УДК 517.98 Дата поступления 14 августа 2008 г.

Кусраев А. Г. Однородное функциональное исчисление в векторных решетках.—Владикавказ, 2008.—34 с.—(Препринт / ИПМИ ВНЦ РАН; № 1).

В работе представлен расширенный вариант однородного функционального исчисления в векторных решетках. Показано, что естественным образом определяется положительно однородная функция от элементов равномерно полной векторной решетки, если эта функция определена на коническом множестве конечномерного пространства и непрерывна на некотором подконусе последнего. Рассмотрена взаимосвязь однородного функционального исчисления и двойственности Минковского. На этой основе в векторных решетках развит метод квазилинеаризации для доказательства неравенств выпуклости и установлены общие формы некоторых классических неравенств. Построенная техника применяется к вычислению однородных функций от регулярных линейных и билинейных операторов в векторных решетках и получены некоторые полезные оценки для них. Рассмотрены также положительно однородные функции от элементов векторных решеток непрерывных и измеримых сечений непрерывных и измеримых расслоений банаховых решеток.

Библиогр. 36.

Received August 14, 2008

Kusraev A. G. Homogeneous Functional Calculus on Vector Lattices.—Vladikavkaz, 2008.—34 p.—(Preprint / IAMI VSC RAS; № 1).

The paper extends homogeneous functional calculus on vector lattices. It is shown that the function of elements of a relatively uniformly complete vector lattice can naturally be defined if the positively homogeneous function is defined on some conic set and is continuous on some subcone. An interplay between Minkowski duality and homogeneous functional calculus is considered. The quasilinearization method for proving convexity inequalities in vector lattices is develop and general forms of some classical and new inequalities are proved. This machinery is applied to compute and estimate homogeneous functions of linear and bilinear regular operators on vector lattices. Homogeneous functions on vector lattices of continuous or measurable sections of bundles of Banach lattices is also considered.

Mathematics Subject Classification (2000): 46A40, 47A50, 47A60, 47A63, 47B65.

Keywords: vector lattice, functional calculus, Minkowski duality, sublinear and superlinear operators, dominated operators, quasilinearization, convexity inequalities, measurable Banach bundle.

Institute of Applied Mathematics and Informatics
Vladikavkaz Science Center of the RAS
Vladikavkaz, 362027, RUSSIA
E-mail: kusraev@smath.ru

HOMOGENEOUS FUNCTIONAL CALCULUS ON VECTOR LATTICES

A. G. Kusraev

1. Introduction

For any finite sequence $(x_1, \dots, x_N)$ $(N \in \mathbb{N})$ in a relatively uniformly complete vector lattice the expression of the form $\varphi(x_1, \dots, x_N)$ can be correctly defined provided that φ is a positively homogeneous continuous function on $\mathbb{R}^N$. The study of such expressions, called *homogeneous functional calculus*, provides a useful tool in a variety of areas, see [9, 18, 19, 26, 27, 28, 36]. At the same time it is of importance in certain problems to deal with $\varphi(x_1, \dots, x_N)$ even if φ is defined on a conic subset of $\mathbb{R}^N$ [5, 28, 29]. The aim of this paper is to extend homogeneous functional calculus and consider an interplay between Minkowski duality and functional calculus on vector lattices as well as to develop the quasilinearization method for proving convexity inequalities in vector lattices.

In Section 2 the extended homogeneous functional calculus is defined. It is shown that the expression $\varphi(x_1, \dots, x_N)$ can naturally be defined in any relatively uniformly complete vector lattice if a positively homogeneous function φ is defined on some conic set $\mathrm{dom}(\varphi) \subset \mathbb{R}^N$ and is continuous on some subcone of $\mathrm{dom}(\varphi)$. Section 3 contains some examples of computing $\widehat{\varphi}(u_1, \dots, u_N)$ whenever $u_1, \dots, u_N$ are continuous or measurable vector-valued functions or φ is a Kobb–Duglas type function. In Section 4 Minkowski duality is transplanted to vector lattice by means of extended functional calculus. In Section 5, using this machinery, the quasilinearization method for proving inequalities is developed in vector lattice setting and the general forms of some classical inequalities (Jensen, Holder, Minkowski) are also given. In Section 6 a Maligranda type inequality for positive bilinear operators on uniformly complete vector latices is deduced. In Sections 7 and 8 formulas for computing $\varphi(T_1, \dots, T_N)$ for linear and bilinear regular operators $T_1, \dots, T_N$ are derived and some operator inequalities are proved. Section 9 deals with homogeneous functions on vector lattice of continuous and measurable sections. Section 10 contains further examples.

There are different ways to define homogeneous functional calculus on vector lattices [6, 18, 26, 30]. We follow the approach of G. Buskes, B. de Pagter, and A. van Rooij [6] going back to G. Ya. Lozanovskiĭ [30]. Theorem 1.1 below see in [6, 19, 26, 36].

For the theory of vector lattices and positive operators we refer to the books [1] and [19]. All vector lattices in this paper are real and Archimedean.

Denote by $\mathscr{H}(\mathbb{R}^N)$ the vector lattice of all continuous functions $\varphi : \mathbb{R}^N \to \mathbb{R}$ which are *positively homogeneous* ($\equiv \varphi(\lambda t) = \lambda\varphi(t)$ for $\lambda \geq 0$ and $t \in \mathbb{R}^N$). Let dt_k stands for the kth coordinate function on $\mathbb{R}^N$, i.e. $dt_k : (t_1, \dots, t_N) \mapsto t_k$.

1.1.Theorem. *Let E be a relatively uniformly complete vector lattice. For any $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$ there exists a unique lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \dots, x_N) \quad \big(\varphi \in \mathscr{H}(\mathbb{R}^N)\big)$$

of $\mathscr{H}(\mathbb{R}^N)$ into E with $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1, \dots, N)$.

If the vector lattice E is *universally σ-complete* ($\equiv$ Dedekind σ-complete and laterally σ-complete) and has an order unit, then Borel functional calculus is also available on E. Let $\mathscr{B}(\mathbb{R}^N)$ denotes the vector lattice of all Borel measurable functions $\varphi : \mathbb{R}^N \to \mathbb{R}$. The following result can be found in [19, Theorem 8.2.14].

1.2. Theorem. *Let E be a universally σ-complete vector lattice with a fixed weak order unit $\mathbb{1}$. For any $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$ there exists a unique sequentially order continuous lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \dots, x_N) \quad \big(\varphi \in \mathscr{B}(\mathbb{R}^N)\big)$$

of $\mathscr{B}(\mathbb{R}^N)$ into E such that $\widehat{\mathfrak{x}}(1_{\mathbb{R}^N}) = \mathbb{1}$ and $\widehat{\mathfrak{x}}(dt_k) = x_k$ ($k := 1, \dots, N$).

Let $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ denote the vector sublattice of $\mathscr{B}(\mathbb{R}^N)$ consisting of all positively homogeneous Borel functions $\varphi : \mathbb{R}^N \to \mathbb{R}$.

1.3. Theorem. *Let E be a universally σ-complete vector lattice and $\widehat{\mathfrak{x}} := (x_1, \dots, x_N) \in E^N$. Then there exists a unique sequentially order continuous lattice homomorphism*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N) \quad \big(\varphi \in \mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)\big)$$

of $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ into E such that $\widehat{\mathfrak{x}}(dt_k) = x_k$ ($k := 1, \dots, N$).

$\lhd$ Put $\mathbb{1} := |x_1| + \dots + |x_N|$ and denote by E_0 the band in E generated by $\mathbb{1}$. Then E_0 is a universally σ-complete vector lattice with order unit $\mathbb{1}$ and one can take $\widehat{\mathfrak{x}} : \mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N) \to E_0$ as in Theorem 1.2. Since $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ is an order σ-closed vector sublattice of $\mathscr{B}(\mathbb{R}^N)$, the restriction of $\widehat{\mathfrak{x}}$ onto $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ is also an order σ-continuous lattice homomorphism. If $h : \mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N) \to E$ is another order σ-continuous lattice homomorphism with $h(dt_k) = \widehat{\mathfrak{x}}(dt_k)$ ($k := 1, \dots, N$), then h and $\widehat{\mathfrak{x}}(\cdot)$ coincide on $\mathscr{H}(\mathbb{R}^N)$ by Theorem 1.1. Afterwards, we infer that h and $\widehat{\mathfrak{x}}(\cdot)$ coincide on the whole $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ due to order σ-continuity. $\rhd$

2. Functional Calculus

In this section we define extended homogeneous functional calculus on relatively uniformly complete vector lattices. Everywhere below $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$.

2.1. Consider a finite collection $x_1, \dots, x_N \in E$ and a vector sublattice $L \subset E$. Denote by $\langle x_1, \dots, x_N \rangle$ and $\mathrm{Hom}(L)$ respectively the vector sublattice of E generated by $\{x_1, \dots, x_N\}$ and the set of all $\mathbb{R}$-valued lattice homomorphisms on L. Put

$$[\mathfrak{x}] := [x_1, \dots, x_N] := \{(\omega(x_1), \dots, \omega(x_N)) \in \mathbb{R}^N : \ \omega \in \mathrm{Hom}(\langle x_1, \dots, x_N \rangle)\}.$$

Let $e := |x_1| + \ldots + |x_N|$ and $\Omega := \{\omega \in \mathrm{Hom}(\langle x_1, \dots, x_N \rangle) : \ \omega(e) = 1\}$. Then e is a strong order unit in $\langle x_1, \dots, x_N \rangle$ and Ω separates the points of $\langle x_1, \dots, x_N \rangle$. Moreover, Ω may be endowed with a compact Hausdorff topology so that the functions $\widehat{x}_k : \Omega \to \mathbb{R}$ defined by $\widehat{x}_k(\omega) := \omega(x_k)$ ($k := 1, \dots, N$) are continuous and $x \mapsto \widehat{x}$ is a lattice isomorphism of $\langle x_1, \dots, x_N \rangle$ into $C(\Omega)$. Put

$$\Omega(x_1, \dots, x_N) := \{(\omega(x_1), \dots, \omega(x_N)) \in \mathbb{R}^N : \ \omega \in \Omega\},$$

and observe that $[x_1, \dots, x_N] := \mathrm{cone}(\Omega(x_1, \dots, x_N))$, where $\mathrm{cone}(A)$ is the conic hull of A defined as $\bigcup\{\lambda A : \ 0 \le \lambda \in \mathbb{R}\}$. Evidently, $\Omega(x_1, \dots, x_N)$ is a compact subset of $\mathbb{R}^N$, since it is the image of the compact set Ω under the continuous map

$\omega \mapsto (\widehat{x}_1(\omega), \dots, \widehat{x}_N(\omega))$. Therefore, $[x_1, \dots, x_N]$ is a *compactly generated* conic set in $\mathbb{R}^N$. (The conic set $[x_1, \dots, x_N]$ is closed if $0 \notin \Omega(x_1, \dots, x_N)$.) A set $C \subset \mathbb{R}^N$ is called *conic* if $\lambda C \subset C$ for all $\lambda \geq 0$ while a convex conic set is referred to as a *cone*. The reasoning similar to [6, Lemma 3.3] shows that $[x_1, \dots, x_N]$ is uniquely determined by any point separating subset Ω_0 of $\mathrm{Hom}(\langle x_1, \dots, x_N \rangle)$. Indeed, if $\Omega_0' := \{\omega(e)^{-1}\omega : 0 \neq \omega \in \Omega_0\}$, then Ω_0' is a dense subset of Ω and $[x_1, \dots, x_N] = \mathrm{cone}\big(\mathrm{cl}(\Omega_0'(x_1, \dots, x_N))\big)$, where $\Omega_0'(x_1, \dots, x_N)$ is the set of all $(\omega(x_1), \dots, \omega(x_N)) \in \mathbb{R}$ with $\omega \in \Omega_0'$.

