Lemmas 4.11(2), 5.2 and 5.3 that $\hat{I}_\mu^o : L_{ow}^1(\mu) \to X$ is order continuous one-to-one lattice homomorphism. Moreover, by Lemma 7.5 the measure μ is semi-finite and localizable.

To complete the proof we have to show that $\hat{I}_\mu^o$ is onto. Given arbitrary $x \in X_+$, there exists an increasing net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in X_Γ order convergent to x, since X_Γ is an order dense ideal of X. Put $f_\alpha := I_\mu^{o-1}(x_\alpha)$ and observe that $(f_\alpha)_{\alpha\in\mathrm{A}}$ is an increasing net in $L_o^1(\mu)$ with $x = \sup_\alpha I_\mu^o(f_\alpha)$. Hence, in view of Lemma 5.12 there exists $f \in L_{ow}^1(\mu)$ such that $0 \leqslant f_\alpha \uparrow f$ and $\hat{I}_\mu^o(f) = x$. $\rhd$

8. Bartle–Dunford–Schwartz Integration

In this section, we introduce Bartle–Dunford–Schwartz integration with respect to a measure taking values in a quasi-Banach lattice, consider some of its elementary properties, and establish the connection with Kantorovich–Wright integration.

As in Section 4, Ω is a nonempty set, $\mathscr{R}$ is a δ-ring of subset of Ω, $\mathscr{R}^{\rm loc}$ is a σ-algebra defined by (1), and $X := (X, \|\cdot\|)$ is a quasi-Banach lattice.

Definition 8.1. A mapping $\mu : \mathscr{R} \to X_+$ is called a *measure* (or τ-measure), if it has the properties:

(1) $\mu(\varnothing) = 0$.

(2) If $(A_n)_{n=1}^\infty$ is a sequence of pairwise disjoint sets in $\mathscr{R}$ with $\bigcup_{n=1}^\infty A_n \in \mathscr{R}$,

then the series $\sum_{n=1}^{\infty}\mu(A_n)$ converges topologically and

$$\mu\Big(\bigcup_{n=1}^{\infty}A_n\Big)=\sum_{n=1}^{\infty}\mu(A_n). \tag{4}$$

Say that a triple $(\Omega,\mathscr{R},\mu)$ is a *τ-measure space* if Ω is a nonempty set, $\mathscr{R}$ is a δ-ring of subsets of Ω, and $\mu:\mathscr{R}\to X_+$ a τ-measure. We will drop τ when it is clear from the context which type of measure is meant (cf. Definition 4.1).

Remark 8.2. Thus, if X is a Dedekind σ-complete quasi-Banach lattice then the σ-additivity of an X-valued measure may be understood in two different ways: either the series in (4) is order convergent as in Definition 4.1, or topologically convergent as in Definition 8.1. It follows from Lemma 2.6 that topological σ-additivity implies order σ-additivity. Of course, if X is order continuous then these two concepts coincide. In this section measure is understood according to Definition 8.1.

Lemma 8.3. *Let $A,B\in\mathscr{R}$ and $(A_n)_{n=1}^{\infty}$ be a sequence in $\mathscr{R}$. Then for a measure $\mu:\mathscr{R}\to X_+$ the following hold:*

(1) *If $A\subset B$ then $\mu(B\setminus A)=\mu(B)-\mu(A)$ and $\mu(A)\leqslant\mu(B)$.*

(2) *If $A_n\uparrow A$ then $\mu(A_n)\uparrow$ and $\mu(A)=\lim_n\mu(A_n)$.*

(3) *If $A_n\downarrow\varnothing$ then $\mu(A_n)\downarrow$ and $\lim_n\mu(A_n)=0$.*

(4) *If $\sum_{n=1}^{\infty}\mu(A_n)$ exists in X and $A=\bigcup_{n=1}^{\infty}A_n$, then $\mu(A)\leqslant\sum_{n=1}^{\infty}\mu(A_n)$.*

$\lhd$ This can be proved by standard arguments from measure theory. $\rhd$

Definition 8.4. Given a measure $\mu:\mathscr{R}\to X_+$, μ-negligible sets, simple functions, the vector lattice of simple function $S(\mathscr{R})$, and the integration operator $I_\mu^o:S(\mathscr{R})\to X$ are defined exactly as in Definitions 4.3 and 4.5.

Remark 8.5. In view of Remark 8.2 and Lemma 4.4 there is no loss of generality in assuming that the δ-ring $\mathscr{R}$ contains all μ-negligible sets and the measure of a μ-negligible set equals zero. Moreover, by Lemma 4.6, I_μ^o is a positive linear operator and $I_\mu^o(|f|)=0$ implies $f=0$ μ-a.e. for each $f\in S(\mathscr{R})$.

Definition 8.6. A positive $\mathscr{R}^{\mathrm{loc}}$-measurable function $f:\Omega\to\mathbb{R}_+\cup\{+\infty\}$ is called *integrable*, if there exists a sequence $(f_n)_{n=1}^{\infty}$ of $\mathscr{R}$-simple functions such that $0\leqslant f_n\uparrow f$ μ-a.e. and there exists $\lim_n\int f_n\,d\mu$ in $(X,\|\cdot\|)$. In this event we denote

$$I_\mu^\tau(f):=\tau\text{-}\!\int f\,d\mu:=\lim_n\int f_n\,d\mu. \tag{5}$$

An arbitrary $\mathscr{R}^{\mathrm{loc}}$-measurable function $f:\Omega\to\mathbb{R}\cup\{\pm\infty\}$ is *integrable*, if so are f^+ and f^- and $I_\mu^\tau(f)$ is defined by $I_\mu^\tau(f):=I_\mu^\tau(f^+)-I_\mu^\tau(f^-)$.

Remark 8.7. (1) If a function f is integrable in the sense of Definition 8.6 then, according to Lemma 2.6, f is integrable in the sense of Definition 4.9 as well and $I_\mu^\tau(f)=I_\mu^o(f)$. It follows that $I_\mu^\tau(f)$ is well defined, that is, do not depend on the choice of an increasing sequence of simple function converging f μ-a.e., see Lemma 4.10. In particular, $\lim_n\|I_\mu^o(f_n)\|=0$ for any sequence (f_n) in $S(\mathscr{R})$ with $f_n\downarrow 0$ μ-a.e.

(2) In the context of Definition 8.6 the set $\mathscr{L}^0(\mu)$, the equivalence relation $\sim$, and the vector lattice $L^0(\mu)$ have the same meaning as in Section 4. Let $\mathscr{L}^1_\tau(\mu)$ be the part of $\mathscr{L}^1_o(\mu)$ consisting of topologically integrable function, while $L^1_\tau(\mu)$ the set of all equivalence classes of members of $\mathscr{L}^1_\tau(\mu)$. Given $f \in \mathscr{L}^1_\tau(\mu)$, the equivalence classes of f in $\mathscr{L}^1_\tau(\mu)$ and $\mathscr{L}^1_o(\mu)$ coincide. Therefore, $L^1_\tau(\mu)$ is a sublattice of $L^1_o(\mu)$.

(3) The integration operator $I^\tau_\mu : \tilde{f} \mapsto \int f\,d\mu$ acting from $L^1_\tau(\mu)$ to X is linear and strictly positive. The latter means that I^τ_μ is positive and $I^\tau_\mu(|\tilde{f}|) = 0$ implies $f = 0$ μ-a.e.

Lemma 8.8. *Let $(f_n)_{n=1}^\infty$ be a decreasing sequence of $\mathscr{R}$-simple functions converging to zero μ-a.e. Then the sequence $\big(I^\tau_\mu(f_n)\big)$ decreases and converges to zero in $(X, \|\cdot\|_X)$.*

$\lhd$ There is no loss of generality in assuming that $f_n(t) \downarrow 0$ for all $t \in \Omega$. Fix an arbitrary $\varepsilon > 0$ and put $M := \sup_{t\in\Omega} f_1(t)$, $B := \{f_1 > 0\}$, and $A_n := \{f_n \geqslant \varepsilon\}$ for all $n \in \mathbb{N}$. Clearly, $B \in \mathscr{R}$ and $A_n \downarrow \varnothing$. By Lemma 8.3(3) there exists $m \in \mathbb{N}$ such that $\|\mu(A_n)\| \leqslant \varepsilon$ for all $n, m \in \mathbb{N}$, $m \leqslant n$. For the same m and n we have

$$0 \leqslant f_n \leqslant f_m = f_m\chi_B = f_m\chi_{A_m} + f_m\chi_{B\setminus A_m} \leqslant M\chi_{A_m} + \varepsilon\chi_B.$$

If C is a quasi-triangle constant of X then

$$\|I^\tau_\mu(f_n)\| \leqslant C(M\|\mu(A_m)\| + \varepsilon\|\mu(B)\|) \leqslant C\varepsilon(M + \|\mu(B)\|) \quad (m \leqslant n).$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\int \phi_n\,d\mu \downarrow 0$. $\rhd$

Lemma 8.9. *If (f_n) is a decreasing sequence of $\mathscr{L}^1_\tau(\mu)$ converging to zero μ-a.e., then $\lim_n \|\int f_n\,d\mu\|_X = 0$.*

