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Цель статьи — дать обзор недавних результатов об инъективных банаховых решетках; излагается булевозначный подход к проблеме и формулируются нерешенные задачи. Центральная идея исследования — булевозначный принцип переноса с AL-пространств на инъективные банаховы решетки: каждая инъективная банахова решетка допускает погружение в подходящую булевозначную модель, превращаясь при этом в AL-пространство. В качестве приложения дается описание инъективных банаховых решеток, аналогичное описанию AL-пространств.

Ключевые слова: AL-пространство, AM-пространство, инъективная банахова решетка, булевозначная модель, булевозначный принцип переноса, однородная банахова решетка, представление инъективных банаховых решеток.

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The aim of this work is to survey recent results on injective Banach lattices, outline a Boolean-valued approach, and pose some open problems. The central idea to the investigation is a Boolean-valued transfer principle from AL-spaces to injective Banach lattices: Every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space. To illustrate the method, a concrete description of injective Banach lattices similar to that of AL-spaces is presented.

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Keywords: AL-space, AM-space, injective Banach lattice, Boolean-valued model, Boolean-valued transfer principle, homogeneous Banach lattice, representation of injective Banach lattices.

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BOOLEAN-VALUED AL-SPACES AND INJECTIVE BANACH LATTICES[1]

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1. Introduction

The aim of this work is to survey recent results on injective Banach lattices obtained in [20, 21, 22], outline a Boolean-valued approach, and pose some open problems. The central idea to the investigation is a *Boolean-valued transfer principle* from AL-spaces to injective Banach lattices: It was announced in [22] and proved in [20] that every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space. According to this fact and fundamental principles of Boolean-valued models, each theorem about the AL-space within Zermelo–Fraenkel set theory has its counterpart for the original injective Banach lattice interpreted as the Boolean-valued AL-space. To illustrate the method we present a concrete description of injective Banach lattices similar to that of AL-spaces which relays upon Maharam's representation of measure algebras [11, 37].

2. Banach Lattices

A *Banach lattice* is a Banach space over the reals that is equipped with a partial order $\leqslant$ for which the *supremum* $x \vee y$ and the *infimum* $x \wedge y$ exist for all vectors $x, y \in X$, and such that the positive cone $X_+ := \{x \in X : 0 \leqslant x\}$ is closed under addition and multiplication by nonnegative real numbers and the order is connected to the norm by the condition that $|x| \leqslant |y| \implies \|x\| \leqslant \|y\|$, where the *absolute value* is defined by $|x| := x \vee (-x)$. All classical Banach spaces (L_p, l_p, $C(K)$, c, c_0) are Banach lattices. A *band* in a Banach lattice X is a subset of the form $A^\perp := \{x \in X : (\forall a \in A)\, |x| \wedge |a| = 0\}$. A band B in X that satisfies $X = B \oplus B^\perp$ is referred to as a *projection band,* while the associated projection is called a *band projection.* Let $\mathbb{P}(X)$ stand for the Boolean algebra of band projections in X.

A linear mapping T from a Banach lattice X to a Banach lattice Y is called *positive* if it sends positive vectors to positive vectors, i. e., $T(X_+) \subset Y_+$. If a positive operator preserves the lattice operations, it is called a *lattice homomorphism.* A one-to-one surjective lattice homomorphism is called a *lattice isomorphism.* A *lattice isometry* is a lattice isomorphism which is also an isometry.

Two classes of Banach lattices play a significant role in the Banach lattice theory.

Definition 1. A Banach lattice X is said to be an *AL-space* (*AM-space*) if $\|x + y\| = \|x\| + \|y\|$ (resp. $\|x \vee y\| = \max\{\|x\|, \|y\|\}$) whenever $|x| \wedge |y| = 0$. An AM-space has a (strong order) *unit* $u \geqslant 0$ if the unit ball of X is the order interval $[-u, u] := \{x : -u \leqslant x \leqslant u\}$.

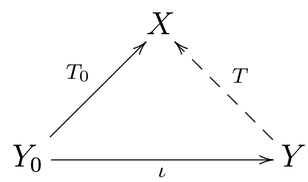
Kakutani Representation Theorem. *An arbitrary AL-space is lattice isometric to $L_p(\mu)$ for some measure μ.*

Kreĭns–Kakutani Representation Theorem. *An AM-space is lattice isometric to a sublattice of $C(K)$ for some compact Hausdorff space K. Moreover, if the AM-space has a strong order unit then it is lattice isometric to $C(K)$ itself.*

Remark 1. Banach lattices were first considered by Kantorovich [17]. For an extensive treatment of Banach lattices see [2, 25, 30, 34, 35].

3. Injective Banach Lattices

Definition 2. A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ with $\|T_0\| = \|T\|$. This definition is illustrated by the commutative ($T_0 = T \circ \iota$) diagram:



Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a positive contractive projection from Y onto X.

Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [3, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms. More details concerning injective Banach lattices see in Lotz [28], Cartwright [7], Haydon [15], Buskes [6], and Wickstead [44].

Lotz [28] was the first who introduced this concept and proved among other things the following two results.

Theorem 1 (Lotz, [28]). *A Dedekind complete AM-space with unit is an injective Banach lattice.*

Taking into account the Kreĭns–Kakutani Representation Theorem one can state Theorem 1 equivalently: The Banach lattice of continuous function $C(K)$ is injective, whenever K is an extremally disconnected Hausdorff compact topological space.

Theorem 2 (Lotz, [28]). *Every AL-space is an injective Banach lattice.*

The result shows that there is an essential difference between injective Banach lattices and injective Banach spaces, since $C(K)$ with extremally disconnected compactum K is the only (up to isometric isomorphism) injective object in the category of Banach spaces and linear contractions.