2.2. For a conic set C in $\mathbb{R}^N$ denote by $\widehat{C} \subset E^N$ the set of all $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$ with $[\,\mathfrak{x}\,] \subset C$. Consider a conic set $K \subset C$. Let $\mathscr{H}(C; K)$ denotes the vector lattice of all positively homogeneous functions $\varphi : C \to \mathbb{R}$ with continuous restriction to K. Fix $(x_1, \dots, x_N) \in \widehat{C}$ and take $\varphi \in \mathscr{H}(C; [\,\mathfrak{x}\,])$. We say that $\widehat{\varphi}(x_1, \dots, x_N)$ *exists* or *is well defined* in E and write $y = \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N)$ if there is an element $y \in E$ such that $\omega(y) = \varphi(\omega(x_1), \dots, \omega(x_N))$ for every $\omega \in \mathrm{Hom}(\langle x_1, \dots, x_N, y \rangle)$. This definition is correct, since for any given $(x_1, \dots, x_N) \in \widehat{C}$ and $\varphi \in \mathscr{H}(C; [\,\mathfrak{x}\,])$ there exists at most one $y \in E$ such that $y = \widehat{\varphi}(x_1, \dots, x_N)$. It is immediate from the definition that $\widehat{\varphi}(\lambda_1 x, \dots, \lambda_N x)$ is well defined for any $(\lambda_1, \dots, \lambda_N) \in C$ and $\widehat{\varphi}(\lambda_1 x, \dots, \lambda_N x) = \widehat{\varphi}(\lambda_1, \dots, \lambda_N)x$ whenever $0 \leq x \in E$. The following proposition can be proved as [6, Lemma 3.3].

Assume that L is a vector sublattice of E containing $\{x_1, \dots, x_N, y\}$ and $\varphi \in \mathscr{H}(C; [x_1, \dots, x_N])$. If $\omega(y) = \varphi(\omega(x_1), \dots, \omega(x_N))$ $(\omega \in \Omega_0)$ for some point separating set Ω_0 of $\mathbb{R}$-valued lattice homomorphisms on L, then $y = \widehat{\varphi}(x_1, \dots, x_N)$.

2.3. Theorem. *Let E be a relatively uniformly complete vector lattice and $\mathfrak{x} \in E^N$, $\mathfrak{x} = (x_1, \dots, x_N)$. Assume that $C \subset \mathbb{R}^N$ is a conic set and $[\,\mathfrak{x}\,] \subset C$. Then $\widehat{\mathfrak{x}}(\varphi) := \widehat{\varphi}(x_1, \dots, x_N)$ exists for every $\varphi \in \mathscr{H}(C; [\,\mathfrak{x}\,])$ and the mapping*

$$\widehat{\mathfrak{x}} : \varphi \mapsto \widehat{\mathfrak{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N) \quad \big(\varphi \in \mathscr{H}(C; [\,\mathfrak{x}\,])\big)$$

is a unique lattice homomorphism from $\mathscr{H}(C; [\,\mathfrak{x}\,])$ into E with $\widehat{dt_j}(x_1, \dots, x_N) = x_j$ for $j := 1, \dots, N$.

◁ Let $\mathscr{H}([\,\mathfrak{x}\,])$ denotes the vector lattice of all positively homogeneous continuous functions defined on $[\,\mathfrak{x}\,]$. Then $\mathscr{H}([\,\mathfrak{x}\,])$ is isomorphic to $C(Q)$, where $Q := [\,\mathfrak{x}\,] \cap \mathbb{S}$ and $\mathbb{S} := \{s \in \mathbb{R}^N : \|s\| := \max\{|s_1|, \dots, |s_N|\} = 1\}$. Much the same reasoning as in [6, Proposition 3.6, Theorem 3.7] shows the existence of a unique lattice homomorphism h from $\mathscr{H}([\,\mathfrak{x}\,])$ into E such that $\widehat{dt_j}(x_1, \dots, x_N) = x_j$ $(j := 1, \dots, N)$. Denote by ρ the restriction operator $\varphi \mapsto \varphi|_{[\mathfrak{x}]}$ $\big(\varphi \in \mathscr{H}(C; [\,\mathfrak{x}\,])\big)$. Then $\rho \circ h$ is the required lattice homomorphism. ▷

Observe that if $\varphi, \psi \in \mathscr{H}(C; [\,\mathfrak{x}\,])$ and $\varphi(t) \leq \psi(t)$ for all $t \in [\,\mathfrak{x}\,]$, then $\widehat{\varphi}(x_1, \dots, x_N) \leq \widehat{\psi}(x_1, \dots, x_N)$. Evidently, $|\varphi(t)| \leq |||\varphi||| \cdot \|t\|$ for all $t \in [\,\mathfrak{x}\,]$ with $|||\varphi||| := \sup\{\varphi(t) : t \in Q\}$ and hence

$$|\widehat{\varphi}(x_1, \dots, x_N)| \leq |||\varphi||| \, (|x_1| \vee \dots \vee |x_N|)\,.$$

In particular, the kernel $\ker(\widehat{\mathfrak{x}})$ of $\widehat{\mathfrak{x}}$ consists of all $\varphi \in \mathscr{H}(C; [\,\mathfrak{x}\,])$ vanishing on $[\,\mathfrak{x}\,]$.

2.4. Let $K, M, N \in \mathbb{N}$ and consider two conic sets $C \subset \mathbb{R}^N$ and $D \subset \mathbb{R}^M$. Let $x_1, \dots, x_N \in E$, $\mathfrak{x} := (x_1, \dots, x_N)$, $[\mathfrak{x}] \subset C$, $\varphi_1, \dots, \varphi_M \in \mathscr{H}(C; [\,\mathfrak{x}\,])$, and denote $\varphi := (\varphi_1, \dots, \varphi_M)$ and $\mathfrak{y} := (y_1, \dots, y_N)$ with $y_k = \widehat{\varphi}_k(x_1, \dots, x_N)$ $(k := 1, \dots, M)$. Suppose that $[\,\mathfrak{y}\,] \subset D$, $\varphi(C) \subset D$, and $\varphi([\,\mathfrak{x}\,]) \subset [\,\mathfrak{y}\,]$. If $\psi := (\psi_1, \dots, \psi_K)$

with $\psi_1,\dots,\psi_K\in\mathscr{H}(D;[\mathfrak{y}])$, then $\psi_1\circ\varphi,\dots,\psi_K\circ\varphi\in\mathscr{H}(C;[\mathfrak{x}])$. Moreover, $\widehat{\varphi}(\mathfrak{x}):=(\widehat{\varphi}_1(\mathfrak{x}),\dots,\widehat{\varphi}_M(\mathfrak{x}))\in E^M$, $\widehat{\psi}(\mathfrak{y}):=(\widehat{\psi}_1(\mathfrak{y}),\dots,\widehat{\psi}_K(\mathfrak{y}))\in E^K$, and $\widehat{\psi\circ\varphi}(\mathfrak{x}):=(\widehat{\psi_1\circ\varphi}(\mathfrak{x}),\dots,\widehat{\psi_K\circ\varphi}(\mathfrak{x}))\in E^K$ are well defined and

$$(\widehat{\psi\circ\varphi})(\mathfrak{x})=\widehat{\psi}(\widehat{\varphi}(\mathfrak{x})).$$

2.5. Theorem. *Let C and K are conic sets in $\mathbb{R}^N$ with K closed and $K\subset C$ and let $\varphi\in\mathscr{H}(C;K)$. Then for every $\varepsilon>0$ there exists a number $R_\varepsilon>0$ such that*

$$|\widehat{\varphi}(\mathfrak{x}+\mathfrak{y})-\widehat{\varphi}(\mathfrak{x})|\le\varepsilon|||\mathfrak{x}|||+R_\varepsilon|||\mathfrak{y}|||$$

for any finite collections $\mathfrak{x}=(x_1,\dots,x_N)\in E^N$ and $\mathfrak{y}=(y_1,\dots,y_N)\in E^N$, provided that $\mathfrak{x},\mathfrak{y}\in\widehat{K}$, $\mathfrak{x}+\mathfrak{y}\in\widehat{K}$ and $|||(u_1,\dots,u_N)|||$ stands for $|u_1|\vee\dots\vee|u_N|$.

$\lhd$ The proof is a duly modification of arguments from [9, Theorem 7]. Denote $K^\times:=\{(s,t)\in K\times K:\ s+t\in K\}$ and define A as the set of all $(s,t)\in K^\times$ with $\max\{\|s\|,\|t\|\}=1$ and $\tau(s,t):=|\varphi(s+t)-\varphi(s)|\ge\varepsilon\|s\|$, where $\|s\|:=\max\{|s_1|,\dots,|s_N|\}$. Then A is a compact subset of $K\times K$ and $(s,t)\mapsto(\tau(s,t)-\varepsilon\|s\|)/\|t\|$ is a continuous function on A, since $\|t\|\ne0$ for $(s,t)\in A$. Therefore,

$$R_\varepsilon:=\sup\left\{\frac{\tau(s,t)-\varepsilon\|s\|}{\|t\|}:\ (s,t)\in A\right\}<\infty.$$

Hence $\tau(s,t)\le\varepsilon\|s\|+R_\varepsilon\|t\|=:\sigma(s,t)$ for all $(s,t)\in K^\times$. Evidently, $\tau\in\mathscr{H}(C^\times,K^\times)$, $\sigma\in\mathscr{H}(\mathbb{R}^N\times\mathbb{R}^N)$, and $\tau\le\sigma$ on $K^\times$. It remains to observe that $(\mathfrak{x},\mathfrak{y})\in\widehat{K^\times}$ and apply 2.3 and the desired inequality follows. $\rhd$

2.6. Proposition. *Let E and F be uniformly complete vector lattices, E_0 a uniformly closed sublattice of E, and $h:E_0\to F$ a lattice homomorphism. Let C be a conic set in $\mathbb{R}^N$, $x_1,\dots,x_N\in E_0$, and $\varphi\in\mathscr{H}(C;[x_1,\dots,x_N])$. Then $[h(x_1),\dots,h(x_N)]\subset[x_1,\dots,x_N]$ and*

$$h(\widehat{\varphi}(x_1,\dots,x_N))=\widehat{\varphi}(h(x_1),\dots,h(x_N)).$$

In particular, if h is the inclusion map $E\hookrightarrow F$ and $x_1,\dots,x_N\in E$, then the element $\widehat{\varphi}(x_1,\dots,x_N)$ relative to F is contained in E and its meaning relative to E is the same.