$\lhd$ Fix an arbitrary $\varepsilon > 0$. For every $n \in \mathbb{N}$ pick a simple function g_n such that $0 \leqslant g_n \leqslant f_n$ μ-a.e. and $\|\int (f_n - g_n)\,d\mu\|_X \leqslant \varepsilon 2^{-n} C^{-(n+1)}$, where C is a quasi-triangle constant. Denote $h_n := \bigwedge_{k=1}^n g_k$ for all $n \in \mathbb{N}$. Then the relations $0 \leqslant h_n \leqslant f_n$ μ-a.e. and $f_n \downarrow 0$ μ-a.e. imply $h_n \downarrow 0$ μ-a.e. By virtue of Lemma 8.8 there exists a natural n_0 such that $\|\int h_n\,d\mu\|_X \leqslant C^{-1}\varepsilon$ for all $n \geqslant n_0$. Observe that for each $n \in \mathbb{N}$

$$0 \leqslant f_n - h_n = f_n - \bigwedge_{i=1}^n g_i = \bigvee_{i=1}^n (f_n - g_i) \leqslant \sum_{i=1}^n (f_i - g_i)$$

holds μ-a.e. Consequently, by virtue of Lemma 2.4 the following estimates fulfill

$$\begin{gathered}\left\|\int f_n\,d\mu\right\|_X \leqslant C\left(\left\|\int (f_n - h_n)\,d\mu\right\|_X + \left\|\int h_n\,d\mu\right\|_X\right) \leqslant \\ C\left(\left\|\int \sum_{i=1}^n (f_i - g_i)\,d\mu\right\|_X + C^{-1}\varepsilon\right) = C\left\|\sum_{i=1}^n \int (f_i - g_i)\,d\mu\right\|_X + \varepsilon \leqslant \\ C\sum_{i=1}^n C^i\left\|\int (f_i - g_i)\,d\mu\right\|_X + \varepsilon \leqslant \sum_{i=1}^n \varepsilon 2^{-i} + \varepsilon \leqslant 2\varepsilon\end{gathered}$$

for all $n \geqslant n_0$. It follows that $\lim_n \|\int f_n\,d\mu\|_X = 0$, as claimed. $\rhd$

Definition 8.10. The space of *topologically integrable* functions is the vector lattice $L^1_\tau(\mu)$ endowed with the quasi-norm

$$\|f\|_\tau := \left\| \int |f|\, d\mu \right\|_X \quad \big(f \in L^1_\tau(\mu)\big). \tag{6}$$

It follows from the linearity and strict positivity of I^τ_μ that $\|\cdot\|_\tau$ is a quasi-norm with the same triangle constant as that of X.

Theorem 8.11. *The space $L^1_\tau(\mu)$ is an order dense ideal in $L^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$. In particular, $L^1_\tau(\mu)$ is a Dedekind σ-complete vector lattice.*

$\lhd$ Note that $L^1_\tau(\mu)$ is a vector subspace of $L^0(\mu)$ and, by definition, a coset $\tilde{f} \in L^0(\mu)$ belongs to $L^1_\tau(\mu)$ if and only if $|\tilde{f}| \in L^1_\tau(\mu)$. Therefore, it suffices to show that if $f, g \in \mathscr{L}^0(\mu)$, $0 \leqslant g \leqslant f$ μ-a.e., and $f \in \mathscr{L}^1_\tau(\mu)$, then $g \in \mathscr{L}^1_\tau(\mu)$.

Observe first that all components of a coset $\tilde{f} \in L^1_\tau(\mu)$ belong to $L^1_\tau(\mu)$, or in other words $\chi_A f \in \mathscr{L}^1_\tau(\mu)$ for all $A \in \mathscr{R}^{\mathrm{loc}}$. Indeed, take an increasing sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{R}$-simple function with $0 \leqslant f_n \uparrow f$ μ-a.e. and $\lim_n \|f - f_n\|_\tau = 0$. Using the easy relation $\chi_{A_1}\chi_{A_2} = \chi_{A_1\cap A_2}$, we conclude that the sequence $(\chi_A f_n)_{n=1}^\infty$ consists of $\mathscr{R}$-simple function. Moreover, $0 \leqslant \chi_A f_n \uparrow \chi_A f$ μ-a.e. and employing the monotonicity property of the quasi-norm $\|\cdot\|_\tau$ yields

$$\|\chi_A f_n - \chi_A f_m\|_\tau = \|\chi_A(f_n - f_m)\|_\tau \leqslant \|(f_n - f_m)\|_\tau \to 0$$

as $n, m \to \infty$. Consequently, $\chi_A f \in \mathscr{L}^1_\tau(\mu)$ and $\chi_A f = \lim_n \chi_A f_n$.

Since $L^0(\mu)$ is a Dedekind σ-complete vector lattice, according to Freudenthal Spectral Theorem [4, Theorem 2.8] there exists a sequence $(\tilde{g}_n)_{n\in\mathbb{N}}$ of linear combinations of components of $\tilde{f}$ such that $0 \leqslant g_n \uparrow g$ μ-a.e. and $0 \leqslant \tilde{g} - \tilde{g}_n \leqslant n^{-1}\tilde{f}$ for all $n \in \mathbb{N}$. Thus, $0 \leqslant \tilde{g}_m - \tilde{g}_n \leqslant 2n^{-1}\tilde{f}$ for $m \geqslant n$ and, using again the monotonicity of $\|\cdot\|_\tau$, we have $\|\tilde{g}_m - \tilde{g}_n\|_\tau \leqslant 2n^{-1}\|\tilde{f}\|_\tau$ for all $m \geqslant n$. It follows that $(\tilde{g}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and so $\lim_n \tilde{g}_n = \tilde{g}$ and $\tilde{g} \in L^1_\tau(\mu)$, as claimed.

Prove now that $L^1_\tau(\mu)$ is order dense in $L^0(\mu)$. Take $0 < \tilde{f} \in L^0(\mu)$ and note that $\{f \geqslant n^{-1}\} \in \mathscr{R}^{\mathrm{loc}}$ for all $n \in \mathbb{N}$. Then $\{f > 0\} = \bigcup_{n=1}^\infty \{f \geqslant n^{-1}\}$ and there exist $n_0 \in \mathbb{N}$ and $B \in \mathscr{R}$ such that $\mu(B \cap \{f \geqslant n_0^{-1}\}) > 0$. Otherwise, $\{f \geqslant n^{-1}\}$ would be μ-negligible and so $\mu(\{f \geqslant n^{-1}\}) = 0$ for all $n \in \mathbb{N}$ by Remark 8.5. Consequently, $\{f > 0\}$ would be μ-negligible, which would contradict the relation $f > 0$. If g is the characteristic function of $B \cap \{f \geqslant n_0^{-1}\}$, then $0 < \tilde{g} \in L^1_\tau(\mu)$ and $n_0^{-1}\tilde{g} \leqslant \tilde{f}$. $\rhd$

Theorem 8.12. *The quasi-normed space $(L^1_\tau(\mu), \|\cdot\|_\tau)$ is an order continuous quasi-Banach lattice.*

$\lhd$ To ensure that $L^1_\tau(\mu)$ is a complete metric space, apply 2.7 (2). Take a sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ in $L^1_\tau(\mu)_+$ with $\sum_{n=1}^\infty C^n \|\tilde{f}_n\|_\tau < \infty$. Then by Definition 8.10 we have $\sum_{n=1}^\infty C^n \|\int f_n\, d\mu\|_X < \infty$ and, since X is a quasi-Banach lattice, the series $\sum_{n=1}^\infty \int f_n\, d\mu$ converges to some $x \in X_+$ by Theorem 2.7 (2). Denote $S_m := \sum_{n=1}^m f_n \in \mathscr{L}^1_\tau(\mu)$ for all $m \in \mathbb{N}$ and note that $0 \leqslant S_m \uparrow$ μ-a.e. and

$$x = \lim_m \int S_m\, d\mu = \bigvee_{m=1}^\infty \int S_m\, d\mu.$$

In view of Theorem 4.12 there exists $f \in \mathscr{L}^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ such that $0 \leqslant S_m \uparrow f$ μ-a.e. Consequently, $\tilde{f} \in L^1_\tau(\mu)$ and $\int f\, d\mu = \lim_m \int S_m\, d\mu$. Thus, $\tilde{f} = \sum_{n=1}^{\infty} \tilde{f}_n$ and $L^1_\tau(\mu)$ is quasi-Banach lattice by Theorem 2.7 (2). Order continuity of $L^1_\tau(\mu)$ follows now from Theorem 2.10, Lemma 8.9, and Theorem 8.11. $\rhd$

Theorem 8.13. *The quasi-normed space $(L^1_\tau(\mu), \|\cdot\|_\tau)$ is a super Dedekind complete order continuous quasi-Banach lattice and an order dense ideal of $L_0(\mu)$. The integration operator I^τ_μ from $L^1_\tau(\mu)$ to X is order continuous and strictly positive.*

$\lhd$ The first part follows from Lemma 2.9 and Theorems 8.11 and 8.12. By Definition 8.10 the integration operator I^τ_μ is isometric on $L^1_\tau(\mu)_+$ and hence quasi-norm continuous as $\|I^\tau_\mu(f)\|_X \leqslant \|f\|_\tau$ for all $f \in L^1_\tau(\mu)$. Thus, the order continuity of the quasi-norm $\|\cdot\|_\tau$ implies the order continuity of I^τ_μ by Lemma 2.6. $\rhd$

9. Representation of Quasi-Banach Lattices

It is important to know under which condition a quasi-Banach lattice is order isometric to some quasi-Banach function space. Different aspects of this problem has been studied by variety of authors. Recent achievements are related with the vector measure integration. In this section we demonstrate that the order based integration enables one to cover a series of results based on Bartle–Dunford–Schwartz type integration. Below, unless otherwise indicated, $X := (X, \|\cdot\|_X)$ is a Dedekind σ-complete quasi-Banach lattice and $(\Omega, \mathscr{R}, \mu)$ is a vector measure space with a semi-finite measure $\mu : \mathscr{R} \to X_+$.