Remark 2. A Banach lattice X is called λ-*injective* if $\|T\| \leqslant \lambda\|T_0\|$ in Definition 2. In what follows injective means 1-injective; λ-injective Banach lattices ($\lambda > 1$) are not considered. For λ-injective Banach lattices ($\lambda > 1$) see [26, 27, 29].

4. Characterization of Injective Banach Lattices

Definition 3. A Banach lattice X has the *splitting property* (or the *Cartwright property*) if, given $x_1, x_2, y \in X_+$ with $\|x_1\| \leqslant 1$, $\|x_2\| \leqslant 1$, and $\|x_1 + x_2 + y\| \leqslant 2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant 1$, and $\|x_2 + y_2\| \leqslant 1$.

Theorem 3 (Cartwright, [7]). *A Banach lattice has the splitting property if and only if its second dual is injective.*

Definition 4. A Banach lattice X is said to have: the *property* (P) if there exists a positive contractive projection in X'' onto X [30, p. 47]; the *Levi property* if $0 \leqslant x_\alpha \uparrow$ and $\|x_\alpha\| \leqslant 1$ imply that $\sup_\alpha x_\alpha$ exists in X [1, Definition 7 (2)]; the *Fatou property* if $0 \leqslant x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$ [1, Definition 7 (3)]. A Banach lattice with the Levi (Fatou) property is also called *order semicontinuous* (resp. *monotonically complete*) [30].

A Dedekind complete Banach lattice X with a separating order continuous dual has property (P) if and only if it has the Levi and Fatou properties [35, Propositions 7.6 and 7.10]. Cartwright [7, Corollary 3.8] proved that a Banach lattice is injective if and only if it has the Cartwright property and the property (P). Haydon demonstrated that the property (P) may be replaced with the intrinsic 'completeness' property.

Theorem 4 (Haydon, [15]). *A Banach lattice is injective if and only if it has the Cartwright, Fatou, and Levi properties.*

Definition 5. A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ for all $x \in X$, where $\pi^\perp := I_X - \pi$. The collection of all M-projections forms a subalgebra $\mathbb{M}(X)$ of the Boolean algebra $\mathbb{P}(X)$. The f-subalgebra of the center $\mathscr{Z}(X)$ generated by $\mathbb{M}(X)$ is called the *M-center* of X and denoted by $\mathscr{Z}_m(X)$.

Observe that $\mathbb{M}(X)$ is an order closed subalgebra of $\mathbb{P}(X)$ whenever X has the Fatou and Levi properties. In this event the relations $\mathbb{B} \simeq \mathbb{M}(X)$ and $\Lambda(\mathbb{B}) \simeq \mathscr{Z}_m(X)$ are equivalent.

Theorem 5 (Haydon, [15]). *An injective Banach lattice X is an AL-space if and only if there is no M-projection in it other than zero and identity, i. e., $\mathbb{M}(X) = \{0, I_X\}$ (if and only if its M-center is one-dimensional).*

Remark 3. Haydon proved three representation theorems for injective Banach lattices, see [15, Theorems 5C, 6H, and 7B]. These results may be also deduced from our representation theorem (see Theorem 10 below).

5. Boolean-Valued Models

In 1963 P. Cohen discovered his *method of 'forcing'* and also proved the independence of the Continuum Hypothesis. A comprehensive presentation of the Cohen forcing method gave rise to the *Boolean-valued models of set theory*, which were first introduced by D. Scott and R. Solovay (see Scott [36]) and P. Vopěnka [43]. A systematic account of the theory of Boolean-valued models can be found in [4, 42].

The term *Boolean-valued analysis,* coined by G. Takeuti (see [39, 40, 41]), signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras.

As these models, the classical Cantorian paradise in the shape of the von Neumann universe V and a specially-trimmed Boolean-valued universe $V^{(\mathbb{B})}$ are usually taken. Comparative analysis is carried out by means of some interplay between V and $V^{(\mathbb{B})}$.

Boolean-valued analysis stems from the fact that each internal field of reals of a Boolean-valued model descends into a universally complete vector lattice. This remarkable fact was discovered by E. Gordon [12, 13]. Two important particular cases were intensively studied by G. Takeuti [39], who observed that the vector lattice of (equivalence classes of) measurable function and a commutative algebra of (unbounded) self-adjoint operators in Hilbert space can be considered as instances of Boolean-valued reals. A detailed presentation of Boolean-valued analysis can be found in [23, 24], see also [19].

6. The Universe of Boolean-Valued Sets

Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \big\{ x : x \text{ is a function, } (\exists\, \beta)\big(\beta < \alpha,\ \operatorname{dom}(x) \subset V_\beta^{(\mathrm{B})},\ \operatorname{Im}(x) \subset \mathbb{B}\big)\big\}.$$

After this recursive definition the *Boolean-valued universe* $V^{(\mathbb{B})}$ or, in other words, the *class of* $\mathbb{B}$*-sets* is introduced by

$$V^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two element Boolean algebra $\mathbb{2} := \{\mathbb{O}, \mathbb{1}\}$ this procedure yields a version of the classical *von Neumann universe* V (cp. [24, Theorem 4.2.8]).

Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the complexity of φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ (for instance, $[\![\varphi_1 \vee \varphi_2]\!] := [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$ and $[\![\forall x \varphi(x)]\!] = \bigwedge\{[\![\varphi(u)]\!] : u \in V^{(\mathbb{B})}\}$) and taking into consideration the way in which a formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$ (with x, y assumed to be elements of $V^{(\mathrm{B})}$) are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t \in \operatorname{dom}(y)} \big(y(t) \wedge [\![t = x]\!]\big),$$

$$[\![x = y]\!] = \bigvee_{t \in \operatorname{dom}(x)} \big(x(t) \Rightarrow [\![t \in y]\!]\big) \wedge \bigvee_{t \in \operatorname{dom}(y)} \big(y(t) \Rightarrow [\![t \in x]\!]\big).$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i. e., $(a \Rightarrow b) := (a^* \vee b)$, where a^* is as usual the *complement* of a.