$\lhd$ Put $y_i:=h(x_i)$ $(i:=1,\dots,N)$. If $\omega\in\mathrm{Hom}(\langle y_1,\dots,y_N\rangle)$, then $\bar\omega:=\omega\circ h$ belongs to $\mathrm{Hom}(\langle x_1,\dots,x_N\rangle)$ and $(\omega(y_1),\dots,\omega(y_N))=(\bar\omega(x_1),\dots,\bar\omega(x_N))\in[x_1,\dots,x_N]$. Therefore, $[y_1,\dots,y_N]$ is contained in $[x_1,\dots,x_N]$. Now, if $y=\widehat{\varphi}(y_1,\dots,y_N)$, $x=\widehat{\varphi}(x_1,\dots,x_N)$, and $\omega\in\mathrm{Hom}(\langle y,y_1,\dots,y_N\rangle)$, then $\bar\omega\in\mathrm{Hom}(\langle x,x_1,\dots,x_N\rangle)$ and by definition

$$\omega(y)=\varphi(\bar\omega(x_1),\dots,\bar\omega(x_N))=\bar\omega(\widehat{\varphi}(x_1,\dots,x_N)=\omega(h(x)),$$

so that $y=h(x)$. $\rhd$

Denote $\mathscr{H}^\infty_{\mathrm{Bor}}(\mathbb{R}^N,[\mathfrak{x}]):=\big\{\varphi\in\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N):\ \sup\{|\varphi(s)|:\ s\in\mathbb{S}\cap[\mathfrak{x}]\}<+\infty\big\}$.

2.7. Theorem. *Let E be a Dedekind σ-complete vector lattice. For $\widehat{\mathfrak{x}}:=(x_1,\dots,x_n)$ in E^N there exists a unique sequentially order continuous lattice homomorphism*

$$\widehat{\mathfrak{x}}:\varphi\mapsto\widehat{\mathfrak{x}}(\varphi)=\widehat{\varphi}(x_1,\dots,x_N)\quad\big(\varphi\in\mathscr{H}^\infty_{\mathrm{Bor}}(\mathbb{R}^N,[\mathfrak{x}])\big)$$

of $\mathscr{H}^\infty_{\mathrm{Bor}}(\mathbb{R}^N,[\mathfrak{x}])$ into E such that $\widehat{\mathfrak{x}}(dt_k)=x_k$ $(k:=1,\dots,N)$.

$\lhd$ Let E_0 be the order ideal in E generated by $x_1, \dots, x_N$. According to 1.3 there exists a unique sequentially order continuous lattice homomorphism $\widehat{\mathfrak{x}}$ of $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ into $(E_0)^{u\sigma}$, a universal σ-completion of E_0, with $\widehat{\mathfrak{x}}(dt_k) = x_k$ $(k := 1, \dots, N)$. Clearly, the image of $\mathscr{H}^{\infty}_{\mathrm{Bor}}(\mathbb{R}^N, [\,\mathfrak{x}\,])$ under $\widehat{\mathfrak{x}}$ is contained in E_0. $\rhd$

3. Examples

Now, we consider extended functional calculus on some special vector lattices E and for some special functions φ. Everywhere in the section $\varphi \in \mathscr{H}(C; K)$.

3.1. Proposition. *Let Q be a Hausdorff topological space, X a Banach lattice, and $C_b(Q, X)$ the Banach lattice of norm bounded continuous functions from Q to X. Assume that $u_1, \dots, u_N \in C_b(Q, X)$ and $[u_1, \dots, u_N] \subset K$. Then $[u_1(q), \dots, u_N(q)] \subset K$ for all $q \in Q$ and*

$$\widehat{\varphi}(u_1, \dots, u_N)(q) = \widehat{\varphi}(u_1(q), \dots, u_N(q)) \quad (q \in Q).$$

$\lhd$ Indeed, for $q \in Q$ the map $\widehat{q} : C_b(Q, X) \to X$ defined by $\widehat{q} : u \mapsto u(q)$ is a lattice homomorphism. Therefore, given $u_1, \dots, u_N \in C_b(Q, X)$, by Proposition 2.6 we have $[\widehat{q}(u_1), \dots, \widehat{q}(u_N)] \subset [u_1, \dots, u_N]$ and $\widehat{q}(\widehat{\varphi}(u_1, \dots, u_N)) = \widehat{\varphi}(\widehat{q}(u_1), \dots, \widehat{q}(u_N))$ from which the required is immediate. $\rhd$

3.2. Suppose now that Q is compact and extremally disconnected. Let $u : D \to X$ be a continuous function defined on a dense subset $D \subset Q$. Denote by $\bar{D}$ the totality of all points in Q at which u has limit and put $\bar{u}(q) := \lim_{p \to q} u(p)$ for all $q \in \bar{D}$. Then the set $\bar{D}$ is comeager in Q and the function $\bar{u} : \bar{D} \to X$ is continuous. Recall that a set is called *comeager* if its complement is meager. Thus, the function $\bar{u}$ is the "widest" continuous extension of u i.e., the domain of every continuous extension of u is contained in $\bar{D}$ and, moreover, $\bar{u}$ is an extension of every continuous extension of u. The function $\bar{u}$ is called the maximal extension of u and denoted by $\mathrm{ext}(u)$. A continuous function $u : D \to X$ defined on a dense subset $D \subset Q$ is said to be extended, if $\mathrm{ext}(u) = u$. Note that all extended functions are defined on comeager subsets of Q.

Let $C_\infty(Q, X)$ stands for the set of all extended X-valued functions. The totality of all bounded extended functions is denoted by $C^b_\infty(Q, X)$. Observe that $C_\infty(Q, X)$ can be represented also as the set of cosets of continuous functions u that act from comeager subsets $\mathrm{dom}(u) \subset Q$ into X. Two vector-valued functions u and v are equivalent if $u(t) = v(t)$ whenever $t \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$.

The set $C_\infty(Q, X)$ is endowed, in a natural way, with the structure of a lattice ordered module over the f-algebra $C_\infty(Q)$. Moreover, $C_\infty(Q, X)$ is uniformly complete and for any $u_1, \dots, u_N \in C_\infty(Q, X)$ the element $\widehat{\varphi}(u_1, \dots, u_N)$ is well defined in $C_\infty(Q, X)$ provided that $[u_1, \dots, u_N] \subset K$.

3.3. Proposition. *Let Q be a extremally disconnected conpact space and X a Banach lattice. Let $u_1, \dots, u_N \in C_\infty(Q, X)$ and $[u_1, \dots, u_N] \subset K$. Then there exists a comeager subset $Q_0 \subset Q$ such that $Q_0 \subset \mathrm{dom}(u_i)$ for all $i := 1, \dots, N$, $[u_1(q), \dots, u_N(q)] \subset K$ for every $q \in Q_0$, and $\widehat{\varphi}(u_1, \dots, u_N)$ is the maximal extension of the continuous function $q \mapsto \widehat{\varphi}(u_1(q), \dots, u_N(q))$ $(q \in Q_0)$, i. e.*

$$\widehat{\varphi}(u_1, \dots, u_N)(q) = \widehat{\varphi}(u_1(q), \dots, u_N(q)) \quad (q \in Q_0).$$

$\lhd$ Put $Q' := \mathrm{dom}(u_1) \cap \dots \cap \mathrm{dom}(u_N)$ and observe that Q' is comeager. There exists a unique function $e \in C_\infty(Q)$ such that $e'(q) := \|u_1(q)\| + \dots + \|u_N(q)\|$

$(q \in Q')$. Let E be the order ideal in $C_\infty(Q)$ generated by e and define the sublattice $E(X) \subset C_\infty(Q, X)$ by

$$E(X) := \big\{u \in C_\infty(Q, X) :\ (\exists\, 0 \le C \in \mathbb{R})\, (\forall\, q \in \mathrm{dom}(u))\, \|u(q)\| \le Ce(q)\big\}.$$

In the Boolean algebra of clopen subsets of Q there exists a partition of unity $\big(Q(\xi)\big)_{\xi\in\Xi}$ with $\chi_{Q(\xi)}e \in C(Q)$ for all $\xi \in \Xi$. Put $Q'_\xi := Q' \cap Q_\xi$ and $Q_0 := \bigcup_{\xi\in\Xi} Q'_\xi$ and observe that Q_0 is comeager in Q. Let π_ξ stands for the band projection in $C_\infty(Q, X)$ defined by $\pi_\xi : u \mapsto \chi_{Q(\xi)}u$. Then $\pi_\xi(E(X)) \subset C_b(Q, X)$ and $(\pi_\xi u_i)(q) = u_i(q)$ $(q \in Q'_\xi;\ i = 1, \dots, N)$. Finally, given $q \in Q'_\xi$, in view of Propositions 2.6 and 3.1 we have $[u_1(q), \dots, u_N(q)] = [(\pi_\xi u_1)(q), \dots, (\pi_\xi u_N)(q)] \subset K$ and

$$(\pi_\xi\widehat{\varphi}(u_1, \dots, u_N))(q)\widehat{\varphi}((\pi_\xi u_1)(q), \dots, (\pi_\xi u_N)(q)) = \\ = \widehat{\varphi}(\pi_\xi u_1, \dots, \pi_\xi u_N)(q) = \widehat{\varphi}(u_1(q), \dots, u_N(q))$$

and the proof is complete. $\triangleright$

3.4. Let (Ω, Σ, μ) be a measure space with the direct sum property and X be a Banach lattice. Let $\mathscr{L}^0(\mu, X) := \mathscr{L}^0(\Omega, \Sigma, \mu, X)$ be the set of all Bochner measurable functions defined almost everywhere on Ω with values in X and $L^0(\mu, X) := \mathscr{L}^0(\mu, X)/\sim$ the space of all equivalence classes (of almost everywhere equal) functions from $\mathscr{L}^0(\mu, X)$. Then $L^0(\mu, X)$ is a Banach lattice and hence $\widehat{\varphi}(u_1, \dots, u_N)$ is well defined in $L^0(\mu, X)$ for $\varphi \in \mathscr{H}(C; K)$ and $u_1, \dots, u_N \in L^0(\mu, X)$ with $[u_1, \dots, u_N] \subset K$. Denote by $\tilde{u}$ the equivalence class of $u \in \mathscr{L}^0(\mu, X)$.