Definition 9.1. The space of (*order*) *integrable* functions is the vector lattice $L^1_o(\mu)$ endowed with the quasi-norm

$$\|f\|_o := \|I^o_\mu(|f|)\|_X := \left\| \int |f|\, d\mu \right\|_X \quad \big(f \in L^1_o(\mu)\big). \tag{7}$$

It follows from the linearity and strict positivity of I^o_μ that $\|f\|_o$ is a quasi-norm with the same triangle constant as that of X. The following result collects some important properties of the space $L^1_o(\mu)$.

Theorem 9.2. *Let X be a Dedekind σ-complete quasi-Banach lattice with the quasi-triangle constant C. Then the the following assertions hold:*

(1) *$L^1_o(\mu)$ is an order dense ideal of $L^0(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$.*

(2) *$(L^1_o(\mu), \|\cdot\|_o)$ is a Dedekind σ-complete quasi-Banach lattice with the quasi-triangle constant C.*

(3) *If X is p-normable for some $0 < p < +\infty$, then so is $L^1_o(\mu)$.*

(4) *$(L^1_\tau(\mu), \|\cdot\|_\tau)$ is a super Dedekind complete quasi-Banach sublattice and an order dense ideal of $(L^1_o(\mu), \|\cdot\|_o)$.*

(5) *If X is order continuous, then $L^1_o(\mu) = L^1_\tau(\mu)$.*

$\lhd$ (1) By Lemma 4.11 $L^1_o(\mu)$ is an order dense ideal of a Dedekind σ-complete vector lattice $L^0(\mu)$.

(2) It suffices to prove the metric completeness of $L^1_o(\mu)$. Take a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1_o(\mu)_+$ such that $\sum_{n=1}^{\infty} C^n \|f_n\|_o < \infty$. By Definition 9.1 we have

$\sum_{n=1}^{\infty} C^n \|I_\mu^o(f_n)\|_X < \infty$ and, as X is a quasi-Banach lattice, there exists $x \in X_+$ with $x = o\text{-}\sum_{n=1}^{\infty} I_\mu^o(f_n)$. Put $g_m := \sum_{n=1}^{m} f_n$ $(m \in \mathbb{N})$ and observe that $0 \leqslant g_m \uparrow$ and $x = \bigvee_{m=1}^{\infty} \int g_m \, d\mu$. By Theorem 4.12 there exists $f \in L_o^1(\mu)$ such that $0 \leqslant g_m \uparrow f$. Consequently, $f = o\text{-}\sum_{n=1}^{\infty} f_n$ and hence $L_o^1(\mu)$ is complete by Theorem 2.7(3).

(3) If X is p-normable, then evidently $\|f+g\|_o^p = \|I_\mu^o(f)+I_\mu^o(g)\|_X^p \leqslant \|I_\mu^o(f)\|_X^p + \|I_\mu^o(g)\|_X^p = \|f\|_o^p + \|g\|_o^p$.

(4) This follows from Remark 8.7 (1), Theorem 8.13, and Definition 9.1.

(5) Assume that X is order continuous and $0 \leqslant \tilde{f} \in L_o^1(\mu)$. Then there exists a sequence of $\mathscr{R}$-simple functions $(f_n)_{n\in\mathbb{N}}$ such that $0 \leqslant f_n \uparrow f$ μ-a.e. and the sequence $(I_\mu^o(f_n))_{n\in\mathbb{N}}$ is increasing and order convergent in X. Thus, $(I_\mu^o(f_n))_{n=1}^{\infty}$ is metric convergent and $f \in L_\tau^1(\mu)$ by definition. ▷

Theorem 9.3. *Let X be a Dedekind σ-complete quasi-Banach lattice and $\Gamma \subset X_+$ a complete disjoint set. Then there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with a semi-finite X_Γ-valued measure μ such that the integration operator I_μ^o is a lattice isomorphism and linear isometry of $L_o^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X_Γ. If X is Dedekind complete then μ is localizable.*

◁ This is immediate from Theorem 7.4, Lemma 7.5, and Definition 9.1. ▷

We now introduce two new quasi-normed lattices $L_{ow}^1(\mu) := L_{ow}^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ and $L_{\tau w}^1(\mu) := L_{\tau w}^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ useful for representing abstract quasi-Banach lattices.

Definition 9.4. The vector lattice $L_{ow}^1(\mu)$ of *weakly integrable* function was introduced in Definition 5.6 and now we endow it by a quasi-norm

$$\|f\|_{ow} := \|\hat{I}_\mu^o(|f|)\|_X \quad (f \in L_{ow}^1(\mu)), \tag{8}$$

where $\hat{I}_\mu^o : L_{ow}^1(\mu) \to X$ is the smallest extension of the integration operator I_μ^o, see Definition 5.1. Denote by $\hat{I}_\mu^\tau : L_{\tau w}^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu) \to X$ the smallest extension of the topological integration operator I_μ^τ. The quasi-normed lattice $L_{\tau w}^1(\mu) := L_{\tau w}^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ of *weakly τ-integrable* function is defined as the vector lattice $L_{\tau w}^1(\mu)$ endowed with the quasi-norm

$$\|f\|_{\tau w} := \|\hat{I}_\mu^\tau(|f|)\|_X \quad (f \in L_{\tau w}^1(\mu)). \tag{9}$$

Some basic properties of we $L_{\tau w}^1(\mu)$ and $L_{ow}^1(\mu)$ are listed in the following result.

Theorem 9.5. *Let $X := (X, \|\cdot\|_X)$ be a Dedekind complete quasi-Banach lattice with the triangle constant C. Then the following assertions hold:*

(1) *$L_{\tau w}^1(\mu)$ and $L_{ow}^1(\mu)$ are order dense ideals in $L^0(\mu)$.*

(2) *$L_{\tau w}^1(\mu)$ and $L_{ow}^1(\mu)$ are quasi-Banach lattices with the triangle constant C.*

(3) *If X is p-normable for some $0 < p < +\infty$, then so are $L_{\tau w}^1(\mu)$ and $L_{ow}^1(\mu)$.*

(4) *$L_{ow}^1(\mu)$ is an order dense ideal of $L_{\tau w}^1(\mu)$ and $\hat{I}_\mu^\tau \leqslant \hat{I}_\mu^o$.*

(5) *The integration operator $I_\mu^o : L_o^1(\mu) \to X$ is order continuous if and only if $L_{ow}^1(\mu) = L_{\tau w}^1(\mu)$ and $\|\cdot\|_{ow} = \|\cdot\|_{\tau w}$.*

◁ (1) According to Theorem 8.11 $L_\tau^1(\mu)$ is an order dense ideal of $L^0(\mu)$ and hence so is $L_{\tau w}^1(\mu)$ by Definition 5.1. Similarly, $L_o^1(\mu)$ is an order dense ideal of $L^0(\mu)$ by Lemma 4.11 and hence so is $L_{ow}^1(\mu)$ by Definition 5.1.

(2) The proof runs along the lines of the proof of 9.2 (2). Take a sequence $(\tilde{f}_n)_{n=1}^{\infty}$ in $L^1_{\tau w}(\mu)_+$ such that $\sum_{n=1}^{\infty} C^n \|\tilde{f}_n\|_{\tau w} < \infty$. In view of the definition of the quasi-norm $\|\cdot\|_{\tau w}$ (see formula (9)) we have $\sum_{n=1}^{\infty} C^n \|\hat{I}^{\tau}_{\mu}(\tilde{f}_n)\|_X < \infty$ and, as X is a quasi-Banach lattice, there exists $x := \sum_{n=1}^{\infty} \hat{I}^{\tau}_{\mu}(\tilde{f}_n)$ in X_+ by Theorem 2.7. Put $S_m := \sum_{n=1}^{m} f_n$ for all $m \in \mathbb{N}$ and note that the sequence $(S_m)_{m\in\mathbb{N}}$ in $\mathscr{L}^1_{\tau w}(\mu)$ is positive and increasing μ-a.e. and

$$x = \lim_m \int S_m \, d\mu = \bigvee_{m=1}^{\infty} \hat{I}^{\tau}_{\mu}(S_m).$$

Define a $\mathscr{R}^{\mathrm{loc}}$-measurable function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ by putting

$$f(t) := \sup_{m\in\mathbb{N}} S_m(t) \quad (t \in \Omega).$$

If f is finite μ-a.e. then it follows from Lemma 5.4 that $\tilde{f} \in L^1_{\tau w}(\mu)$ and $\hat{I}^{\tau}_{\mu}(\tilde{f}) = x$ so that,

$$\left\| f - \sum_{n=1}^{m} \tilde{f}_n \right\|_{\tau w} = \left\| x - \sum_{n=1}^{m} \hat{I}^{\tau}_{\mu}(\tilde{f}_n) \right\|_X \to 0$$

as $m \to +\infty$ and we are done by Theorem 2.7.