We say that the *statement* $\varphi(x_1,\dots,x_n)$ *is valid or the elements* $x_1,\dots,x_n$ *possess the property* φ *inside* $\mathrm{V}^{(\mathbb{B})}$ if $[\![\varphi(x_1,\dots,x_n)]\!]=\mathbb{1}$. In this event, we write also

$$\mathrm{V}^{(\mathbb{B})}\vDash\varphi(x_1,\dots,x_n).$$

The universe $\mathrm{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled:

Transfer Principle. *For every theorem* φ *of ZFC, we have* $[\![\varphi]\!]=\mathbb{1}$ (*also in ZFC*)*; i. e.,* φ *is true inside the Boolean-valued universe* $\mathrm{V}^{(\mathbb{B})}$.

Maximum Principle. *Let* $\varphi(x)$ *be a formula of ZFC. Then* (*in ZFC*) *there is a* $\mathbb{B}$*-valued set* x_0 *satisfying* $[\![(\exists\,x)\varphi(x)]\!]=[\![\varphi(x_0)]\!]$.

Corollary. *If* $\mathrm{V}^{(\mathbb{B})}\models(\exists\,x)\varphi(x)$, *then* $\mathrm{V}^{(\mathbb{B})}\models\varphi(x_0)$ *for some* $x_0\in\mathrm{V}^{(\mathbb{B})}$.

7. Ascending and Descending

As was mentioned above, a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models V and $\mathrm{V}^{(\mathrm{B})}$ is needed. The relevant *ascending-and-descending technique* rests on the functors of *canonical embedding*, *descent*, and *ascent*.

Standard name. Given $X\in\mathrm{V}$, we denote by $X^{\wedge}\in\mathrm{V}^{(\mathbb{B})}$ the *standard name* of X. The standard name is an embedding of V into $\mathrm{V}^{(\mathbb{B})}$. Moreover, the standard name sends V onto $\mathrm{V}^{(2)}$, i. e., $\mathrm{V}\simeq\mathrm{V}^{(2)}\subset\mathrm{V}^{(\mathbb{B})}$, where $2:=\{\mathbb{0},\mathbb{1}\}\subset\mathbb{B}$.

A formula is *restricted* provided that each bound variable in it is restricted by a bounded quantifier; i. e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall\,x\in y)$ or $(\exists\,x\in y)$.

Restricted Transfer Principle. *Let* $\varphi(x_1,\dots,x_n)$ *be a bounded formula of ZFC. Then* (*in ZFC*) *for every collection* $x_1,\dots,x_n\in\mathrm{V}$ *we have*

$$\varphi(x_1,\dots,x_n)\iff\mathrm{V}^{(\mathbb{B})}\models\varphi(x_1^{\wedge},\dots,x_n^{\wedge}).$$

Descent. Given an arbitrary element X of the Boolean-valued universe $\mathrm{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}$ of X as $X{\downarrow}:=\{y\in\mathrm{V}^{(\mathbb{B})}:\ [\![y\in x]\!]=\mathbb{1}\}$. The class $X{\downarrow}$ is a set, i. e., $X{\downarrow}\in\mathrm{V}$ for all $X\in\mathrm{V}^{(\mathbb{B})}$. If $[\![X\neq\varnothing]\!]=\mathbb{1}$ then $X{\downarrow}$ is nonempty.

Suppose that $X,Y,f\in\mathrm{V}^{(\mathbb{B})}$ are such that $[\![f:X\to Y]\!]=\mathbb{1}$, i. e., f is a mapping from X into Y inside $\mathrm{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ into $Y{\downarrow}$ satisfying $[\![f{\downarrow}(x)=f(x)]\!]=\mathbb{1}$ for all $x\in X{\downarrow}$. The descent of a mapping is *extensional*:

$$[\![x_1=x_2]\!]\leqslant[\![f(x_1)=f(x_2)]\!]\quad(x_1,x_2\in X{\downarrow}).$$

Assume that P is an n-ary relation on X inside $\mathrm{V}^{(\mathbb{B})}$; i. e., $X,P\in\mathrm{V}^{(\mathbb{B})}$ and $[\![P\subset X^{n^{\wedge}}]\!]=1$ $(n\in\mathbb{N})$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that

$$(x_1,\dots,x_n)\in P'\iff[\![(x_1,\dots,x_n)^B\in P]\!]=\mathbb{1}.$$

We denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

Ascent. Let $X \in \mathrm{V}$ and $X \subset \mathrm{V}^{(\mathbb{B})}$; i. e., let X be some set composed of $\mathbb{B}$-valued sets or, in other words, $X \in \mathscr{P}(\mathrm{V}^{(\mathbb{B})})$. There exists a unique $X{\uparrow} \in \mathrm{V}^{(\mathbb{B})}$ such that $[\![y \in X{\uparrow}]\!] = \bigvee\{[\![x = y]\!] : x \in X\}$ for all $y \in \mathrm{V}^{(\mathbb{B})}$. The element $X{\uparrow}$ is called the *ascent* of X. Observe that the ascent extend the standard name in the sense that $Y^{\wedge}$ is the ascent of $\{y^{\wedge} : y \in Y\}$ whenever $Y \in \mathrm{V}$.

Let $X, Y \in \mathscr{P}(\mathrm{V}^{(\mathbb{B})})$, and $f : X \to Y$. There exists a unique function $f{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathrm{V}^{(\mathbb{B})}$ such that $f{\uparrow}(A{\uparrow}) = f(A){\uparrow}$ is valid for every subset $A \subset X$ if and only if f is extensional.