Let $\mathscr{L}^\infty(\mu, X)$ stand for the part of $\mathscr{L}^0(\mu, X)$ comprising all essentially bounded functions and $L^\infty(\mu, X) := \mathscr{L}^\infty(\mu, X)/\sim$. Put $\mathscr{L}^\infty(\mu) := \mathscr{L}^\infty(\mu, \mathbb{R})$ and $L^\infty(\mu) := L^\infty(\mu, \mathbb{R})$. Denote by $\mathbb{L}^\infty(\mu)$ the part of $\mathscr{L}^\infty(\mu)$ consisting of all function defined everywhere on Ω. Then $\mathbb{L}^\infty(\mu)$ is a vector lattice with point-wise operations and order. Recall that a lattice homomorphism $\rho : L^\infty(\mu) \to \mathbb{L}^\infty(\mu)$ is said to be a *lifting* of $L^\infty(\mu)$ if $\rho(f) \in f$ for every $f \in L^\infty(\mu)$ and $\rho(\mathbb{1})$ is the identically one function on Ω. (Here $\mathbb{1}$ is the coset of the identically one function on Ω). Clearly, a lifting is a right-inverse of the quotient homomorphism $\phi : f \mapsto \tilde{f}$ $(f \in \mathscr{L}^\infty(\mu)$. The space $L^\infty(\mu)$ admits a lifting if and only if (Ω, Σ, μ) possesses the direct sum property. If $f \in \mathscr{L}^\infty(\mu)$, then the function $\rho(\tilde{f})$ is also denoted by $\rho(f)$.

3.5. Proposition. *Let $u_1, \dots, u_N \in \mathscr{L}^0(\Omega, \Sigma, \mu, F)$, and $[\widetilde{u}_1, \dots, \widetilde{u}_N] \subset K$. Then there exists a measurable set $\Omega_0 \subset \Omega$ such that $\mu(\Omega\backslash\Omega_0) = 0$, $[u_1(\omega), \dots, u_N(\omega)] \subset K$ for all $\omega \in \Omega_0$, and $\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N)$ is the equivalence class of the measurable function $\omega \mapsto \widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))$ $(\omega \in \Omega_0)$.*

$\triangleleft$ The problem can be reduced to Proposition 3.2 by means of Gutman's approach to vector-valued measurable functions. Let ρ be a lifting of $L^\infty(\Omega, \Sigma, \mu)$ and $\tau : \Omega \to Q$ be the corresponding canonical embedding of Ω into the Stone space Q of the Boolean algebra $B(\Omega, \Sigma, \mu)$, see [16]. The preimage $\tau^{-1}(V)$ of any meager set $V \subset Q$ is measurable and μ-negligible. Moreover τ is Borel measurable and $v \circ \tau$ is Bochner measurable for every $v \in C_\infty(Q, X)$. Denote by τ^* the mapping which sends each function $v \in C_\infty(Q, X)$ to the equivalence class of the measurable function $v \circ \tau$. The mapping τ^* is a linear and order isomorphism of $C_\infty(Q, X)$ onto $L^0(\Omega, \Sigma, \mu, X)$. If σ is the inverse of τ^*, then $[\sigma(\tilde{u}_1), \dots, \sigma(\tilde{u}_N)] \subset K$ and $\sigma\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N) = \widehat{\varphi}(\sigma(\tilde{u}_1), \dots, \sigma(\tilde{u}_N))$ by Proposition 2.6. According to Proposition 3.3 there exists a comeager subset $Q_0 \subset Q$ such that $[\sigma(\tilde{u}_1)(q), \dots, \sigma(\tilde{u}_N)(q)] \subset K$ for all $q \in Q_0$ and

$$\widehat{\varphi}(\sigma(\tilde{u}_1), \dots, \sigma(\tilde{u}_N))(q) = \widehat{\varphi}\big(\sigma(\tilde{u}_1)(q), \dots, \sigma(\tilde{u}_N)(q)\big) \quad (q \in Q_0).$$

Clearly, the functions $u'_i := \sigma(\tilde{u}_i) \circ \tau$ and u_i are equivalent and $\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N)$ is the equivalence class of $\sigma(\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N)) \circ \tau$. Let Ω' stands for the set of all $\omega \in \Omega$ with $u'_i(\omega) = u_i(\omega)$ for all $i = 1, \dots, N$. Then $\Omega_0 := \tau^{-1}(Q_0) \cap \Omega'$ is measurable and $\mu(\Omega \setminus \Omega_0) = 0$. Substituting $q = \tau(\omega)$ we get $[u'_1(\omega), \dots, u'_N(\omega)] \subset K$ for all $\omega \in \Omega_0$ and

$$\sigma\widehat{\varphi}(\tilde{u}_1, \dots, \tilde{u}_N)(\tau(\omega)) = \widehat{\varphi}\big(u'_1(\omega), \dots, u'_N(\omega)\big) \quad (\omega \in \Omega_0),$$

which is equivalent to the required statement. $\rhd$

3.6. A conic set $C \subset \mathbb{R}^N$ is said to be *multiplicative* if $st := (s_1t_1, \dots, s_Nt_N) \in C$ for all $s := (s_1, \dots, s_N) \in C$ and $t := (t_1, \dots, t_N) \in C$. A function $\varphi : C \to \mathbb{R}$ is called *multiplicative* if $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in C$.

Take a subset $I \subset \{1, \dots, N\}$ and define $\mathbb{R}^N_I$ as the cone in $\mathbb{R}^N$ consisting of 0 and $(s_1, \dots, s_N) \in \mathbb{R}^N_+$ with $s_i > 0$ $(i \in I)$. We will write $x_i \gg 0$ $(i \in I)$ if $[x_1, \dots, x_N] \subset \mathbb{R}^N_I$. The general form of a positively homogeneous multiplicative function $\varphi : \mathbb{R}^N_I \to \mathbb{R}$ other that $\varphi \equiv 0$ is given by

$$\varphi(t_1, \dots, t_N) = 0 \quad (t_1 \cdot \ldots \cdot t_N = 0),$$
$$\varphi(t_1, \dots, t_N) = \exp(g_1(\ln t_1)) \cdot \ldots \cdot \exp(g_N(\ln t_N)) \quad (t_1 \cdot \ldots \cdot t_N \neq 0),$$

where $g_1, \dots, g_N$ are some additive functions in $\mathbb{R}$ with $\sum_{i=1}^N g_i = I_{\mathbb{R}}$. If φ is continuous at any interior point of $\mathbb{R}^N_+$ or bounded on any ball contained in $\mathbb{R}^N_I$, then we get a Kobb–Duglas type function and if, in addition, φ is nonnegative, then $\varphi(t_1, \dots, t_N) = t_1^{\alpha_1} \cdot \ldots \cdot t_N^{\alpha_N}$ with $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ and $\sum_{i=1}^N \alpha_i = 1$.

By definition $x_i \gg 0$ $(i \in I)$ implies that $\widehat{\varphi}(x_1, \dots, x_N)$ is well defined for every $\varphi \in \mathscr{H}(\mathbb{R}^N_I, [x_1, \dots, x_N])$. Thus, the expression $x_1^{\alpha_1} \cdot \ldots \cdot x_N^{\alpha_N}$ is well defined in E provided that $x_k \gg 0$ for all k with $\alpha_k < 0$. At the same time $\varphi \in \mathscr{H}(\mathbb{R}^N_+)$ whenever $I = \varnothing$ and in this case $x_1^{\alpha_1} \cdot \ldots \cdot x_N^{\alpha_N}$ is well defined in E for arbitrary $x_k \geq 0$ and $\alpha_k \geq 0$ $(k = 1, \dots, N)$.

3.7. Proposition. *Let E, F and G be vector lattices with E and F uniformly complete and $b : E \times F \to G$ a lattice bimorphism. Let $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$, $\mathfrak{y} := (y_1, \dots, y_N) \in F^N$, and $[\mathfrak{x}] \cup [\mathfrak{y}] \subset K$ for some multiplicative closed conic set $K \subset \mathbb{R}^N$. If $\phi \in \mathscr{H}(C, K)$ is multiplicative on K, then $\widehat{\phi}(b(x_1, y_1), \dots, b(x_N, y_N))$ exists in G and*

$$\widehat{\phi}(b(x_1, y_1), \dots, b(x_N, y_N)) = b(\widehat{\phi}(x_1, \dots, x_N), \widehat{\phi}(y_1, \dots, y_N)).$$

$\lhd$ Put $u = \widehat{\phi}(x_1, \dots, x_N)$ and $v = \widehat{\phi}(y_1, \dots, y_N)$. Let E_0 and F_0 be the vector sublattices in E and F generated by $\{u, x_1, \dots, x_N\}$ and $\{v, y_1, \dots, y_N\}$, respectively. Let G_0 be the order ideal in G generated by $b(e, f)$ where $e := |u| + |x_1| + \dots + |x_N|$ and $f := |v| + |y_1| + \dots + |y_N|$. Observe that $\mathrm{Hom}(G_0)$ separates the points of G_0. By [23, Theorem 3.2] every $\mathbb{R}$-valued lattice bimorphism on $E_0 \times F_0$ is of the form $\sigma \otimes \tau : (x, y) \mapsto \sigma(x)\tau(y)$ with $\sigma \in \mathrm{Hom}(E_0)$ and $\tau \in \mathrm{Hom}(F_0)$. Denote by b_0 the restriction of b to $E_0 \times F_0$. Given an $\mathbb{R}$-valued lattice homomorphism ω on G_0, we have the representation $\omega \circ b = \sigma \otimes \tau$ for some lattice homomorphisms $\sigma : E_0 \to \mathbb{R}$ and $\tau : F_0 \to \mathbb{R}$. Since K is multiplicative, we have

$$\big(\omega(b(x_1, y_1), \dots, \omega(b(x_N, y_N))\big) = \big(\sigma(x_1)\tau(y_1), \dots, \sigma(x_N)\tau(y_N)\big)$$
$$= (\sigma(x_1), \dots, \sigma(x_N)) \cdot (\tau(y_1), \dots, \tau(y_N)) \in K,$$

and thus $[b(x_1, y_1), \dots, b(x_N, y_N)] \subset K$. Now, making use of 2.6 and multiplicativity of ϕ we deduce

$$\begin{aligned}\omega \circ b(u, v) &= \sigma(\widehat{\phi}(x_1, \dots, x_N))\tau(\widehat{\phi}(y_1, \dots, y_N)) \\ &= \phi(\sigma(x_1), \dots, \sigma(x_N))\phi(\tau(y_1), \dots, \tau(y_N)) \\ &= \phi(\sigma(x_1)\tau(y_1), \dots, \sigma(x_N)\tau(y_N)) \\ &= \phi(\omega \circ b(x_1, y_1), \dots, \omega \circ b(x_N, y_N)) \\ &= \omega \circ \widehat{\phi}(b(x_1, y_1), \dots, b(x_N, y_N)),\end{aligned}$$

as required by definition 2.2. $\rhd$

3.8. In particular, we can take $G := F \mathbin{\bar{\otimes}} F$, the Fremlin tensor product of E and F, or $E^{\odot}$, the square of E, and put $b := \otimes$ or $b := \odot$ in 3.7. Thus, under the hypotheses of 3.7 we have

$$\begin{aligned}\widehat{\phi}(x_1 \otimes y_1, \dots, x_N \otimes y_N) &= \widehat{\phi}(x_1, \dots, x_N) \otimes \widehat{\phi}(y_1, \dots, y_N), \\ \widehat{\phi}(x_1 \odot y_1, \dots, x_N \odot y_N) &= \widehat{\phi}(x_1, \dots, x_N) \odot \widehat{\phi}(y_1, \dots, y_N).\end{aligned}$$