Ensure that f is finite μ-a.e. Observe that for $A := \{f = \infty\}$ we have $A \in \mathscr{R}^{\mathrm{loc}}$ and $0 \leqslant S_m \wedge k\chi_A \uparrow_m k\chi_A$ μ-a.e. for all $k \in \mathbb{N}$. Since $\sup_m \hat{I}^{\tau}_{\mu}(S_m \wedge k\chi_A) \leqslant x$, again Lemma 5.4 shows that $k\chi_A \in L^1_{\tau w}(\mu)$ and $\hat{I}^{\tau}_{\mu}(k\chi_A) \leqslant x$ for all $k \in \mathbb{N}$. Thus, $\hat{I}^{\tau}_{\mu}(\chi_A) = 0$ and by strict positivity of $\hat{I}^{\tau}_{\mu}$ we conclude that $\chi_A = 0$ μ-a.e. Now, if an arbitrary $B \in \mathscr{R}$ is contained in A, then $\chi_B = 0$ μ-a.e. It follows that $\mu(B) = 0$ and hence $\mu(A) = 0$ as μ is semi-finite.

For $L^1_{ow}(\mu)$ the proof given above carries over verbatim replacing $\|\cdot\|_{\tau w}$ by $\|\cdot\|_{ow}$, $\hat{I}^{\tau}_{\mu}$ by $\hat{I}^{o}_{\mu}$, and metric limits by order limits.

(3) This is trivial, see 9.2 (3).

(4) Assume that $0 \leqslant f \in L^1_{ow}(\mu)$, $g \in L^1_{\tau}(\mu)$, and $0 \leqslant g \leqslant f$. By the assertion 9.2 (4) $g \in L^1_o(\mu)$ and $I^{\tau}_{\mu}(g) = I^{o}_{\mu}(g) \leqslant \hat{I}^{o}_{\mu}(f)$ for all $g \in L^1_{\tau}(\mu)$ with $0 \leqslant g \leqslant f$. It follows from Definition 5.1 that $f \in L^1_{\tau w}(\mu)$ and $\hat{I}_{\mu}(f)^{\tau} \leqslant \hat{I}^{o}_{\mu}(f)$. Thus, we get $L^1_{ow}(\mu) \subset L^1_{\tau w}(\mu)$ and $\hat{I}^{\tau}_{\mu}(f) \leqslant \hat{I}^{o}_{\mu}(f)$ for all $f \in L^1_{ow}(\mu)$. Moreover, $L^1_{ow}(\mu)$ is an order dense ideal of $L^1_{\tau w}(\mu)$ by (1).

(5) If $L^1_{ow}(\mu) = L^1_{\tau w}(\mu)$ then $\hat{I}^{\tau}_{\mu} = \hat{I}^{o}_{\mu}$ by Remark 8.7 (1), so that $\hat{I}^{o}_{\mu}$ is order continuous according to Theorem 8.13. Conversely, suppose that the integration operator I^{o}_{μ} from $L^1_o(\mu)$ to X is order continuous. Take $0 \leqslant f \in L^1_{\tau w}(\mu)$ and $g \in L^1_o(\mu)$ with $0 \leqslant g \leqslant f$. By the assertion 9.2 (4) there exists a net $(g_{\alpha})_{\alpha\in\mathrm{A}}$ in $L^1_{\tau}(\mu)$ such that $0 \leqslant g_{\alpha} \uparrow g$. Making use of Remark 8.7 (1) and order continuity of the integration operator I^{o}_{μ} we deduce $I^{o}_{\mu}(g) = \sup_{\alpha} I^{o}_{\mu}(g_{\alpha}) = \sup_{\alpha} I^{\tau}_{\mu}(g_{\alpha}) \leqslant I^{\tau}_{\mu}(f)$ for all $g \in L^1_o(\mu)$ with $0 \leqslant g \leqslant f$. By definition of the smallest extension we have $f \in L^1_{ow}(\mu)$ and $\hat{I}^{o}_{\mu}(f) \leqslant \hat{I}^{\tau}_{\mu}(f)$, whence $L^1_{\tau w}(\mu) \subset L^1_{ow}(\mu)$ and $\hat{I}^{o}_{\mu}(f) \leqslant \hat{I}^{\tau}_{\mu}(f)$ for all $0 \leqslant f \in L^1_{\tau w}(\mu)$. The latter fact together with 9.5 (4) yields the desired result. $\rhd$

Theorem 9.6. *Let X be a Dedekind complete quasi-Banach lattice. Then there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with an X_+-valued semi-finite localizable* *measure μ such that the smallest extension $\hat{I}_\mu^o$ of the integration operator I_μ^o implements an isometric lattice isomorphism of $L^1_{ow}(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X. Moreover, for each complete disjoint set $\Gamma \subset X_+$ one can choose μ so that I_μ^o maps $L^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X_Γ.*

$\lhd$ This follows from Theorem 7.6 and Definition 9.4. $\rhd$

Lemma 9.7. *Let $X := (X, \|\cdot\|_X)$ be a Dedekind complete quasi-Banach lattice and $\mu : \mathscr{R} \to X_+$ a semi-finite localizable measure. Assume that $(f_\alpha)_{\alpha\in\mathrm{A}}$ is an increasing net in $L^1_{\tau w}(\mu)$ and there exists $y = \sup_{\alpha\in\mathrm{A}} \hat{I}_\mu^\tau(f_\alpha)$ in X_+. Then there is $f \in L^1_{\tau w}(\mu)$ such that $0 \leqslant f_\alpha \uparrow f$ and $\hat{I}_\mu^\tau(f) = y$.*

$\lhd$ According to Theorems 9.5.(1) and Theorem 5.10 $L^1_{\tau w}(\mu)$ is an order dense ideal of universally complete vector lattice $L^0(\mu)$. By Theorems 8.13 and Lemmas 5.2 and 5.3(2) the operator $\hat{I}_\mu^\tau$ is order continuous and strictly positive. In view of Theorem 5.10 we can apply Lemma 5.5 to $\hat{I}_\mu^\tau$. $\rhd$

10. Order Continuous Parts of Quasi-Banach Lattices

According to Remarks 8.2 and 8.7 (1), order and topological integration theories with respect to a vector measure coincide whenever the measure takes values in an order continuous quasi-Banach lattice. In this section we consider a special case of measures with values in order continuous parts of quasi-Banach lattices.

Definition 10.1. The *order continuous part* X_{an} of a quasi-Banach lattice X is the largest order continuous ideal in X or, more explicitly, the collection of all $x \in X$ such that, given a net (x_α) in X, the relation $|x| \geqslant x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$. The *order σ-continuous part* X_a of X is defined similarly as the largest order σ-continuous ideal in X, that is, using sequences instead of more nets. Clearly, $X_{an} \subset X_a$

Lemma 10.2. *If $X = (E, \|\cdot\|)$ is a quasi-Banach lattice then X_a is a closed sublattice of X. In particular, X is a quasi-Banach lattice.*

$\lhd$ Take a sequence (x_n) in X_a converging to $x \in X$. We may assume without loss of generality that x and all x_n are positive. Suppose that a sequence (y_k) in X obeys the conditions $x \geqslant y_k \downarrow 0$. In order to ensure the closedness of X_a, it suffices to show that $\|y_k\| \downarrow 0$. Fix an arbitrary $\varepsilon > 0$. Observe that the sequence $(x_n \wedge y_1)_{n\in\mathbb{N}}$ converges to y_1 by Lemma 2.6 and hence there exists $n_0 \in \mathbb{N}$ with $\|x_{n_0} \wedge y_1 - y_1\| < \varepsilon/(2C)$ where C is a quasi-triangle constant of X. The last inequality remains valid if we replace y_1 by y_k for any $k \in \mathbb{N}$. Indeed,

$$0 \leqslant y_k - x_{n_0} \wedge y_k = (y_k - x_{n_0}) \vee 0 \leqslant (y_1 - x_{n_0}) \vee 0 = y_1 - x_{n_0} \wedge y_1$$

and the monotonicity of the quasi-norm yields $\|y_k - x_{n_0} \wedge y_k\| \leqslant \|y_1 - x_{n_0} \wedge y_1\| \leqslant \varepsilon/2C$. The sequence $(x_{n_0} \wedge y_k)_{k\in\mathbb{N}}$ is contained in X_a and $x_{n_0} \wedge y_k \downarrow 0$ as $k \to \infty$. By definition 10.1 $\|x_{n_0} \wedge y_k\| \downarrow 0$ so that there exists $k_0 \in \mathbb{N}$ such that $\|x_{n_0} \wedge y_k\| \leqslant \varepsilon/(2C)$ for all $k \geqslant k_0$. It follows that

$$\|y_k\| \leqslant C(\|y_k - x_{n_0} \wedge y_k\| + \|x_{n_0} \wedge y_k\|) \leqslant \varepsilon$$

for all $k \geqslant k_0$. Thus, $\|y_k\| \downarrow 0$ and we are done. $\rhd$

Corollary 10.3. *If X is a Dedekind σ-complete quasi-Banach lattice then*

$$X_a = X_{an}.$$

$\lhd$ Indeed, X_a is also Dedekind σ-complete, possesses a σ-continuous quasi-norm, and is a quasi-Banach lattice by Lemma 10.2. In view of Theorem 2.10 X_a is order continuous and hence $X_a = X_{an}$. $\rhd$

Lemma 10.4. *Let X be a Dedekind σ-complete quasi-Banach lattice and $(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ a τ-measure space with X_+-valued measure. Then*

$$[L_o^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)]_a = [L_o^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)]_{an} = L_\tau^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu).$$