8. Boolean-Valued Reals

Recall the well-known assertion of ZFC: *There exists a field of reals that is unique up to isomorphism.* Denote by $\mathbb{R}$ the field of reals (in the sense of V). Successively applying the transfer and maximum principles, we find an element $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ for which $[\![\,\mathscr{R}$ is a field of reals$\,]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}' \in \mathrm{V}^{(\mathbb{B})}$ satisfies the condition $[\![\,\mathscr{R}'$ is a field of reals$\,]\!] = \mathbb{1}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic$\,]\!] = \mathbb{1}$. In other words, there exists an internal field of reals $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. We call $\mathscr{R}$ the *internal reals* in $\mathrm{V}^{(\mathbb{B})}$.

Consider another well-known assertion of ZFC: *If $\mathbb{P}$ is an Archimedean ordered field then there is an isomorphic embedding h of the field $\mathbb{P}$ into $\mathbb{R}$ such that the image $h(\mathbb{P})$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(\mathbb{P})$ is dense in $\mathbb{R}$.*

Note also that $\varphi(x)$, presenting the conjunction of the axioms of an Archimedean ordered field x, is bounded; therefore, $[\![\,\varphi(\mathbb{R}^{\wedge})\,]\!] = \mathbb{1}$, i. e., $[\![\,\mathbb{R}^{\wedge}$ is an Archimedean ordered field$\,]\!] = \mathbb{1}$. ‘Pulling’ the above assertion through the transfer principle, we conclude that $[\![\,\mathbb{R}^{\wedge}$ is isomorphic to a dense subfield of $\mathscr{R}\,]\!] = \mathbb{1}$. We further assume that $\mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{R}$. It is easy to see that the elements $0^{\wedge}$ and $1^{\wedge}$ are the zero and unity of $\mathscr{R}$.

Look now at the descent $\mathbf{R} := \mathscr{R}{\downarrow}$ of the algebraic structure $\mathscr{R}$. In other words, consider the descent of the underlying set of the structure $\mathscr{R}$ together with the descended operations and order. For simplicity, we denote the operations and order in $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\leqslant$.

The fundamental result of Boolean-valued analysis is the Gordon Theorem which describes an interplay between $\mathbb{R}$, $\mathscr{R}$, and $\mathbf{R}$ and reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean-valued model.*

Theorem 6 (Gordon Theorem, [12])**.** *Let $\mathscr{R}$ be a field of reals in $\mathrm{V}^{(\mathbb{B})}$ and $\mathbf{R} = \mathscr{R}{\downarrow}$. Then the following assertions hold:*

(1) *The algebraic structure $\mathbf{R}$ (with the descended operations and order) is an universally complete vector lattice.*

(2) *The internal field $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ can be chosen so that*

$$[\![\,\mathbb{R}^{\wedge} \text{ is a dense subfield of the field } \mathscr{R}\,]\!] = \mathbb{1}.$$

(3) *There is a Boolean isomorphism* χ *from* $\mathbb{B}$ *onto* $\mathbb{P}(\mathbf{R})$ *such that*

$$\chi(b)x = \chi(b)y \Longleftrightarrow b \leqslant [\![\, x = y \,]\!],$$
$$\chi(b)x \leqslant \chi(b)y \Longleftrightarrow b \leqslant [\![\, x \leqslant y \,]\!]$$
$$(x, y \in \mathbf{R};\ b \in \mathbb{B}).$$

Let $\Lambda \subset \mathbf{R} = \mathscr{R}{\downarrow}$ be the order ideal in $\mathbf{R}$ generated by $1^\wedge$ equipped with the order-unit norm $\|\cdot\|_\infty$:

$$\Lambda := \{x \in \mathbf{R} :\ (\exists C \in \mathbb{B}) - C1^\wedge \leqslant x \leqslant C1^\wedge\};$$
$$\|x\|_\infty := \inf\{C > 0 :\ -C1^\wedge \leqslant x \leqslant C1^\wedge\} \quad (x \in \Lambda).$$

Write $\Lambda = \Lambda(\mathbb{B})$, since Λ is uniquely defined by $\mathbb{B}$. Clearly, Λ is a Dedekind complete AM-space with unit $1^\wedge$. By Kreĭns–Kakutani Representation Theorem $\Lambda \simeq C(K)$ with K being an extremally disconnected compact Hausdorff space.

Remark 4. The version of the Gordon Theorem for complexes is also true: *Each complex universally complete vector lattice is an interpretation of the complexes in an appropriate Boolean-valued model.*

9. Boolean-Valued Banach Lattices

What kind of category is produced by applying the descending procedure to the category of Banach lattices in $\mathbb{V}^{(\mathbb{B})}$? The answer is given in terms of $\mathbb{B}$-cyclicity.

Let $(\mathscr{X}, \|\cdot\|)$ be a Banach lattice in $\mathbb{V}^{(\mathbb{B})}$. Define the map $N : \mathscr{X}{\downarrow} \to \mathbf{R} := \mathscr{R}{\downarrow}$ as the descent $N(\cdot) := (\|\cdot\|){\downarrow}$ of the norm $\|\cdot\|$. Then N is an $\mathbf{R}$-valued norm.

Definition 6. The *bounded descent* $\mathscr{X}{\Downarrow}$ of $\mathscr{X}$ is defined as the set

$$\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} :\ N(x) \in \Lambda\}$$

equipped with the descended operations and considered the mixed norm:

$$|\!|\!| x |\!|\!| := \|N(x)\|_\infty \ \ (x \in \mathscr{X}{\Downarrow}).$$

Proposition. $(\mathscr{X}{\Downarrow}, |\!|\!|\cdot|\!|\!|)$ *is a Banach lattice for any internal Banach lattice* $(\mathscr{X}, \|\cdot\|)$.