Taking 3.6 into consideration we get the following: If $0 \le \alpha_1, \dots, \alpha_N \in \mathbb{R}$, $\alpha_1 + \dots + \alpha_N = 1$, then $|x_1 \otimes y_1|^{\alpha_1} \cdot \ldots \cdot |x_N \otimes y_N|^{\alpha_N}$ exists in $E \mathbin{\bar{\otimes}} F$ for all $x_1, \dots, x_N \in E$ and $y_1, \dots, y_N \in F$ and

$$\prod_{i=1}^{N} |x_i \otimes y_i|^{\alpha_i} = \left(\prod_{i=1}^{N} |x_i|^{\alpha_i}\right) \otimes \left(\prod_{i=1}^{N} |y_i|^{\alpha_i}\right);$$

if, in addition, $E = F$, then we also have

$$\prod_{i=1}^{N} |x_i \odot y_i|^{\alpha_i} = \left(\prod_{i=1}^{N} |x_i|^{\alpha_i}\right) \odot \left(\prod_{i=1}^{N} |y_i|^{\alpha_i}\right).$$

3.10. Proposition. *Let E be a uniformly complete vector lattice, $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$, $\mathfrak{p} := (\pi_1, \dots, \pi_N) \in \mathrm{Orth}(E)^N$, and $[\mathfrak{x}] \cup [\mathfrak{p}] \subset K$ for some multiplicative closed conic set $K \subset C \subset \mathbb{R}^N$. If $\phi \in \mathscr{H}(C, [\mathfrak{x}]) \cap \mathscr{H}(C, [\mathfrak{p}])$ is multiplicative on K, then $\widehat{\phi}(\pi_1 x_1, \dots, \pi_N x_N))$ exists in E and*

$$\widehat{\phi}(\pi_1 x_1, \dots, \pi_N x_N) = \widehat{\phi}(\pi_1, \dots, \pi_N)\big(\widehat{\phi}(x_1, \dots, x_N)\big).$$

$\lhd$ The bilinear operator b from $E \times \mathrm{Orth}(E)$ to E defined by $b(x, \pi) := \pi(x)$ is a lattice bimorphism and all we need is to apply Proposition 3.7. $\rhd$

4. Minkowski Duality

The *Minkowski duality* is the mapping that assigns to a sublinear function its support set or, in other words, its subdifferential (at zero). For any Hausdorff locally convex spaces X the Minkowski duality is a bijection between the collections of all lower semicontinuous sublinear functions on X and all closed convex subsets of the conjugate space X', see [25, 34]. The extended functional calculus (Theorems 1.3, 2.3, and 2.7) allows to transplant the Minkowski duality to vector lattice setting.

4.1. A function $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is called *sublinear* if it is *positively homogeneous*, i.e. $\varphi(0) = 0$ and $\varphi(\lambda t) = \lambda\varphi(t)$ for all $\lambda > 0$ and $t \in \mathbb{R}^N$, and *subadditive*, i.e. $\varphi(s+t) \leq \varphi(s)+\varphi(t)$ for all $s,t \in \mathbb{R}^N$. A function $\psi : \mathbb{R}^N \to \mathbb{R}\cup\{-\infty\}$ is called *superlinear* if $-\psi$ is sublinear. We say that φ is *lower semicontinuous* (ψ is *upper semicontinuous*) if the *epigraph* $\mathrm{epi}(\varphi) := \{(t,\alpha) \in \mathbb{R}^N \times \mathbb{R} : \varphi(t) \leq \alpha\}$ (the *hypograph* $\mathrm{hypo}(\varphi) := \{(t,\alpha) \in \mathbb{R}^N \times \mathbb{R} : \varphi(t) \geq \alpha\}$) is a closed subset of $\mathbb{R}^N \times \mathbb{R}$. The *effective domain* of a sublinear φ (superlinear ψ) is $\mathrm{dom}(\varphi) := \{t \in \mathbb{R}^N : \varphi(t) < +\infty\}$ $\big(\mathrm{dom}(\psi) := \{t \in \mathbb{R}^N : \psi(t) > -\infty\}\big)$. The *subdifferential* $\underline{\partial}\varphi$ of a sublinear function φ and the *superdifferential* $\overline{\partial}\psi$ of a superlinear function ψ are defined by

$$\underline{\partial}\varphi := \{t \in \mathbb{R}^N : \langle s,t\rangle \leq \varphi(s)\,(s \in \mathbb{R}^N)\},$$
$$\overline{\partial}\psi := \{t \in \mathbb{R}^N : \langle s,t\rangle \geq \psi(s)\,(s \in \mathbb{R}^N)\},$$

where $s = (s_1 \dots, s_N)$, $t = (t_1 \dots, t_N)$, $\langle s,t\rangle := \sum_{k=1}^N s_k t_k$. Denote by $\mathscr{H}_\vee(\mathbb{R}^N, K)$ and $\mathscr{H}_\wedge(\mathbb{R}^N, K)$ respectively the sets of all lower semicontinuous sublinear functions $\varphi : \mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ and upper semicontinuous superlinear functions $\psi : \mathbb{R}^N \to \mathbb{R}\cup\{-\infty\}$ which are finite and continuous on a fixed cone $K \subset \mathbb{R}^N$. Put $\mathscr{H}_\vee(\mathbb{R}^N) := \mathscr{H}_\vee(\mathbb{R}^N, \{0\})$ and $\mathscr{H}_\wedge(\mathbb{R}^N) := \mathscr{H}_\wedge(\mathbb{R}^N, \{0\})$. We shall consider $\mathscr{H}_\vee(\mathbb{R}^N)$ and $\mathscr{H}_\wedge(\mathbb{R}^N)$ as subcones of the vector lattice of Borel measurable functions $\mathscr{H}_{\mathrm{Bor}}(\mathbb{R}^N)$ with the convention that all infinite values are replaced by zero value.

4.2. Theorem. *Let $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$ and $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$. Then there exist countable subsets $A \subset \underline{\partial}\varphi$ and $B \subset \overline{\partial}\psi$ such that the representations hold:*

$$\varphi(s) = \sup\{\langle s,t\rangle : t \in A\} \quad (s \in \mathbb{R}^N),$$
$$\psi(s) = \inf\{\langle s,t\rangle : t \in B\} \quad (s \in \mathbb{R}^N).$$

$\lhd$ The claim is true for $A = \underline{\partial}\varphi$ and $B = \overline{\partial}\psi$ in any locally convex space X. The sets $\underline{\partial}\varphi$ and $\overline{\partial}\psi$ may be replaced by their countable subsets A and B provided that X is a separable Banach space, say $X = \mathbb{R}^N$ (see [17, Proposition A.1]). $\rhd$

4.3. Remark. Let H be a linear (or semilinear) subset of E. The *support set* $\partial_H x$ of $x \in E$ with respect to H is the set of all H-minorants of x: $\partial_H x := \{h \in H : h \leq x\}$. The *$H$-convex hull* of $x \in E$ is defined by $\mathrm{co}_H\, x := \sup\{h \in H : h \in \partial_H x\}$. An element x is called *H-convex* (abstract convex with respect to H) if $\mathrm{co}_H\, x = x$. Now the problem is to examine abstract convex elements, that is elements which can be represented as upper envelopes of subsets of a given set of elementary elements. For this *abstract convexity* see S. S. Kutateladze and A. M. Rubinov [25], as well as A. M. Rubinov [35].

In this section we deal with the description of H-convex elements in E in the event that H is the linear hull of a finite collection $\{x_1, \dots, x_N\} \subset E$. The following two theorems say that under some conditions an element in E is H-convex if and only if it is of the form $\widehat{\mathfrak{x}}(\varphi)$ for some lower semicontinuous sublinear functions φ.

For $A \subset \mathbb{R}^N$ denote by $\langle A, \mathfrak{x}\,\rangle$ the set of all linear combinations $\sum_{k=1}^N \lambda_k x_k$ in E with $(\lambda_1, \dots, \lambda_N) \in A$, so that

$$\sup\langle A, \mathfrak{x}\,\rangle := \sup\left\{\sum_{k=1}^N \lambda_k x_k : (\lambda_1, \dots, \lambda_N) \in A\right\}.$$

4.4. Theorem. *Let E be a σ-complete vector lattice with an order unit, $x_1, \dots, x_N \in E$, and $\mathfrak{x} := (x_1, \dots, x_N)$. Assume that $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$, and $[\mathfrak{x}] \subset \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)$. Then $\mathfrak{x}(\varphi)$ exists in E if and only if $\langle\, \underline{\partial}\varphi, \mathfrak{x}\,\rangle$ is order* *bounded above,* $\mathfrak{x}(\psi)$ *exists in* E *if and only if* $\langle\, \underline{\partial}\psi, \mathfrak{x}\,\rangle$ *is order bounded below, and the representations hold:*

$$\widehat{\mathfrak{x}}(\varphi) = \sup \langle\, \underline{\partial}\varphi, \mathfrak{x}\,\rangle\,, \quad \widehat{\mathfrak{x}}(\psi) = \inf \langle\, \overline{\partial}\psi, \mathfrak{x}\,\rangle\,.$$

Moreover, $\widehat{\varphi}(x_1,\dots,x_N)$ $\big(\widehat{\psi}(x_1,\dots,x_N)\big)$ *is an order limit of an increasing (decreasing) sequence which is comprised of the finite suprema (infima) of linear combinations of the form* $\sum_{i=1}^N \lambda_i x_i$ *with* $(\lambda_1,\dots\lambda_N) \in \underline{\partial}\varphi$ $\big((\lambda_1,\dots\lambda_N) \in \overline{\partial}\psi\big)$.