$\lhd$ According to Theorem 9.2(2) $L_o^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ is a Dedekind σ-complete quasi-Banach lattice so that $[L_o^1(\mu)]_a = [(L_o^1(\mu))]_{an}$ by Corollary 10.3. Therefore, it suffices to ensure the equality $[L_o^1(\mu)]_a = L_\tau^1(\mu)$. By virtue of Theorem 9.2(4) $L_\tau^1(\mu)$ is an order dense ideal of $L_o^1(\mu)$ and has an order continuous norm, whence $[L_o^1(\mu)]_a \supset L_\tau^1(\mu)$. To verify the converse inclusion consider $0 \leqslant \tilde{f} \in [L_o^1(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)]_a$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathscr{R}$-simple functions with $0 \leqslant f_n \uparrow f$ μ-a.e. Then we have $f \geqslant f - f_n \downarrow 0$ μ-a.e. and $\lim_n \|\tilde{f} - \tilde{f}_n\|_o = 0$ because $\tilde{f} \in (L_o^1(\mu))_a$. It follows that

$$\lim_{m,n} \|I_\mu^\tau(\tilde{f}_m) - I_\mu^\tau(\tilde{f}_n)\|_X = \lim_{m,n} \|I_\mu^o(\tilde{f}_m) - I_\mu^o(\tilde{f}_n)\|_X = \lim_{m,n} \|\tilde{f}_m - \tilde{f}_n\|_o = 0,$$

that is, the sequence $\big(I_\mu^\tau(\tilde{f}_n)\big)_{n\in\mathbb{N}}$ is Cauchy in X. From this we conclude that $f(t) := \lim_n f_n(t)$ holds μ-a.e. and $I_\mu^\tau(f) = \lim_n I_\mu^\tau(f_n)$, that is $f \in L_\tau^1(\mu)$ by Definition 8.6. $\rhd$

Lemma 10.5. *Let X be an order continuous quasi-Banach lattice and Γ a complete disjoint set in X_+. Then $X = X_\Gamma$, that is, X can be decomposed into an unconditional direct sum of the family of mutually disjoint bands $X_\gamma := \{\gamma\}^{\perp\perp}$. In particular, for every $x \in X$ there is a sequence $\sigma : \mathbb{N} \to \Gamma$ such that $x = \sum_{n\in\mathbb{N}} x_{\sigma(n)}$ with $x_{\sigma(n)} \in X_{\sigma(n)}$ for all $n \in \mathbb{N}$ and the series converging unconditionally.*

$\lhd$ In view of Lemma 2.9 X is super Dedekind complete. Clearly, $(X_\gamma)_{\gamma\in\Gamma}$ is a complete family of mutually disjoint bands of X. Thus, by Lemma 7.2 each $x \in X_+$ has a unique representation of the form $x = \bigvee_{\gamma\in\Gamma} x_\gamma$ with $0 \leqslant x_\gamma \in X_\gamma$ for all $\gamma \in \Gamma$. Super Dedekind completeness yields a function $\sigma : \mathbb{N} \to \Gamma$ such that $x = \bigvee_{n\in\mathbb{N}} x_{\sigma(n)}$. The series $x = \sum_{n\in\mathbb{N}} x_{\sigma(n)}$ converges unconditionally because $x = \bigvee_{n\in\mathbb{N}} x_{\sigma(\pi(n))}$ for any permutation of indices π. In particular, $x \in X_\Gamma$ and $X = X_\Gamma$, as claimed. $\rhd$

Theorem 10.6. *Let X be a Dedekind complete quasi-Banach lattice whose order continuous part X_a is order dense in X. Then there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with semi-finite localizable $\mu : \mathscr{R} \to X_a$ such that the integration operator I_μ^τ is a lattice isomorphism and linear isometry of $L_\tau^1(\mu)$ onto X_a, while $\hat{I}_\mu^\tau$ is a lattice isomorphism and linear isometry of $L_{\tau w}^1(\mu)$ onto X. Moreover, $L_\tau^1(\mu) = L_o^1(\mu)$ and $L_{\tau w}^1(\mu) = L_{ow}^1(\mu)$ as well as $I_\mu^\tau = I_\mu^o$ and $\hat{I}_\mu^\tau = \hat{I}_\mu^o$.*

$\lhd$ Put $Y := X_a$ and fix a complete disjoint set Γ in Y_+. According to Theorem 9.3 there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with a semi-finite localizable Y_Γ-valued measure μ such that the integration operator I_μ^o is a lattice isomorphism and linear isometry of $L_o^1(\mu)$ onto Y_Γ. By virtue of Lemma 10.5 $Y_\Gamma = Y$ and hence $L_o^1(\mu)$ is order continuous as Y is an order continuous quasi-Banach lattice by Lemma 10.2. By virtue of Lemma 9.2 (5) we have $L^1_o(\mu) = L^1_\tau(\mu)$ and $I^o_\mu = I^\tau_\mu$ and hence I^τ_μ is an isometric lattice isomorphism of $L^1_\tau(\mu)$ onto X_a.

Let $\hat{I}^\tau_\mu : L^1_{\tau w}(\mu) \to X$ stand for the smallest extension of I^τ_μ. Then $\hat{I}^\tau_\mu$ is a one-to-one isometric lattice homomorphism by the formula (9) and Lemma 5.3. Show that $\hat{I}^\tau_\mu$ is onto. Given arbitrary $x \in X$, one can pick an increasing net $(x_\alpha)_{\alpha\in\mathrm{A}}$ in Y_+ such that $x = \sup_\alpha x_\alpha$. Put $f_\alpha := (\hat{I}^\tau_\mu)^{-1}(x_\alpha)$ and note that $(f_\alpha)_{\alpha\in\mathrm{A}}$ is an increasing net in $L^1_\tau(\mu)_+$ with $\sup_\alpha I^\tau_\mu(f_\alpha) = x$. Since μ is semi-finite and localizable, Lemma 9.7 guarantees the existence of $f \in L^1_{\tau w}(\mu)$ such that $0 \leqslant f_\alpha \uparrow f$ and $\hat{I}^\tau_\mu(f) = x$. Thus, $\hat{I}^\tau_\mu$ is a lattice isomorphism of $L^1_{\tau w}(\mu)$ onto X. Moreover, $\hat{I}^\tau_\mu$ is also isometric by the formula (9) in Definition 9.4. Finally, the above mentioned equalities $L^1_o(\mu) = L^1_\tau(\mu)$ and $I^o_\mu = I^\tau_\mu$ imply that $L^1_{ow}(\mu) = L^1_{\tau w}(\mu)$ and $\hat{I}^o_\mu = \hat{I}^\tau_\mu$. $\triangleright$

Let X be a quasi-Banach lattice and $(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ a vector measure space with a semi-finite localizable measure $\mu : \mathscr{R} \to X_+$ which is countable additive in the sense of order or metric convergence depending on the context. By Theorem 5.10 and Corollary 5.11 $L^0(\mu)$ is a universal completion both of quasi-Banach lattices $L^1_o(\mu)$ and $L^1_\tau(\mu)$. According to Definition 3.2 we can construct maximal quasi-normed extensions $(L^1_{o\varkappa}(\mu), \|\cdot\|_{o\varkappa})$ and $(L^1_{\tau\varkappa}(\mu), \|\cdot\|_{\tau\varkappa})$ of $L^1_o(\mu)$ and $L^1_\tau(\mu)$, respectively. By virtue of Theorems 3.10 and 8.3 $L^1_{\tau\varkappa}(\mu)$ is a quasi-Banach lattice and an order dense ideal in $L^0(\mu)$, while $L^1_{o\varkappa}(\mu)$ is a quasi-normed lattice and order dense ideal in $L^0(\mu)$ which is metrically complete under the additional assumption that $L^1_o(\mu)$ has the weak σ-Fatou property.[3] Moreover, $L^1_{\tau\varkappa}(\mu)$ has the Fatou and Levi properties by Corollary 3.13, since $L^1_\tau(\mu)$ is order continuous. The following five propositions present some simple relations between $L^1_{o\varkappa}(\mu)$ and $L^1_{ow}(\mu)$, as well as $L^1_{\tau\varkappa}(\mu)$ and $L^1_{\tau w}(\mu)$.

Proposition 10.7. *Let $X := (X, \|\cdot\|_X)$ be a Dedekind complete quasi-Banach lattice and $\mu : \mathscr{R} \to X_+$ a semi-finite localizable measure. Then $L^1_{ow}(\mu) \subset L^1_{o\varkappa}(\mu)$ and the inequality $\|f\|_{o\varkappa} \leqslant \|f\|_{ow}$ holds for all $f \in L^1_{ow}(\mu)$. If, in addition, X has the Fatou property then $\|f\|_{o\varkappa} = \|f\|_{ow}$ for all $f \in L^1_{ow}(\mu)$.*

$\triangleleft$ Given arbitrary $0 \leqslant f \in L^1_{ow}(\mu)$ and $g \in L^1_o(\mu)$ with $0 \leqslant g \leqslant f$, we have $\|g\|_o = \|I^o_\mu(g)\|_X \leqslant \|\hat{I}^o_\mu(f)\|_X = \|f\|_{ow}$ for all $g \in L^1_o(\mu)$ according to the formulas (7) and (8). It follows that $f \in L^1_{o\varkappa}(\mu)$ and $\|f\|_{o\varkappa} \leqslant \|f\|_{ow}$ for all $f \in L^1_{ow}(\mu)$ by the definition of $L^1_{o\varkappa}(\mu)$.