Definition 7. Say that X is a Banach lattice *with a Boolean algebra of band projections* $\mathbb{B}$ if there is a Boolean isomorphism $\varphi : \mathbb{B} \to \mathbb{P}(X)$ and $\varphi(\mathbb{B})$ is a complete subalgebra in $\mathbb{P}(X)$. In this event $\mathbb{B}$ is identified with $\varphi(\mathbb{B})$ and one write $\mathbb{B} \subset \mathbb{P}(X)$. Let $\mathbb{B} \subset \mathbb{P}(X)$ and $\mathbb{B} \subset \mathbb{P}(Y)$. A *lattice* $\mathbb{B}$-*isometry* is a lattice isometry $T : X \to Y$ with an additional property $b \circ T = T \circ b$ for all $b \in \mathbb{B}$.

Definition 8. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. The set of all partitions of unity in $\mathbb{B}$ is denoted by $\mathrm{Prt}(\mathbb{B})$. Let $(b_\xi)_{\xi\in\Xi} \in \mathbb{B}$ and $(x_\xi)_{\xi\in\Xi} \subset X$. The element $x \in X$ is called a *mixture* of (x_ξ) by (b_ξ) and is denoted by $x := \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi)$, whenever $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$.

Definition 9. A Banach lattice X is said to be $\mathbb{B}$-*cyclic* if $\mathbb{B} \subset \mathbb{P}(X)$ and the closed unit ball B_X of X is mix-complete, i. e., has the property:

$$(x_\xi) \subset B_X,\ (b_\xi) \in \operatorname{Prt}(\mathbb{B}) \implies \exists\ \operatorname{mix}_\xi(b_\xi x_\xi) \in B_X.$$

Theorem 7. *A bounded descent $\mathscr{X}{\Downarrow}$ of a Banach lattice $\mathscr{X}$ from the model $\mathrm{V}^{(\mathbb{B})}$ is a $\mathbb{B}$-cyclic Banach lattice. Conversely, if X is a $\mathbb{B}$-cyclic Banach lattice, then in the model $\mathrm{V}^{(\mathbb{B})}$ there exists up to lattice isometry a unique Banach lattice $\mathscr{X}$ whose bounded descent $\mathscr{X}{\Downarrow}$ is lattice $\mathbb{B}$-isometric to X. Moreover, $\mathbb{B} \simeq \mathbb{M}(X)$ if and only if $[\![$there is no M-projection in $\mathscr{X}$ other than 0 and $I_{\mathscr{X}}]\!] = \mathbb{1}$, i. e.,*

$$\mathbb{B} \simeq \mathbb{M}(X) \iff [\![\mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}.$$

Definition 10. The internal Banach lattice $\mathscr{X}$ in Theorem 7 is called the *Boolean-valued representation* of X.

Remark 5. It follows from Theorem 7 that the descent of a category of Banach lattices and positive operators inside $\mathrm{V}^{(\mathbb{B})}$ is the category of $\mathbb{B}$-cyclic Banach lattices and positive $\mathbb{B}$-linear operators, see [20]. A detailed presentation of the descent of the category of Banach spaces and bounded linear operators see in [23] and [19].

10. Boolean-Valued AL-Spaces

Theorem 8. *Suppose that X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X} \in \mathrm{V}^{(\mathbb{B})}$ is its Boolean-valued representation. Then the following assertions hold:*

(1) *X is injective $\iff [\![\mathscr{X}$ is injective$]\!] = \mathbb{1}$.*

(2) *X is injective and $\mathbb{B} \simeq \mathbb{M}(X)$*

$$\iff [\![\mathscr{X} \text{ is injective and } \mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}.$$

Theorem 9 (Haydon, [15])**.** *Let X is an injective Banach space. Then*

$$X \text{ is an } AL\text{-space} \iff \mathbb{M}(X) = \{0, I_X\}.$$

Now, putting together Theorems 7, 8, and 9 we arrive at our main representation theorem for injective Banach lattice. For further results see [20, 21, 22].

Theorem 10. *A bounded descent $\mathscr{X}{\Downarrow}$ of an AL-space $\mathscr{X}$ from $\mathrm{V}^{(\mathbb{B})}$ is an injective Banach lattice with $\mathbb{B} \simeq \mathbb{M}(\mathscr{X}{\Downarrow})$. Conversely, if X is an injective Banach lattice and $\mathbb{B} \simeq \mathbb{M}(X)$, then there exist an AL-space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ whose bounded descent is lattice $\mathbb{B}$-isometric to X; in symbols, $X \simeq_{\mathbb{B}} \mathscr{X}{\Downarrow}$.*

Remark 6. Theorem 10 implies the transfer principles from AL-spaces to injective Banach spaces which can be stated as follows:

(1) Every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space.

(2) Each theorem about the AL-space within Zermelo–Fraenkel set theory has its counterpart for the original injective Banach lattice interpreted as a Boolean-valued AL-space.

(3) Translation of theorems from AL-spaces to injective Banach lattices is carried out by appropriate general operations and principles of Boolean-valued analysis.

11. Direct Sums of Injective Banach Lattices

Let $(X_\alpha)_{\alpha\in\mathrm{A}}$ be a family of injective Banach lattices. Neither $\Big(\sum^{\oplus}_{\gamma\in\Gamma} X_\alpha\Big)_{l_\infty}$, nor $\Big(\sum^{\oplus}_{\gamma\in\Gamma} X_\alpha\Big)_{l_1}$ is an injective Banach lattice in general. Nevertheless, one can construct the injective sum $\sum^{\mathrm{ins}}_{\alpha\in\mathrm{A}} X_\alpha$ which is always an injective Banach lattice.