$\lhd$ Assume that $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$ and $[x_1,\dots,x_N] \subset \operatorname{dom}(\varphi)$. Let E_0 denotes the band in E generated by $\mathbb{1} := |x_1| + \dots + |x_N|$ and by $\mathbb{1}$ and $E_0^{u\sigma}$ stands for the universally σ-completion E_0. By Theorem 1.3 $\widehat{\mathfrak{x}}(\varphi)$ always exists in E_0 and the required representation holds true in $E_0^{u\sigma}$, since φ is Borel. In more details, let φ_0 vanishes on $\mathbb{R}^N \backslash \operatorname{dom}(\varphi)$ and coincides with φ on $\operatorname{dom}(\varphi)$. Then φ_0 is a Borel function on $\mathbb{R}^N$ and according to 4.2 we may choose an increasing sequence (φ_n) of Borel functions such that φ_n coincides with the finite supremum of linear combinations of the form $\sum_{i=1}^N \lambda_i t_i$ on $\operatorname{dom}(\varphi)$ and (φ_n) converges point-wise to φ_0. By Theorem 1.3 the sequence $(\widehat{\mathfrak{x}}(\varphi_n))$ is increasing and order convergent to $\widehat{\mathfrak{x}}(\varphi_0) = \widehat{\mathfrak{x}}(\varphi)$. Now it is clear that $\langle\, \underline{\partial}\varphi, \mathfrak{x}\,\rangle$ is order bounded above in E if and only if $\mathfrak{x}(\varphi) \in E_0$. $\rhd$

4.5. Theorem. *Let* E *be a relatively uniformly complete vector lattice,* $x_1,\dots,x_N \in E$, *and* $\mathfrak{x} := (x_1,\dots,x_N)$. *If* $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N; [\,\mathfrak{x}\,])$ *and* $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N; [\,\mathfrak{x}\,])$, *then*

$$\widehat{\mathfrak{x}}(\varphi) = \sup \langle\, \underline{\partial}\varphi, \mathfrak{x}\,\rangle\,,$$
$$\widehat{\mathfrak{x}}(\psi) = \inf \langle\, \overline{\partial}\psi, \mathfrak{x}\,\rangle\,.$$

Moreover, $\widehat{\varphi}(x_1,\dots,x_N)$ $\big(\widehat{\psi}(x_1,\dots,x_N)\big)$ *is a relatively uniform limit of an increasing (decreasing) sequence which is comprised of the finite suprema (infima) of linear combinations of the form* $\sum_{i=1}^N \lambda_i x_i$ *with* $\lambda = (\lambda_1,\dots\lambda_N) \in \underline{\partial}\varphi$ $(\lambda \in \overline{\partial}\psi)$.

$\lhd$ Consider $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N; [x_1,\dots,x_N])$ and denote $y = \widehat{\varphi}(x_1,\dots,x_N)$. By 2.3

$$v_\lambda := \lambda_1 x_1 + \ldots + \lambda_N x_N \leq y$$

for an arbitrary $\lambda := (\lambda_1,\dots,\lambda_N) \in \underline{\partial}\varphi$. Assume that $v \in E$ is such that $v \geq v_\lambda$ for all $\lambda \in \underline{\partial}\varphi$. By the Kreĭns–Kakutani Representation Theorem there is a lattice isomorphism $x \mapsto \tilde{x}$ of the principal ideal E_u generated by $u = |x_1| + \ldots + |x_N| + |v|$ onto $C(Q)$ for some compact Hausdorff space Q. Then v, $x_1,\dots,x_N$, v_λ, and y lie in E_u and for any $\lambda \in \underline{\partial}\varphi$ the point-wise inequality $\widetilde{v}(q) \geq \widetilde{v_\lambda}(q)$ $(q \in Q)$ is true. By 3.1 and 2.6 we conclude that

$$\widetilde{y}(q) = \varphi(\widetilde{x_1}(q),\dots,\widetilde{x_N}(q)) = \sup\{\widetilde{v_\lambda}(q) :\ \lambda \in \underline{\partial}\varphi\} \leq \widetilde{v}(q).$$

Thus we have $y \leq v$ and thereby $y = \sup\{v_\lambda :\ \lambda \in \underline{\partial}\varphi\}$.

Put $U := \{v_\lambda :\ \lambda \in \underline{\partial}\varphi\}$ and denote by $U^\vee$ the subset of E consisting of the suprema of the finite subsets of U. Then $U^\vee \subset E_u$ and the set $\widetilde{U^\vee} := \{\widetilde{v} :\ v \in U^\vee\}$ is upward directed in $C(Q)$ and its point-wise supremum equals to $\widetilde{y}$. By Dini Theorem $\widetilde{U^\vee}$ converges to $\widetilde{y}$ uniformly and thus $U^\vee$ is norm convergent to y in E_u. The superlinear case $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N; [x_1,\dots,x_N])$ is considered in a similar way. $\rhd$

4.6. In some situation it is important to know wether the function is the upper or lower envelope of a family of increasing linear functionals. Suppose that $\mathbb{R}^N$ is preordered by a cone $K \subset \mathbb{R}^N$, i. e $s \geq t$ means that $s - t \in K$. The dual cone of positive linear functionals is denoted by K^*. A function $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ is called *increasing* (with respect to K) if $s \geq t$ implies $\phi(s) \geq \phi(t)$. A lower semicontinuous sublinear (an upper semicontinuous superlinear) ϕ is increasing if and only if $\underline{\partial}\phi \subset K^*$ ($\overline{\partial}\phi \subset K^*$) and thus ϕ is an upper envelope of a family of increasing linear functionals (is a lower envelope of a family of increasing linear functionals). If ϕ is increasing only on $\mathrm{dom}(\phi)$, then this claim is no longer true but under some mild conditions it is still valid for the restriction of ϕ onto $\mathrm{dom}(\phi)$, see [25, 35].

Proposition. *Let $\varphi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be the same as in Theorem 4.2. Suppose that, in addition,* $\mathrm{dom}(\varphi) - K = K - \mathrm{dom}(\varphi)$ *and* $\mathrm{dom}(\psi) - K = K - \mathrm{dom}(\psi)$. *Then the following assertions hold:*

(1) φ is increasing on $\mathrm{dom}(\varphi)$ if and only if

$$\varphi(s) = \sup\{\langle s, t\rangle : \ t \in (\underline{\partial}\varphi) \cap K^*\} \quad (s \in \mathrm{dom}(\varphi));$$

(2) ψ is increasing on $\mathrm{dom}(\psi)$ if and only if

$$\psi(s) = \inf\{\langle s, t\rangle : \ t \in (\overline{\partial}\psi) \cap K^*\} \quad (s \in \mathrm{dom}(\psi)).$$

$\lhd$ Indeed, we may assume $\mathbb{R}^N = \mathrm{dom}(\varphi) - K$ and then the function $\varphi^* : \mathbb{R}^N \to \mathbb{R}$ defined by $\varphi^*(s) = \inf\{\varphi(t) : t \in \mathrm{dom}(\varphi), t \geq s\}$ $(s \in \mathbb{R}^N)$ is increasing and sublinear and coincides with φ on $\mathrm{dom}(\varphi)$; moreover $\underline{\partial}\varphi^* = (\underline{\partial}\varphi) \cap K^*$. Similarly, assuming $\mathbb{R}^N = \mathrm{dom}(\psi) - K$, we deduce that the function $\psi_* : \mathbb{R}^N \to \mathbb{R}$ defined by $\psi_*(s) = \sup\{\psi(t) : t \in \mathrm{dom}(\psi), t \leq s\}$ $(s \in \mathbb{R}^N)$ is increasing and superlinear and agrees with ψ on $\mathrm{dom}(\psi)$; moreover, $\overline{\partial}\psi_* = (\overline{\partial}\psi) \cap K^*$. It remains to observe that φ and ψ are increasing if and only if $\varphi = \varphi^*$ and $\psi = \psi_*$. $\rhd$

4.7. Corollary. *Assume that φ is increasing on* $\mathrm{dom}(\varphi)$, *ψ is increasing on* $\mathrm{dom}(\psi)$, $\mathrm{dom}(\varphi) - K = K - \mathrm{dom}(\varphi)$, *and* $\mathrm{dom}(\psi) - K = K - \mathrm{dom}(\psi)$. *If, in addition, the assumptions of either* 4.4 *or* 4.5 *are fulfilled, then in* 4.4 *and* 4.5 *the sets $\underline{\partial}\,\varphi$ and $\overline{\partial}\varphi$ may be replaced by $(\underline{\partial}\,\varphi) \cap K^*$ and $(\overline{\partial}\psi) \cap K^*$.*

4.8. A *gauge* is a sublinear function $\varphi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$. A *co-gauge* is a superlinear function $\psi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{-\infty\}$. The *lower polar* function φ° of a gauge φ and the *upper polar* function $\psi_\circ$ of a co-gauge ψ are defined by

$$\varphi^\circ(t) := \inf\{\lambda \geq 0 : \ (\forall\, s \in \mathbb{R}^N)\, \langle s, t\rangle \leq \lambda\varphi(s)\} \quad (t \in \mathbb{R}^N),$$
$$\psi_\circ(t) := \sup\{\lambda \geq 0 : \ (\forall\, s \in \mathbb{R}^N)\, \langle s, t\rangle \geq \lambda\psi(s)\} \quad (t \in \mathbb{R}^N)$$

(with the conventions $\sup \varnothing = -\infty$, $\inf \varnothing = +\infty$, and $0(+\infty) = 0(-\infty) = 0$). Thus, φ° is a gauge and $\psi_\circ$ is a co-gauge. Observe also that the inequalities hold:

$$\langle s, t\rangle \leq \varphi(s)\varphi^\circ(t) \quad (s \in \mathrm{dom}(\varphi),\, t \in \mathrm{dom}(\varphi^\circ)),$$
$$\langle s, t\rangle \geq \psi(s)\psi_\circ(t) \quad (s \in \mathrm{dom}(\psi),\, t \in \mathrm{dom}(\psi_\circ)).$$

Denote $\varphi^{\circ\circ} := (\varphi^\circ)^\circ$ and $\psi_{\circ\circ} := (\psi_\circ)_\circ$.

4.9. Bipolar Theorem. *Let φ be a gauge and ψ be a co-gauge. Then $\varphi^{\circ\circ} = \varphi$ if and only if φ is lower semicontinuous and $\psi_{\circ\circ} = \psi$ if and only if ψ is upper semicontinuous.*

$\lhd$ See [34]. $\rhd$

4.10. The lower polar function φ° is a gauge and can be also calculate by

$$\varphi^\circ(t) = \sup_{s \in \mathbb{R}^N} \frac{\langle s, t\rangle}{\varphi(s)} = \sup\{\langle s, t\rangle : \ s \in \mathbb{R}^N, \ \varphi(s) \leq 1\} \quad (t \in \mathbb{R}^N),$$

(with the conventions $\alpha/0 = +\infty$ for $\alpha > 0$ and $\alpha/0 = 0$ for $\alpha \le 0$) and

$$\psi_\circ(t) = \inf_{s\in\mathbb{R}^N} \frac{\langle s,t\rangle}{\psi(s)} = \inf\{\langle s,t\rangle:\ s\in\mathbb{R}^N,\ \psi(s)\ge 1 \text{ or } \psi(s)=0\} \quad (t\in\mathbb{R}^N)$$

(with the conventions $\alpha/0 = +\infty$ for $\alpha \ge 0$ and $\alpha/0 = -\infty$ for $\alpha < 0$).