Assume now that X has the Fatou property. Given $0 \leqslant f \in L^1_{ow}(\mu)$, the set $\{g \in L^1_o(\mu) : 0 \leqslant g \leqslant f\}$ is upward directed and so is it image $\{I^o_\mu(g) : g \in L^1_o(\mu), 0 \leqslant g \leqslant f\} \subset X_+$, as I^o_μ is positive. Using the Fatou property we deduce

$$\|f\|_{ow} = \|\hat{I}^o_\mu(f)\|_X = \|\sup\{I^o_\mu(g) : g \in L^1_o(\mu),\ 0 \leqslant g \leqslant f\}\|_X =$$
$$\sup\{\|I^o_\mu(g)\|_X : g \in L^1_o(\mu),\ 0 \leqslant g \leqslant f\} = \|f\|_{o\varkappa}$$

for all $f \in L^1_{ow}(\mu)$, as claimed. $\triangleright$

[3] We do not know whether $L^1_o(\mu)$ is metrically complete (and hence a quasi-Banach lattice) without this additional assumption coming from Theorem 3.10.

Proposition 10.8. *Let $X := (X, \|\cdot\|_X)$ be a Dedekind complete quasi-Banach lattice with the Fatou and Levi properties and $\mu : \mathscr{R} \to X_+$ a semi-finite localizable measure. Then $L^1_{ow}(\mu) = L^1_{o\varkappa}(\mu)$ and $\|f\|_{o\varkappa} = \|f\|_{ow}$ for all $f \in L^1_{ow}(\mu)$.*

$\lhd$ In view of Proposition 10.7 it suffices to show the inclusion $L^1_{ow}(\mu) \supset L^1_{o\varkappa}(\mu)$. Take an arbitrary $0 \leqslant f \in L^1_{o\varkappa}(\mu)$ and observe that $\{I^o_\mu(g) : g \in L^1_o(\mu), 0 \leqslant g \leqslant f\}$ is an upward directed set in X_+. Thus, the relation $f \in L^1_{o\varkappa}(\mu)$ implies

$$\sup\{\|I^o_\mu(g)\|_X : g \in L^1_o(\mu), 0 \leqslant g \leqslant f\} \leqslant \|f\|_{o\varkappa} < \infty$$

and, by the Levi property for X, there exists $x := \sup\{I^o_\mu(g) : g \in L^1_o(\mu), 0 \leqslant g \leqslant f\}$ in X. By the definition of $\hat{I}^o_\mu$ we get $f \in L^1_{ow}(\mu)$ and $\hat{I}^o_\mu(f) = x$. $\rhd$

Proposition 10.9. *If X is a Dedekind complete quasi-Banach lattice and $\mu : \mathscr{R} \to X_+$ is a semi-finite localizable measure then $L^1_{\tau w}(\mu) \subset L^1_{\tau\varkappa}(\mu)$ and $\|f\|_{\tau\varkappa} \leqslant \|f\|_{\tau w}$ $(f \in L^1_{\tau w}(\mu))$. If, in addition, X has the Fatou property then $\|f\|_{\tau\varkappa} = \|f\|_{\tau w}$ $(f \in L^1_{\tau w}(\mu))$.*

$\lhd$ Take $0 \leqslant f \in L_{\tau w}(\mu)$. For arbitrary $g \in L^1_\tau(\mu)$ with $0 \leqslant g \leqslant f$ we have $\|g\|_\tau = \|I^\tau_\mu(g)\|_X \leqslant \|\hat{I}^\tau_\mu(f)\|_X = \|f\|_{\tau w}$ by definition of $\hat{I}^\tau_\mu$. Thus, in view of Definition 3.2 $f \in L^1_{\tau\varkappa}(\mu)$ and $\|f\|_{\tau\varkappa} \leqslant \|f\|_{\tau w}$ for all $f \in L^1_{\tau w}(\mu)$. The set $\{I^\tau_\mu([0, f] \cap L^1_\tau(\mu))\}$ is upward directed in X_+ as I^τ_μ is a positive operator. Now, assuming that X has the Fatou property and using the definition of $\hat{I}^\tau_\mu$ we deduce

$$\begin{aligned}\|f\|_{\tau w} = \|\hat{I}^\tau_\mu(f)\|_X = \|\sup\{I^\tau_\mu(g) : g \in L^1_\tau(\mu), 0 \leqslant g \leqslant f\}\|_X = \\ = \sup\{\|g\|_\tau : g \in L^1_\tau(\mu), 0 \leqslant g \leqslant f\} = \|f\|_{\tau\varkappa},\end{aligned}$$

as claimed. $\rhd$

Proposition 10.10. *If X is a Dedekind complete quasi-Banach lattice with the Fatou and Levi properties and $\mu : \mathscr{R} \to X_+$ is a semi-finite localizable measure, then $L^1_{\tau w}(\mu)$ and $L^1_{\tau\varkappa}(\mu)$ coincide as quasi-Banach lattices.*

$\lhd$ Proposition 10.9 guarantees that $L^1_{\tau w}(\mu) \subset L^1_{\tau\varkappa}(\mu)$ and $\|f\|_{\tau\varkappa} = \|f\|_{\tau w}$ for all $f \in L^1_{\tau w}(\mu)$. Therefore, we only have to show the converse inclusion $L^1_{\tau w}(\mu) \supset L^1_{\tau\varkappa}(\mu)$. Take arbitrary $0 \leqslant f \in L^1_{\tau\varkappa}(\mu)$. As mentioned above, the set $\{I^\tau_\mu([0, f] \cap L^1_\tau(\mu))\}$ is upward directed in X_+. By Definition 3.2 we have

$$\begin{aligned}&\sup\{\|I^\tau_\mu(g)\|_X : g \in L^1_\tau(\mu), 0 \leqslant g \leqslant f\} \\ &\qquad = \sup\{\|g\|_\tau : g \in L^1_\tau(\mu), 0 \leqslant g \leqslant f\} = \|f\|_{\tau\varkappa} < \infty.\end{aligned}$$

By the Levi property $x := \sup\{I^\tau_\mu(g) : g \in L^1_\tau(\mu), 0 \leqslant g \leqslant f\}$ exists in X_+ and hence $f \in L^1_{\tau w}(\mu)$ and $\hat{I}^\tau_\mu(f) = x$ by definition of $\hat{I}^\tau_\mu$. $\rhd$

Proposition 10.11. *If X is a Dedekind complete quasi-Banach lattice and $\mu : \mathscr{R} \to X_+$ is a semi-finite localizable measure, then $L^1_{o\varkappa}(\mu) \subset L^1_{\tau\varkappa}(\mu)$ and $\|f\|_{\tau\varkappa} \leqslant \|f\|_{o\varkappa}$ for all $f \in L^1_{o\varkappa}(\mu)$. Moreover $L^1_{o\varkappa}(\mu)$ and $L^1_{\tau\varkappa}(\mu)$ coincide as quasi-Banach lattices if and only if $L^1_o(\mu)$ has the Fatou property.*

$\lhd$ For arbitrary $0 \leqslant f \in L_{o\varkappa}(\mu)$ and $g \in L^1_\tau(\mu)$ with $0 \leqslant g \leqslant f$ we have $\|g\|_\tau = \|g\|_o \leqslant \|f\|_{o\varkappa}$ by Theorem 9.2 (4). It follows that $f \in L^1_{\tau\varkappa}(\mu)$ and $\|f\|_{\tau\varkappa} \leqslant \|f\|_{o\varkappa}$ for all $f \in L^1_{\tau w}(\mu)$.

To prove the second part of the claim observe that, by Proposition 3.12, $L^1_{o\varkappa}(\mu)$ has the Fatou property if and only if so does $L^1_o(\mu)$. Now, if $L^1_{o\varkappa}(\mu)$ and $L^1_{\tau\varkappa}(\mu)$ coincide as quasi-Banach lattices, then $L^1_{o\varkappa}(\mu)$ has the Fatou properties, since so does $L^1_{\tau\varkappa}(\mu)$. Thus, $L^1_o(\mu)$ has the Fatou property. Conversely, suppose that $L^1_o(\mu)$ has the Fatou property. Take $f \in L^1_{\tau\varkappa}(\mu)$ and $g \in L^1_o(\mu)$ with $0 \leqslant g \leqslant f$. There exists a net (f_α) in $L^1_\tau(\mu)$ such that $0 \leqslant f_\alpha \uparrow f$. Consequently, $0 \leqslant f_\alpha \wedge g \uparrow g$ and $\|g\|_o = \sup_\alpha \|f_\alpha \wedge g\|_o = \sup_\alpha \|f_\alpha \wedge g\|_\tau \leqslant \|f\|_{\tau\varkappa}$ by the Fatou property. It follows that $f \in L^1_{o\varkappa}(\mu)$ and $\|f\|_{o\varkappa} \leqslant \|f\|_{\tau\varkappa}$. Thus, $L^1_{\tau\varkappa}(\mu) \subset L^1_{o\varkappa}(\mu)$ and $\|f\|_{o\varkappa} \leqslant \|f\|_{\tau\varkappa}$ for all $f \in L^1_{\tau\varkappa}(\mu)$. The latter together with the first part of the claim yields $L^1_{o\varkappa}(\mu) = L^1_{\tau\varkappa}(\mu)$ and $\|f\|_{\tau\varkappa} = \|f\|_{o\varkappa}$ for all $f \in L^1_{o\varkappa}(\mu)$. $\rhd$

Corollary 10.12. *Let X be a quasi-Banach lattice with the Fatou and Levi properties whose order continuous part X_a is order dense in X. Then there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with a semi-finite localizable measure μ with values in X_a such that the integration operator I^τ_μ is a lattice isomorphism and linear isometry of $L^1_\tau(\mu)$ onto X_a, while $\hat{I}^\tau_\mu$ is a lattice isomorphism and linear isometry of $L^1_{\tau w}(\mu)$ onto X. Moreover, $L^1_\tau(\mu) = L^1_o(\mu)$, $I^\tau_\mu = I^o_\mu$ and $L^1_{\tau\varkappa}(\mu) = L^1_{o\varkappa}(\mu) = L^1_{\tau w}(\mu) = L^1_{ow}(\mu)$.*

$\lhd$ This is immediate from Theorem 10.6 and Propositions 10.8 and 10.10. $\rhd$

11. Banach Lattice Valued Measures

In this section we stress that some consequences of the above order integration theory are new even in the case of Banach lattices. Throughout this section X is a Banach lattice with the topological dual of X^* and $(\Omega, \mathscr{R}, \mu)$ is a vector measure space with a semi-finite X_+-valued measure $\mu : \mathscr{R} \to X_+$. Denote by B^*_+ the positive part of the unit ball of X.