Denote by $\mathrm{Prt}_\sigma := \mathrm{Prt}_\sigma(\mathbb{B})$ and $\mathscr{P}_{\mathrm{fin}}(X)$ the set of all countable partitions of unity in $\mathbb{B}$ and the collection of all finite subsets of X, respectively. Define $\mathbb{B}\langle X_0\rangle$ by

$$\mathbb{B}\langle X_0\rangle := \big\{x \in X : \ x = \mathrm{mix}_\xi(b_\xi x_\xi),\ (x_\xi) \subset X_0,\ (b_\xi) \in \mathrm{Prt}(\mathbb{B})\big\}.$$

The details concerning the following result see in [21].

Theorem 11. *Let $(X_\alpha)_{\alpha\in\mathrm{A}}$ be a family of injective Banach lattices. Assume that there is a complete Boolean algebra $\mathbb{B}$ and a family $(b_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathbb{B}$ with $\bigvee_{\alpha\in\mathrm{A}} b_\alpha = \mathbb{1}$ and $\mathbb{M}(X_\alpha) \simeq \mathbb{B}_\alpha = [\mathbb{O}, b_\alpha]$ for all $\alpha \in \mathrm{A}$. Then there exists a unique up to a lattice $\mathbb{B}$-isometry injective Banach lattice X such that the following hold:*

(1) $\mathbb{B} \simeq \mathbb{M}(X)$.

(2) *For any $\alpha \in \mathrm{A}$ there is a lattice $\mathbb{B}_\alpha$-isometry $\iota_\alpha : X_\alpha \to X$.*

(3) $\big(\iota_\alpha(X_\alpha)\big)_{\alpha\in\mathrm{A}}$ *is a family of pair-wise disjoint bands in X.*

(4) $\mathbb{B}\big\langle\bigoplus_{\alpha\in\mathrm{A}} \iota_\alpha(X_\alpha)\big\rangle$ *is norm dense in X.*

(5) *For any $\mathbf{x} = (x_\alpha)_{\alpha\in\mathrm{A}} \in X$ we have*

$$\|\mathbf{x}\|_{\mathrm{ins}} = \sup_{\theta\in\mathscr{P}_{\mathrm{fin}}(\mathrm{A})} \ \inf_{(\pi_k)\in\mathrm{Prt}_\sigma} \ \sup_{k\in\mathbb{N}} \ \sum_{\alpha\in\theta} \|\pi_k x_\alpha\|.$$

Definition 11. The Banach lattice $(X, \|\cdot\|_{\mathrm{ins}})$ is called the *injective sum* of injective Banach lattices. Denote $\sum^{\mathrm{ins}}_{\alpha\in\mathrm{A}} X_\alpha := X$.

12. Tensor Products of Injective Banach Lattices

If one of the Banach lattices X or Y is an AL-space then the *projective tensor product* $X\hat{\otimes}_\pi Y$ is a Banach lattice. If one of the Banach lattices X or Y is a Dedekind complete AM-spaces with unit then the *injective tensor product* $X\check{\otimes}_\varepsilon Y$ is a Banach lattice. However, in general, $X\hat{\otimes}_\pi Y$ and $X\check{\otimes}_\varepsilon Y$ need not be Banach lattices, see [5, § 9].

In [9] Fremlin introduced for every two Archimedean vector lattices X and Y a new Archimedean vector lattice $X \bar{\otimes} Y$. The *Fremlin projective tensor product* $X\hat{\otimes}_{|\pi|}Y$ of Banach lattices X and Y is the completion of $X \bar{\otimes} Y$ with respect to the *positive projective norm* $\|\cdot\|_{|\pi|}$, see [10].

The *Wittstock injective tensor product* $X\check{\otimes}_{|\varepsilon|}Y$ of Banach lattices X and Y is the completion of $X \otimes Y$ with respect to the *positive injective norm* $\|\cdot\|_{|\varepsilon|}$, [45].

Let X and Y be injective Banach lattices. No one of the four tensor products $X \otimes_\epsilon Y$, $X \otimes_\pi Y$, $X\check{\otimes}_{|\epsilon|}Y$, $X\hat{\otimes}_{|\pi|}Y$ is in general an injective Banach lattice. But there exists a 'mixed' *positive injective-projective tensor product* $X \otimes_{\epsilon|\pi|} Y$ which is always injective. Details concerning the following result can be found in [21].

Theorem 12. *Let X_1 and X_2 be arbitrary injective Banach lattices. Then there exist a unique up to isomorphism injective Banach lattice $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$ and a lattice bimorphism $\bar{\otimes} : X_1 \times X_2 \to X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$ such that the following hold:*

(1) $\bar{\otimes}$ *induces an embedding* ϕ *of the Fremlin tensor product* $X_1 \bar{\otimes} X_2$ *into* $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$.

(2) *There is a Boolean isomorphism* $\jmath$ *from* $\mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$ *onto* $\mathbb{M}(X_1 \hat{\otimes}_{\epsilon|\pi|} X_2)$ *with* $\jmath(\pi_1 \otimes \pi_2)(x_1 \bar{\otimes} x_2) = \pi_1 x_1 \bar{\otimes} \pi_2 x_2$ *for all* $\pi_i \in \mathbb{M}(X_i)$ *and* $x_i \in X_i$ $(i = 1, 2)$.

(3) $\|x_1 \otimes x_2\|_{\epsilon|\pi|} = \|x_1\| \cdot \|x_2\|$ *for all* $x_1 \in X_1$ *and* $x_2 \in X_2$.

(4) $X_1 \bar{\otimes} X_2$ *is* $\mathbb{B}$*-dense in* $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2$ *with* $\mathbb{B} = \mathbb{M}(X_1 \hat{\otimes}_{\epsilon|\pi|} X_2)$.