Denote by $\mathscr{G}_\vee(\mathbb{R}^N, K)$ and $\mathscr{G}_\wedge(\mathbb{R}^N, K)$ respectively the sets of all lower semicontinuous gauges $\varphi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ and upper semicontinuous co-gauges $\psi : \mathbb{R}^N \to \mathbb{R}_+ \cup \{-\infty\}$ which are finite and continuous on a fixed cone $K \subset \mathbb{R}^N$. Put $\mathscr{G}_\vee(\mathbb{R}^N) := \mathscr{G}_\vee(\mathbb{R}^N, \{0\})$ and $\mathscr{G}_\wedge(\mathbb{R}^N) := \mathscr{G}_\wedge(\mathbb{R}^N, \{0\})$. Observe that $\mathscr{G}_\vee(\mathbb{R}^N) \subset \mathscr{H}_\vee(\mathbb{R}^N)$ and $\mathscr{G}_\wedge(\mathbb{R}^N) \subset \mathscr{H}_\wedge(\mathbb{R}^N)$.

4.11. Corollary. *Assume that either the assumptions of* 4.4 *are fulfilled and, in addition,* $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N)$ *and* $\psi \in \mathscr{G}_\wedge(\mathbb{R}^N)$*, or the assumptions of* 4.5 *are fulfilled and additionally* $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N; [\mathfrak{x}])$ *and* $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N; [\mathfrak{x}])$*. Then in* 4.4 *and* 4.5 *the sets* $\underline{\partial}\,\varphi$ *and* $\overline{\partial}\varphi$ *may be replaced by* $\{t \in \mathbb{R}^N : \varphi^\circ(t) \le 1\}$ *and* $\{t \in \mathbb{R}^N : \psi_\circ(t) \ge 1\}$*, respectively.*

$\lhd$ It is immediate from the Bipolar Theorem and the above definitions, since obviously $\underline{\partial}\,\varphi = \{t \in \mathbb{R}^N : \varphi^\circ(t) \le 1\}$ and, $\overline{\partial}\,\psi = \{t \in \mathbb{R}^N : \psi^\circ(t) \ge 1\}$. $\rhd$

5. Convexity Inequalities

According to Minkowski duality lower semicontinuous sublinear functions and upper semicontinuous superlinear functions are respectively upper and lower envelopes of families of linear functions. This fact can be used for proving inequalities and such approach is often called the *quasilinearization method*, see [2, 32]. Below we show that the same approach works in abstract setting and prove Jensen's, Hölder's, and Minkowski's inequalities in uniformly complete vector lattices.

5.1. Given a cone $K \subset \mathbb{R}^N$, denote by $\mathscr{H}_\curlyvee(\mathbb{R}^N, K)$ ($\mathscr{H}_\curlywedge(\mathbb{R}^N, K)$) the set of all sublinear (superlinear) functions $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ ($\mathbb{R} \cup \{-\infty\}$) with the properties: a) ϕ is lower semicontinuous (upper semicontinuous), b) $K \subset \mathrm{dom}(\phi)$ and ϕ is continuous on K, c) ϕ is increasing on $\mathrm{dom}(\phi)$ with respect to $\mathbb{R}^N_+$, d) $\mathbb{R}^N_+ - \mathrm{dom}(\phi) = \mathrm{dom}(\phi) - \mathbb{R}^N_+$.

Let E and F be vector lattices. An operator $f : E \to F \cup \{+\infty\}$ is said to be sublinear if $f(0) = 0$, $f(\lambda x) = \lambda f(x)$, and $f(x + y) \le f(x) + f(y)$ for all $0 \le \lambda \in \mathbb{R}$ and $x, y \in E$. A superlinear operator $g : E \to F \cup \{-\infty\}$, $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, are defined as in 4.1. We say that f is increasing on $\mathrm{dom}(f)$ if $x \ge y$ implies $f(x) \ge f(y)$ for $x, y \in \mathrm{dom}(f)$. For more details concerning sublinear operators, see [22].

5.2. Theorem (The generalized Jensen inequalities). *Let E and F be relatively uniformly complete vector lattices, $f : E \to F \cup \{+\infty\}$ an increasing sublinear operator, and $g : E \to F \cup \{-\infty\}$ an increasing superlinear operator. Assume that $\varphi \in \mathscr{H}_\curlyvee(\mathbb{R}^N, K)$ and $\psi \in \mathscr{H}_\curlywedge(\mathbb{R}^N, K)$. If $x_1, \dots, x_N \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $[x_1, \dots, x_N] \subset K$, then $\widehat{\varphi}(x_1, \dots, x_N) \in \mathrm{dom}(g)$, $\widehat{\psi}(x_1, \dots, x_N) \in \mathrm{dom}(f)$ and*

$$f\big(\widehat{\psi}(x_1, \dots, x_N)\big) \le \widehat{\psi}\big(f(x_1), \dots, f(x_N)\big),$$
$$g\big(\widehat{\varphi}(x_1, \dots, x_N)\big) \ge \widehat{\varphi}\big(g(x_1), \dots, g(x_N)\big).$$

$\lhd$ According to 4.6 we have the following representation

$$\psi(s) = \inf\big\{\langle s,t\rangle:\ t \in (\overline{\partial}\psi) \cap \mathbb{R}^N_+\big\} \quad (s \in \mathrm{dom}(\psi)),$$

since ψ is increasing on $\operatorname{dom}(\psi)$ and $(\mathbb{R}^N_+)^* = \mathbb{R}^N_+$. Now, using 4.6 and taking into consideration that f is sublinear and increasing, we deduce

$$
\begin{aligned}
&f(\widehat{\psi}(x_1 \dots, x_N)) \\
&\leq \inf\{f(\lambda_1 x_1 + \dots + \lambda_N x_N) : \ (\lambda_1, \dots, \lambda_N) \in (\overline{\partial}\psi) \cap \mathbb{R}^N_+\} \\
&\leq \inf\{\lambda_1 f(x_1) + \dots + \lambda_N f(x_N) : \ (\lambda_1, \dots, \lambda_N) \in (\overline{\partial}\psi) \cap \mathbb{R}^N_+\} \\
&= \widehat{\psi}(f(x_1), \dots, f(x_N)).
\end{aligned}
$$

The second inequality is handled in a similar way. $\rhd$

5.3. Remark. The given simple proof contains some additional possibilities.

(1) The inequalities from 5.2 remain valid if $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N, K)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N, K)$, and $f, g : E \to F$ are positive linear operators. In this case f and g are actually homogeneous (not only positively homogeneous!) and, for $(\lambda_1, \dots, \lambda_N)$ in $\overline{\partial}\psi$ or in $\underline{\partial}\varphi$, there is no need to involve the additional requirement $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N_+$.

(2) If E and F are Dedekind σ-complete then the classes of admissible φ and ψ in 5.2 may be extended: the generalized Jensen inequalities remain valid if $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$ and $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$, provided that $\widehat{\varphi}(x_1 \dots, x_N)$ and $\widehat{\psi}(x_1 \dots, x_N)$ are well defined in E. Indeed we need only to refer to 4.4 instead of 4.5.

(3) Equalities hold in 5.2 in the following cases: (a) in addition to hypotheses of Theorem 5.2, $f, g : E \to F$ are lattice homomorphisms; (b) all hypotheses from 5.3 (2) are fulfilled and $f, g : E \to F$ are sequentially order continuous lattice homomorphisms. Indeed, according to Theorems 4.4 and 4.5 we can choose a decreasing sequence φ_n which consists of the finite infima of linear functions of the form $t \mapsto \langle t, \lambda\rangle$ with $\lambda \in \underline{\partial}\varphi$ such that $\widehat{\varphi}_n(x_1, \dots, x_N)$ converges to $\widehat{\varphi}(x_1, \dots, x_N)$ uniformly in case (a) and in order in case (b). It remains to observe that $f(\widehat{\varphi}_n(x_1, \dots, x_N)) = \widehat{\varphi}_n(f(x_1), \dots, f(x_N))$ and pass to the o-limit or u-limit as $n \to +\infty$.

5.4. Corollary. *Let E be a Dedekind σ-complete Banach lattice, (Ω, Σ, μ) a measure space with the direct sum property, and $x_1, \dots, x_N \in \mathscr{L}^1(\Omega, \Sigma, \mu, E)$. If $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N, K)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N, K)$, and $[\tilde{x}_1, \dots, \tilde{x}_N] \subset K$, then $[x_1(\omega), \dots, x_N(\omega)] \subset K$ for almost all $\omega \in \Omega$ and*

$$
\begin{aligned}
\int_\Omega \widehat{\psi}(x_1(\omega), \dots, x_N(\omega))\, d\mu(\omega) &\leq \widehat{\psi}\bigg(\int_\Omega x_1(\omega)\, d\mu(\omega), \dots, \int_\Omega x_N(\omega)\, d\mu(\omega)\bigg), \\
\int_\Omega \widehat{\varphi}(x_1(\omega), \dots, x_N(\omega))\, d\mu(\omega) &\geq \widehat{\varphi}\bigg(\int_\Omega x_1(\omega)\, d\mu(\omega), \dots, \int_\Omega x_N(\omega)\, d\mu(\omega)\bigg).
\end{aligned}
$$

$\lhd$ The Bohner integral defines a positive linear operator $\tilde{x} \mapsto I_\mu(\tilde{x}) := \int_\Omega x(\omega)\, d\mu(\omega)$ from $L^1(\Omega, \Sigma, \mu, E)$ to E. Therefore, we can apply Theorem 5.2 with $f := I_\mu$ taking Remark 5.3 (2) into consideration and using Proposition 3.5. $\rhd$

5.5. The generalized Hölder inequality. *Let E and F be relatively uniformly complete vector lattices and let $f : E \to F \cup \{+\infty\}$ be an increasing sublinear mapping with $\operatorname{dom}(f) = E_+$. Then for $x_1, \dots, x_N \in E$ and $0 \leq \alpha_1, \dots, \alpha_N \in \mathbb{R}$, with $\alpha_1 + \dots + \alpha_N = 1$ we have*

$$
f\bigg(\prod_{i=1}^N |x_i|^{\alpha_i}\bigg) \leq \prod_{i=1}^N f(|x_i|)^{\alpha_i}.
$$

The reverse inequality holds provided that $f : E \to F\{-\infty\}$ *is superlinear,* $\alpha_1 + \dots + \alpha_N = 1$, $(-1)^k(1-\alpha_1-\dots-\alpha_k)\alpha_1\cdot\ldots\cdot\alpha_k \geq 0$ $(k := 1,\dots,N-1)$, *and* $x_i \gg 0$, $f(x_i) \gg 0$ *for all* i *with* $\alpha_i < 0$.

$\lhd$ Let $\alpha_1 + \dots + \alpha_N = 1$. The function $\phi(t_1,\dots,t_N) = t_1^{\alpha_1}\cdot\dots\cdot t_N^{\alpha_N}$ is superlinear on $\mathbb{R}^N_+$ if $0 \leq \alpha_1,\dots,\alpha_N$ and sublinear on $\mathbb{R}^N_I$ (see 3.6) with $I := \{i \in \{1,\dots,N\} : \alpha_i < 0\}$ whenever α_i, x_i, and $f(x_i)$ obey the latter conditions. $\rhd$

5.6. The generalized Minkowski inequality. *Let* E *and* F *be relatively uniformly complete vector lattices,* $f : E \to F \cup \{+\infty\}$ *be an increasing sublinear mapping with* $\mathrm{dom}(f) = E_+$, *and* $x_1,\dots,x_N \in E$. *If either and* $0 < \alpha \leq 1$ *or* $\alpha < 0$ *and additionally* $x_i \gg 0$ *and* $f(x_i) \gg 0$ *for all* $i := 1,\dots,N$, *then*

$$f\Bigg(\bigg(\sum_{i=1}^N |x_i|^\alpha\bigg)^{1/\alpha}\Bigg) \leq \bigg(\sum_{i=1}^N f(|x_i|)^\alpha\bigg)^{1/\alpha}.$$

The reverse inequality holds if $f : E \to F \cup \{-\infty\}$ *is superlinear and* $\alpha \geq 1$.