Definition 11.1. An $\mathscr{R}^{\mathrm{loc}}$-measurable function $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is called *weakly integrable with respect to μ* or *weakly μ-integrable* if

$$\|f\|_\mu := \sup_{x^* \in B^*_+} \int |f| \, d|x^*\mu| < +\infty,$$

where $|x^*\mu| : \mathscr{R}^{\mathrm{loc}} \to [0, \infty]$ variation of $x^*\mu$ and B^*_+ the positive part of the unit ball in X^*. A weakly integrable function f is *integrable with respect to μ* if for each $A \in \mathscr{R}^{\mathrm{loc}}$ there exists a vector denoted by $\int_A f \, d\mu \in X$, such that

$$x^*\Big(\int_A f \, d\mu\Big) = \int_A f \, dx^*\mu \text{ for all } x^* \in X^*.$$

Denote by $L^1_w(\mu)$ the space of (equivalence classes) all weakly μ-integrable function equipped with the norm $\|\cdot\|_\mu$ and let $L^1(\mu)$ stand for the subspace of $L^1_w(\mu)$ consisting of (equivalence classes) all μ-integrable functions. Note that if $\|f\|_\mu < \infty$ then $|f| < \infty$ μ-a.e. Thus, $L^1_w(\mu)$ and $L^1(\mu)$ can be considered as subspaces of $L^0(\mu)$.

Theorem 11.2. *The spaces $L^1_w(\mu)$ and $L^1(\mu)$ are Banach lattices spaces and order dense ideas of $L^0(\mu)$. Moreover $L^1(\mu)$ is a closed subspace of $L^1_w(\mu)$.*

$\lhd$ See [46, Theorem 3.3] and [53, Theorems 4.5, 4.7, 4.10]. $\rhd$

Remark 11.3. Observe that the space $S(\mathscr{R})$ of $\mathscr{R}$-simple functions (see Section 4) is dense in $L^1(\mu)$. Moreover, for every $\mathscr{R}$-simple function $\varphi = \sum_{i=1}^n a_i\chi_{A_i}$ we have that $I_\mu(f) := \int f\,d\mu = I_\mu^\tau(f)$. It follows that the spaces $L^1(\mu) = (L^1(\mu), \|\cdot\|_\mu)$ and $L^1_\tau(\mu) = (L^1_\tau(\mu), \|\cdot\|_\tau)$ coincide as Banach lattices (see Section 8).

Now we present two representation results for Dedekind complete Banach lattices.

Theorem 11.4. *Let X be a Dedekind complete Banach lattice. Then there exists a measure space $(\Omega, \mathscr{R}, \mu)$ with a semi-finite localizable X_+-valued measure μ such that the smallest extension $\hat{I}_\mu^o$ of the integration operator I_μ^o implements an isometric lattice isomorphism of $L^1_{ow}(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X. Moreover, for each complete disjoint set $\Gamma \subset X_+$ one can choose μ so that I_μ^o maps $L^1_o(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X_Γ.*

$\lhd$ Follows immediately from Theorem 9.6. $\rhd$

Theorem 11.5. *Let X be an order complete Banach lattice whose order continuous part X_a is order dense in X. Then there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with semi-finite localizable $\mu : \mathscr{R} \to X_a$ such that the integration operator I_μ^τ is a lattice isomorphism and linear isometry of $L^1_\tau(\mu)$ onto X_a, while the smallest extension $\hat{I}_\mu^\tau$ of I_μ^τ is a lattice isomorphism and linear isometry of $L^1_{\tau w}(\Omega, \mathscr{R}^{\mathrm{loc}}, \mu)$ onto X. Moreover, $L^1_\tau(\mu) = L^1_o(\mu)$, $L^1_{\tau w}(\mu) = L^1_{ow}(\mu)$ and $I_\mu^\tau = I_\mu$, $\hat{I}_\mu^\tau = \hat{I}_\mu^o$.*

$\lhd$ The proof is follows immediately from Theorem 10.6. $\rhd$

The space $L^1_w(\mu)$ always has the σ-Fatou and σ-Levi properties (see Calabuig, Delgado, Juan, and Sánchez Pérez [6]).[4] It was indicated in [6, § 5] as an open question whether the space $L^1_w(\mu)$ has the Fatou and Levi properties.

Definition 11.6. A measure $\mu : \mathscr{R} \to X_+$ is said to be *$\mathscr{R}$-decomposable* if we can write $\Omega = N \cup \bigcup_{\alpha\in\Delta} \Omega\alpha$ where N is μ-null set and $\{\Omega_\alpha : \alpha \in \Delta\}$ is a family of pairwise disjoint sets in $\mathscr{R}$ satisfying that

(1) if $A_\alpha \in \mathscr{R} \cap 2^{\Omega_\alpha}$ for all $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha \in \mathscr{R}^{\mathrm{loc}}$,

(2) for each $x^* \in X^*$ if $Z_\alpha \in \mathscr{R} \cap 2^{\Omega_\alpha}$ is $|x^*\mu|$-null set for all $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} Z_\alpha$ is $|x^*\mu|$-null set.

It was shown in [6, Theorem 5.8] that if a measure μ is $\mathscr{R}$-decomposable, then $L^1_w(\mu)$ has the Fatou and Levi properties. We now show that $L^1_w(\mu)$ has the Fatou and Levi properties under the weaker assumption that μ is localizable.

Lemma 11.7. *Let Ω be nonempty set, Σ a σ-algebra of subsets of Ω and $\lambda : \Sigma \to \mathbb{R}_+ \cup \{\infty\}$ a semi-finite measure. Assume that $f \in L^0(\lambda)$ and a net $(f_\alpha)_{\alpha\in A}$ in $L_1(\lambda)$ has the properties that $0 \leqslant f_\alpha \uparrow_\alpha f$ and $\sup_\alpha \int f_\alpha\,d\lambda < \infty$. Then $f \in L_1(\lambda)_+$ and there exists an increasing sequence of indices $(\alpha_n)_{n=1}^\infty \subset A$ such that $0 \leqslant f_{\alpha_n} \uparrow_n f$ and $\sup_n \int f_{\alpha_n}\,d\lambda = \int f\,d\mu$.*

$\lhd$ Without loss of generality, we can assume that the set $F := \{f_\alpha : \alpha \in A\}$ contains suprema of all its finite subsets. Choose increasing sequence $(\alpha_n)_{n=1}^\infty \subset A$ such, that $C := \sup_n \int f_{\alpha_n}\,d\lambda = \sup_\alpha \int f_\alpha\,d\lambda$. By monotone convergence theorem there exists $f_0 \in L_1(\lambda)$ with $0 \leqslant f_{\alpha_n} \uparrow_n f_0$ and $\int f_0\,d\lambda = C$. If we prove that f_0 is

[4]The Fatou (σ-Fatou) property in Calabuig, Delgado, Juan, and Sánchez Pérez [6] is understood as the conjunction of the Fatou and Levi (σ-Fatou and σ-Levi) properties. In this work we distinguish between these properties following Fremlin [23, Definitions 23A and 23I], see also Abramovich and Aliprantis [2, Definition 7].

an upper bound of $(f_\alpha)_{\alpha\in A}$ then, in view of $f_0 = \sup_n f_{\alpha_n} \leqslant f$, we get $f = f_0$ and the proof is complete. Fix arbitrary $\beta \in A$ and note that $f_\beta \vee f_{\alpha_n} \uparrow_n f_\beta \vee f_0$. So by dominated convergence theorem $\int f_\beta \vee f_{\alpha_n}\, d\mu \uparrow_n \int f_\beta \vee f_0\, d\mu$. Since $f_\beta \vee f_{\alpha_n} \in F$, we have $\int f_\beta \vee f_{\alpha_n}\, d\mu \leqslant \int f_0\, d\mu$ for all $n \in \mathbb{N}$. Consequently, $\int f_\beta \vee f_0\, d\mu = \sup_n \int f_\beta \vee f_{\alpha_n}\, d\mu \leqslant \int f_0\, d\mu$. At the same time, $\int f_\beta \vee f_0\, d\mu \geqslant \int f_0\, d\mu$. Therefore, $\int f_\beta \vee f_0\, d\mu = \int f_0\, d\mu$ and $f_\beta \vee f_0 = f_0$ for all $\beta \in A$. It follows that f_0 is an upper bound of the net $(f_\alpha)_{\alpha\in A}$. $\rhd$