(5) $X_1 \hat{\otimes}_{\epsilon|\pi|} X_2 = X_0^{\downarrow\uparrow}$, *where* X_0 *comprises all finite sums* $\sum_{k=1}^n \pi_k \phi(u_k)$ *with* $\pi_k \in \mathbb{M}(X_1 \hat{\otimes}_{\epsilon|\pi|} X_2)$ *and* $u_k \in X_1 \bar{\otimes} X_2$ $(k = 1, \dots, n \in \mathbb{N})$.

(6) *For any* $x \in X_1 \bar{\otimes} X_2$ *we have the representation*

$$\|x\|_{\text{inj}} = \inf \left\{ \sup_{k \in \mathbb{N}} \sum_{i=1}^n \|\pi_k u_{i,k}\| \cdot \|\rho_k v_{i,k}\| \right\}$$

where infimum is taken over all $(\pi_k) \in \mathrm{Prt}_\sigma(\mathbb{B}_1)$, $(\rho_k) \in \mathrm{Prt}_\sigma(\mathbb{B}_2)$, *and* $0 \leqslant u_{i,k} \in X_1$, $0 \leqslant v_{i,k} \in X_2$ $(i \leqslant n)$ *with* $|x| \leqslant \sum_{i=1}^n u_{i,k} \otimes v_{i,k}$ $(k \in \mathbb{N})$.

13. Representation of AL-Spaces

For every cardinal γ, there exists a canonical measure on the unit cube $[0,1]^\gamma$ that is the γth power of Lebesgue's measure on $[0,1]$. The associated measure algebra and the corresponding Banach lattice of integrable functions will be denoted by $\mathbb{I}^\gamma$ and $L_1([0,1]^\gamma)$, respectively.

The famous Maharam theorem tells us that the measure algebras are the 'building blocks' for every Maharam algebra ($\equiv$ measure algebra of the mesure space with the direct sum property). More precisely, every atomless nonzero (finite) Maharam algebra is isomorphic to a direct sum of concrete measure algebras $\mathbb{I}^\gamma$, and it is uniquely determined by a family of cardinals, see [11, 321A] and [37, 17.5.3]. Transferring the structure theory of Maharam algebras, yields an important representation theorem for AL-spaces (Theorem 14 below, see [37, 26.4.7].

Definition 12. The *density character* of a subset S of a topological space is the smallest cardinal γ such that S contains a dense subset of cardinality γ.

Definition 13. A Banach lattice X is said to be γ*-homogeneous* if X is non-atomic and whenever $x, y \in X$ with $x \leqslant y$ and $x \neq y$ the density character of the order interval $[x, y]$ is γ.

Theorem 13. *Let* X *be a* γ*-homogeneous* AL*-space. Then there exists a cardinal* δ *such that*

$$X \simeq \delta L_1([0,1]^\gamma),$$

where δY *denotes* l_1*-direct sum of* δ *many copies of* Y.

Every non-atomic Banach lattice can be decomposed into a direct sum of homogeneous Banach lattices and thus the Banach lattices $L_1([0,1]^\gamma)$ are the 'building blocks' for all AL-spaces.

Corollary. *For any nonatomic* AL*-space* X *there exist a family of cardinals* $(\delta_\gamma)_{\gamma\in\Gamma}$ *with* Γ *being a set of cardinals such that the lattice isometry holds:*

$$X \simeq \Big(\sum\nolimits_{\gamma\in\Gamma}^{\oplus} \delta_\gamma L_1\big([0,1]^\gamma\big) \Big)_{l_1}.$$

Theorem 14. *Let X be an AL-space. Then there exists a unique well-ordered family $(\mathfrak{m}_\sigma)_{0\leqslant\sigma<\tau}$ of cardinals with τ an ordinal such that $\{\sigma : \mathfrak{m}_\sigma \neq 0\}$ is cofinal in τ, each $\mathfrak{m}_\sigma$ is either equal to 0, or 1, or is uncountable, and*

$$X \simeq l_1(\gamma) \oplus \sum_{0\leqslant\sigma<\tau}^{\oplus} \mathfrak{m}_\sigma L_1\big([0,1]^{\omega_\sigma}\big),$$

where $\simeq$ denotes lattice isometry, $\oplus$ and $\sum^\oplus$ denote l_1-joins, and $\mathfrak{m}Y$ stands for the l_1-join of $\mathfrak{m}$ copies of Y.

14. Representation of Injective Banach Lattices

Let Λ_γ be a Dedekind complete AM-space with unit and L_γ be an AL-space. Then $M_\gamma \hat{\otimes}_{\epsilon|\pi|} L_\gamma$ is an injective Banach lattice by Theorem 12. Moreover, in view of Theorem 11, $\sum_{\gamma\in\Gamma}^{\mathrm{ins}} M_\gamma \hat{\otimes}_{\epsilon|\pi|} L_\gamma$ is also an injective Banach lattice. Actually, every injective Banach lattice have a similar representation, so that Dedekind complete AM-spaces with unit and AL-spaces are the *'building blocks'* for any injective Banach lattice. This follows from Theorems 10 and 13.

Definition 14. A subset $X_0 \subset X$ is said to be $\mathbb{B}$-*dense* in X whenever $\mathbb{B}\langle X_0\rangle$ is norm dense in X.

Definition 15. The $\mathbb{B}$-*density character* of a subset S of a $\mathbb{B}$-cyclic Banach lattice is the smallest cardinal γ such that S contains a $\mathbb{B}$-dense subset of cardinality γ.

Definition 16. A $\mathbb{B}$-cyclic Banach lattice X is said to be $(\mathbb{B},\gamma)$-*homogeneous* if X has no $\mathbb{B}$-atom (see [20, Definition 8.1]) and whenever $x, y \in X$ with $x \leqslant y$ and $x \neq y$ the $\mathbb{B}$-density character of the order interval $[x, y]$ is γ, while X is $\mathbb{B}$-homogeneous, whenever X is $(\mathbb{B},\gamma)$-homogeneous for some γ.

Theorem 15. *Let X be a $\mathbb{B}$-homogeneous injective Banach lattice with $\mathbb{B} = \mathbb{M}(X)$. Then there are the sets of cardinals Γ and Δ a partition of unity $(\pi_{\gamma\delta})_{(\gamma,\delta)\in\Gamma\times\Delta}$ in $\mathbb{B}$ such that*

$$X \simeq_{\mathbb{B}} \sum_{(\gamma,\delta)\in\Gamma\times\Delta}^{\mathrm{ins}} \Lambda_{\gamma\delta} \hat{\otimes}_{\epsilon|\pi|} \delta L_1([0,1]^\gamma).$$

Remark 7. Neither the decomposition in Theorem 13 nor a decomposition of an injective Banach lattice into $\mathbb{B}$-homogeneous bands is unique in general, cf. [19, Ch. 8]. The reason for this is the so-called *cardinal collapsing* phenomena: it is possible for two infinite cardinals $\kappa < \lambda$ to satisfy $\mathrm{V}^{(\mathbb{B})} \models |\kappa^\wedge| = |\lambda^\wedge|$. In this event we say that λ has been collapsed to κ in $\mathrm{V}^{(\mathbb{B})}$, see [4, Ch. 5].

Remark 8. It should be emphasized that, according to Theorem 10, our representation theorem 15 is just an interpretation of Theorem 13 in the Boolean-valued model $\mathrm{V}^{(\mathbb{B})}$ with $\mathbb{B} \simeq \mathbb{M}(X)$. This should be compared with the representation of Kaplansky–Hilbert modules and AW^*-algebras of [31, 32, 33], see also [19].

Corollary. *If $\mathbb{B}$ is associated with the measure algebra (Ω,Σ,μ) then there exists a family $(\Omega_{\gamma\delta})_{(\gamma,\delta)\in\Gamma\times\Delta}$ of pair-wise disjoint measurable sets $\Omega_{\gamma\delta} \subset \Omega$ such that $\mu(\Omega_{\gamma\delta}) > 0$ for all γ and δ, $\Omega = \bigcup_{\gamma\delta} \Omega_{\gamma\delta}$, and*

$$X \simeq_{\mathbb{B}} \sum_{(\gamma,\delta)\in\Gamma\times\Delta}^{\mathrm{ins}} L_\infty\big(\Omega_{\gamma\delta},\, \delta L_1([0,1]^\gamma)\big).$$

Remark 9. The concept of $\mathbb{B}$-atomic Banach lattice was introduced in [21]. It was proved that if X is a $\mathbb{B}$-atomic injective Banach lattice with $\mathbb{B} = \mathbb{M}(X)$, then there is a partition of unity $(\pi_\gamma)_{\gamma\in\Gamma}$, with Γ being a set of cardinals, such the following lattice $\mathbb{B}$-isometry holds:

$$X \simeq_{\mathbb{B}} \Big(\sum\nolimits^{\oplus}_{\gamma\in\Gamma} \Lambda_\gamma \hat{\otimes}_{\varepsilon|\pi|} l_1(\gamma)\Big)_{l_\infty},$$

where $\Lambda_\gamma := \pi_\gamma\Lambda$ and $\Lambda = \Lambda(\mathbb{B})$.

Now, every injective Banach lattice is decomposable into an injective sum of a $\mathbb{B}$-atomic band and a family of $\mathbb{B}_\alpha$-homogeneous bands. Therefore Theorem 15 and [21, Theorem 8.11] enables us to obtain a complete description of an arbitrary injective Banach lattice. To do this we have to interpret Theorem 14 in $\mathrm{V}^{(\mathbb{B})}$ making use of Theorem 10.

15. Open Problems

A real Banach lattice X is said to be λ-*injective*, if for every Banach lattice Y, closed sublattice $Y_0 \subset Y$, and positive $T_0 : Y_0 \to X$ there exists a positive extension $T : Y \to X$ with $\|T\| \leqslant \lambda\|T_0\|$. It was proved in [26] that every finite-dimensional λ-injective Banach lattice is lattice isomorphic to $\big(\sum^{\oplus}_{j\leqslant k} l_1(n_j)\big)_{l_\infty}$, while it was shown in [29] that every order continuous λ-injective Banach lattice is lattice isomorphic to $L_1(\mu)$ space. But the general question, as far as I know, is still open:

Problem 1: *Is every λ-injective Banach lattice order isomorphic to 1-injective Banach lattice?*

One of the intriguing problems, dating from the work [14], is the classification of the Banach space whose duals are isometric to an *AL*-space, see also [27]. I believe that the injective version of this problem deserves an independent study.

Problem 2: *Classify and characterize the Banach spaces whose duals are injective Banach lattices.*

As is seen from Theorem 10 an injective Banach lattice X has a mixed *LM*-structure. Thus, the dual X' should have, in a sense, an *ML*-structure. Hence a natural question arises:

Problem 3: *What kind of duality theory is there for injective Banach lattices?*

Every Banach space has an injective envelope, see [8, 18]. Following [8] we can give the definition: An *injective envelope* of a Banach lattice X is a pair $({}^{\mathfrak{e}}X, \iota)$ with ${}^{\mathfrak{e}}X$ an injective Banach lattice and $\iota : X \to {}^{\mathfrak{e}}X$ a lattice isometry such that the only sublattice of ${}^{\mathfrak{e}}X$ that is injective and contains $\iota(X)$ is ${}^{\mathfrak{e}}X$ itself, cf. [8].

Problem 4: *Does every Banach lattice have an injective envelope?*

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