Lemma 11.8. *Let $X := (X, \|\cdot\|_X)$ be a Banach lattice and $(\Omega, \mathscr{R}, \mu)$ a vector measure space with a semi-finite measure $\mu : \mathscr{R} \to X_+$. Assume that $f \in L^0(\mu)$ and a net $(f_\alpha)_{\alpha\in A}$ in $L^1_w(\mu)$ is such that $0 \leqslant f_\alpha \uparrow_\alpha f$ and $\sup_\alpha \|f_\alpha\|_\mu < \infty$. Then $f \in L^1_w(\mu)_+$ and $\sup_\alpha \|f_\alpha\|_\mu = \|f\|_\mu$. Moreover, there exists increasing sequence $(\alpha_n)_{n=1}^\infty \subset A$ such that $0 \leqslant f_{\alpha_n} \uparrow_n f$ and $\sup_n \|f_{\alpha_n}\|_\mu = \|f\|_\mu$. In particular, $L^1_w(\mu)$ has the Fatou property.*

$\lhd$ By virtue of Rybakov's theorem there exists $x^* \in B^*_+$ such that $|x^*\mu|$-negligible sets and μ-negligible sets from $\mathscr{R}^{\mathrm{loc}}$ coincide. Consequently, $L^0(\mu)$ and $L^0(|x^*\mu|)$ coincide as vector lattices. By virtue of Lemma 11.7 $f \in L^1(|x^*\mu|)$ and there exists increasing sequence $(\alpha_n)_{n=1}^\infty \subset A$ with $0 \leqslant f_{\alpha_n} \uparrow_n f$ in $L^1(|x^*\mu|)$. Then $0 \leqslant f_{\alpha_n} \uparrow_n f$ in $L_0(\mu)$ and $\sup_n \|f_n\|_\mu < \infty$. Since $L^1_w(\mu)$ has σ-Fatou and σ-Levi properties, we have $f \in L^1_w(\mu)$ and $\|f\|_\mu = \sup_n \|f_{\alpha_n}\|_\mu$. At the same time, $\sup_n \|f_{\alpha_n}\|_\mu \leqslant \sup_\alpha \|f_\alpha\|_\mu$ and hence $\|f\|_\mu = \sup_\alpha \|f_\alpha\|_\mu$. $\rhd$

Theorem 11.9. *Let X be a Banach lattice and $(\Omega, \mathscr{R}, \mu)$ a vector measure space with a semi-finite localizable X_+-valued measure μ. Then the Banach lattice $L^1_w(\mu)$ has the Fatou and Levi properties.*

$\lhd$ Suppose a net $(f_\alpha)_{\alpha\in A}$ in $L^1_w(\mu)$ is such that $0 \leqslant f_\alpha \uparrow_\alpha$ and $\sup_\alpha \|f_\alpha\|_\mu < \infty$. Since μ is semi-finite and localizable, $L^0(\mu)$ is a universally complete vector lattice. By virtue of Lemma 3.8 there exists a band projection π in $L^0(\mu)$ such that $f := \sup_\alpha \pi f_\alpha$ exists in $L^0(\mu)$, while the complimentary projection $\pi' := I_{L^0(\mu)} - \pi$ obeys the equation $N\pi'(g) = \sup_\alpha \pi'(f_\alpha \wedge Ng)$ for all $g \in L^0(\mu)$ and $N \in \mathbb{N}$. Clearly, if $\pi' = 0$ then $f_\alpha \uparrow_\alpha f$ and $f \in L_0(\mu)$ so that we are done by Lemma 11.8. Assuming that $\pi' \neq 0$, we can choose $0 \leqslant g \in L^1_w(\mu)$ with $\pi'(g) > 0$ as $L^1_w(\mu)$ is an order dense ideal in $L^0(\mu)$. Applying again Lemma 11.8 we deduce $\|N\pi'(g)\|_\mu = \sup_\alpha \|\pi'(f_\alpha \wedge Ng)\|_\mu \leqslant \sup_\alpha \|f_\alpha\|_\mu < \infty$ for all $N \in \mathbb{N}$, a contradiction. $\rhd$

According to Definition 3.2 we can make the maximal quasi-normed extension $L^1_{\tau\varkappa}(\mu) := (L^1_{\tau\varkappa}(\mu), \|\cdot\|_{\tau\varkappa})$ of the Banach lattice $L^1_\tau(\mu) := (L^1_\tau(\mu), \|\cdot\|_\tau)$. The quasi-normed lattice $L^1_{\tau\varkappa}(\mu)$ is a Banach lattice with the Levi and Fatou properties and an order dense ideal of $L_0(\mu)$ (see the remark after Theorem 10.6). The following result can be considered as an alternative description of $L^1_w(\mu)$.

Theorem 11.10. *Let $X := (X, \|\cdot\|_X)$ be a Banach lattice and $(\Omega, \mathscr{R}, \mu)$ a vector measure space with a semi-finite localizable measure $\mu : \mathscr{R} \to X_+$. Then $L^1_w(\mu)$ and $L^1_{\tau\varkappa}(\mu)$ coincide as Banach lattices.*

$\lhd$ Let $f \in L^1_w(\mu)$ and take arbitrary $g \in L^1_\tau(\mu)$ with $0 \leqslant g \leqslant f$. Then $\|g\|_\tau = \|g\|_\mu \leqslant \|f\|_\mu < \infty$ so that $f \in L^1_{\tau\varkappa}(\mu)$ and we get $L^1_w(\mu) \subset L^1_{\tau\varkappa}(\mu)$. To ensure the converse inclusion, observe that $L^1_\tau(\mu)$ is order dense in $L^1_{\tau\varkappa}(\mu)$ and $L_{\tau\varkappa}(\mu)$ has the Fatou and Levi properties (see the remark after Theorem 11.9). It follows that for $f \in L^1_{\tau\varkappa}(\mu)$ there exists a net $(f_\alpha)_{\alpha\in A}$ in $L_\tau(\mu)$ such that $0 \leqslant f_\alpha \uparrow f$ and $\sup_\alpha \|f_\alpha\|_\tau = \|f\|_{\tau\varkappa} < \infty$. By Lemma 11.8 $f \in L^1_w(\mu)$ and $\|f\|_\mu = \sup_\alpha \|f_\alpha\|_\mu = \sup_\alpha \|f_\alpha\|_{\tau\varkappa} = \|f\|_{\tau\varkappa}$. Consequently, $L^1_w(\mu) = L^1_{\tau\varkappa}(\mu)$ and $\|f\|_\mu = \|f\|_{\tau\varkappa}$ for all $f \in L^1_{\tau\varkappa}(\mu)$, as claimed. $\rhd$

Corollary 11.11. *Let X be Banach lattice with the Fatou and Levi properties whose order continuous part X_a is order dense in X. Then there exists a vector measure space $(\Omega, \mathscr{R}, \mu)$ with a semi-finite localizable measure μ with values in X_a such that the integration operator I^τ_μ is a lattice isomorphism and linear isometry of $L^1_\tau(\mu)$ onto X_a, while $\hat{I}^\tau_\mu$ is a lattice isomorphism and linear isometry of $L^1_{\tau\varkappa}(\mu)$ onto X. Moreover, $L^1_\tau(\mu) = L^1(\mu)$, and $L^1_{\tau\varkappa}(\mu) = L^1_{o\varkappa}(\mu) = L^1_{\tau w}(\mu) = L^1_{ow}(\mu) = L^1_w(\mu)$ where all equalities mean the coincidence of Banach lattices.*

$\lhd$ This is immediate from Corollary 10.12, Remark 11.3, and Theorem 11.10. $\rhd$

The following example taken from [6, Example 4.1] shows that $L^1_\tau(\mu)$ and $L^1_o(\mu)$ may differ essentially.

Example 11.12. Let Γ be an uncountable abstract set. We recall that $c_0(\Gamma)$ denotes the space of all real functions $f \in \mathbb{R}^\Gamma$ for witch $\{t \in \Gamma : |f(t)| > \varepsilon\}$ is finite for all $\varepsilon > 0$, $l^\infty(\Gamma)$ denotes the space of all real bounded functions $f \in \mathbb{R}^\Gamma$ and $l^\infty_0(\Gamma)$ denotes the space of all real bounded functions $f \in l^\infty(\Gamma)$ with countable support. All these spaces are Banach lattices with uniform norm $\|f\|_\infty := \{\sup |f(t)| : t \in \Gamma\}$. It is clearly that $c_0(\Gamma) \subset l^\infty_0(\Gamma) \subset l^\infty(\Gamma)$. Let $\mathscr{R}$ be the δ-ring of finite subsets of Γ and $\mu : \mathscr{R} \to l^\infty_0(\Gamma)$ the vector measure defined by $\mu(A) = \chi_A$ far all $A \in \mathscr{R}$. Then $L^1(\mu) = L^1_\tau(\mu) = c_0(\Gamma)$, $L^1_o(\mu) = l^\infty_0(\Gamma)$, $L^1_{ow}(\mu) = l^\infty(\Gamma) = L^1_{\tau w}(\mu)$.

Remark 11.13. (1) Theorems 11.4 and 11.5 have no parallel versions in terms of Banach lattices $L^1(\mu)$ and $L^1_w(\mu)$.

(2) It can be easily seen that $\mathscr{R}$-decomposable measure is localizable (cf. Fremlin [24, Theorem 211L (d)]). Thus, Theorem 11.9 is a generalization of [6, Theorem 5.8] stating that $L^1_w(\mu)$ has the Fatou and Levi properties, whenever μ is $\mathscr{R}$-localizable.

(3) Corollary 11.11 contains the following result obtained in Delgado and Juan [18]: *Every Banach lattice having the Fatou and Levi properties and having its order continuous part as an order dense subset, can be represented as the space $L^1_w(\mu)$ of weakly integrable functions with respect to some vector measure μ defined on a δ-ring.* Some application of this result see in Juan and Sánchez Pérez [30].

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