$\lhd$ The function $\phi(t_1,\dots,t_N) = \big(t_1^{\alpha_1} + \dots + t_N^{\alpha_N}\big)^{1/\alpha}$ is superlinear on $\mathbb{R}^N_+$ if $0 < \alpha < 1$, superlinear on $\mathrm{int}(\mathbb{R}^N_+)$ if $\alpha < 0$, and sublinear on $\mathbb{R}^N_+$ if $\alpha \geq 1$. $\rhd$

5.7. Remark. The generalized Hölder and Minkowski inequalities as they stand in 5.5 and 5.6 were obtained in [20] making use of the representations ($0 < \alpha < 1$):

$$s^\alpha t^{1-\alpha} = \inf\{\alpha\lambda^{1/\alpha}s + (1-\alpha)\lambda^{-1/(1-\alpha)}t :\ 0 < \lambda \in \mathbb{Q}\},$$
$$(s^\alpha + t^\alpha)^{1/\alpha} = \inf\{\lambda^{-1/\alpha}s + (1-\lambda)^{-1/\alpha}t :\ 0 < \lambda < 1,\ \lambda \in \mathbb{Q}\}.$$

Equalities hold in 5.5 and 5.6 if f and g are lattice homomorphisms, see 5.3 (3). In the special case of vector lattices of measurable functions the first inequality from 5.2 as well as 5.5 ($0 < \alpha_k < 1$) and 5.6 ($0 < \alpha < 1$) were established by M. Haase [17, Proposition 1.1, Remarks 1.2 (5) and 1.2 (6)]. Some special cases of 5.3 (and other interesting results) were also obtained by R. Drnovšek and A. Peperko in [10]. Various classical and recent inequalities are related to Hölder's or Minkowski's inequality (see E. F. Beckenbach, R. Bellman [2]; D. S. Mitrinović, J. E. Pečarić, A. M. Fink [32]). Some of them can naturally be transferred into the environment of vector lattice. By way of example we consider one more result that generalizes the inequality obtained by J. E. Pečarić and P. R. Beesack (see [32, Ch. VI, §4, Theorem 4]).

5.8. Proposition. *Let* E *and* F *be relatively uniformly complete vector lattices,* $f : E \to F \cup \{+\infty\}$ *be an increasing sublinear mapping,* $g : E \to \mathbb{R} \cup \{-\infty\}$ *an increasing superlinear function, and* $\mathrm{dom}(f) = dom(g) = E_+$. *Suppose that* $x_1,\dots,x_N \in E$ *and* $y_1,\dots,y_N \in E$ *with* $g(|y_i|) > 0$ $(i := 1,\dots,N)$. *Then for* $\alpha,\beta \in \mathbb{R}$, $0 < \alpha < 1 \leq \beta$, *we have*

$$\frac{f\Big(\big(\sum_{i=1}^N |x_i|^\alpha\big)^{\frac{1}{\alpha}}\Big)}{g\Big(\big(\sum_{i=1}^N |y_i|^\beta\big)^{\frac{1}{\beta}}\Big)} \leq \Bigg(\sum_{i=1}^N \bigg(\frac{f(|x_i|)}{g(|y_i|)}\bigg)^{\frac{\alpha\beta}{\beta-\alpha}}\Bigg)^{\frac{\beta-\alpha}{\alpha\beta}}.$$

$\lhd$ Let A stands for the left-hand side of the required inequality. Put $\gamma := \alpha\beta/(\beta-\alpha)$, $\sigma := \beta/(\beta-\alpha)$, and $\tau := -\alpha/(\beta-\alpha)$. Denote $u_i = f(|x_i|)^\gamma$ and $a_i = g(|y_i|)^{-\gamma}$ and observe that u_i is well defined in the universal completion F^u of F with a fixed order unit. Now, first apply Minkowski inequality to f and $0 < \alpha < 1$ and reverse

Minkowski inequality to g and $\beta \geq 1$ (see 5.6), and then use reverse Hölder inequality (see 5.5) in F^u for the sum $\sum_{i=1}^N a_i u_i$ with powers σ and τ taking into account that $\sigma/\gamma = 1/\alpha$ and $\tau/\gamma = -1/\beta$:

$$A \leq \left(\sum_{i=1}^N f(|x_i|)^\alpha \right)^{\frac{1}{\alpha}} \cdot \left(\sum_{i=1}^N g(|y_i|)^\beta \right)^{-\frac{1}{\beta}}$$

$$= \left(\left(\sum_{i=1}^N u^{\frac{1}{\sigma}} \right)^\sigma \cdot \left(\sum_{i=1}^N a^{\frac{1}{\tau}} \right)^\tau \right)^{\frac{1}{\gamma}} \leq \left(\sum_{i=1}^N a_i u_i \right)^{\frac{1}{\gamma}} = \left(\sum_{i=1}^N \left(\frac{f(|x_i|)}{g(|y_i|)} \right)^\gamma \right)^{\frac{1}{\gamma}}.$$

Thus, the required inequality is true in F^u and hence in F, since both sides are well defined in F. ⊳

6. Inequalities for Bilinear Operators

In this section we deduce a Maligranda type inequality for positive bilinear operators on uniformly complete vector latices using the above machinery.

6.1. The Fremlin tensor product $E \bar{\otimes} F$ need not be uniformly complete even for uniformly complete E and F. Therefore, the expressions of the form $\widehat{\varphi}(x_1 \otimes y_1, \dots, x_N \otimes y_N)$ with continuous positively homogeneous φ are generally meaningless in $E \bar{\otimes} F$. Denote by $E \widetilde{\otimes} F$ the uniform completion of $E \bar{\otimes} F$, see for example [33, Theorem 2.13]. Of course, $\widehat{\varphi}(x_1 \otimes y_1, \dots, x_N \otimes y_N)$ is well defined in $E \widetilde{\otimes} F$ provided that $[x_1 \otimes y_1, \dots, x_N] \subset \operatorname{dom}(\varphi)$.

It is quite natural to consider $E \widetilde{\otimes} F$ as the tensor product in the category of uniformly complete vector lattices and positive (or regular) operators. One can easily prove that $E \widetilde{\otimes} F$ shares the important universal property of Fremlin's tensor products $E \bar{\otimes} F$ (see [11, Theorem 5.3]): *If G is a uniformly complete vector lattice, then for every positive bilinear operator $b : E \times F \to G$ there exists a unique positive linear operator $T : E \widetilde{\otimes} F \to G$ such that $b = T\otimes$. Moreover, b is a lattice bimorphism if and only if T is a lattice homomorphism.*

6.2. Now we are going to prove a general Maligranda type inequality for positive bilinear operators. Consider a multiplicative conic set $K \subset \mathbb{R}^N$, see 3.8. A triple of functions $(\varphi_0, \varphi_1, \varphi_2)$ is called *C-submultiplicative* (*C-supermultiplicative*) on K if $K \subset \operatorname{dom}(\varphi_i)$ $(i := 0, 1, 2)$ and

$$\varphi_1(s)\varphi_2(t) \geq C\varphi_0(st) \quad \big(\varphi_1(s)\varphi_2(t) \leq C\varphi_0(st)\big)$$

for some positive $0 < C \in \mathbb{R}$ and all $s, t \in K$. In the special case $N = 2$ and $K = \mathbb{R}^N_+$ these inequalities are equivalent to

$$\varphi_1(1,s)\varphi_2(1,t) \geq C\varphi_0(1,st), \quad \big(\varphi_1(1,s)\varphi_2(1,t) \leq C\varphi_0(1,st)\big) \quad (0 < s, t \in \mathbb{R}).$$

Maligranda [31, Theorem 1] proved that if $\varphi, \varphi_0, \varphi_1$ are continuous gauges on $\mathbb{R}^2$ and the triple $(\varphi, \varphi_0, \varphi_1)$ is C-supermultiplicative, then for any positive bilinear operator $T : (E + F) \times (E + F) \to L^0(\Omega, \Sigma, \mu)$ with E and F ideal spaces on (Ω, Σ, μ) the inequality holds

$$T(\varphi_0(|x_0|, |x_1|), \varphi_1(|y_0|, |y_1|)) \leq C\varphi(T(|x_0|, |x_1|), T(|y_0|, |y_2|))$$

for all $x_0, y_0 \in E$ and $x_1, y_1 \in F$.

6.3. Lemma. *Let E, F, and G be uniformly complete vector lattices, $b : E\times F \to G$ be a positive bilinear operator and a positive linear operator $\Phi_b : E\widetilde{\otimes}F \to G$ be the linearization of b via tensor product, i.e., $b = \Phi_b\otimes$. Let $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N, K)$ and $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N, K)$ for some cone $K \subset \mathbb{R}^N$. Then for $x_1, \dots, x_N \in E$ and $y_1, \dots, y_N \in F$ with $[b(x_1, y_1), \dots, b(x_N, y_N)] \subset K$ and $[x_1 \otimes y_1, \dots, x_N \otimes y_N] \subset K$ we have*

$$\widehat{\varphi}(b(x_1, y_1), \dots, b(x_N, y_N)) \le \Phi_b\big(\widehat{\varphi}(x_1 \otimes y_1, \dots, x_N \otimes y_N)\big),$$
$$\widehat{\psi}(b(x_1, y_1), \dots, b(x_N, y_N)) \ge \Phi_b\big(\widehat{\psi}(x_1 \otimes y_1, \dots, x_N \otimes y_N)\big).$$

Equalities hold whenever b is a lattice hamomorphism.

$\lhd$ Apply Jensen's inequalities 5.2 with $f = g = \Phi_b$ taking into consideration Remarks 5.3 (2) and 5.3 (3). $\rhd$

6.4. Lemma. *Let E and F be uniformly complete vector lattices, $x_1, \dots, x_N \in E$ and $y_1, \dots, y_N \in E$. Suppose that $\psi_0, \psi_1, \psi_2 \in \mathscr{G}_\vee(\mathbb{R}^N, K)$, $\varphi_0, \varphi_1, \varphi_2 \in \mathscr{G}_\wedge(\mathbb{R}^N, K)$ with a multiplicative close conic set $K \subset \mathbb{R}^N$, and the triple $(\varphi_0, \varphi_1, \varphi_2)$ and (ψ_0, ψ_1, ψ_2) are C-supermultiplicative and C-submultiplicative on K, respectively. If $[x_1, \dots, x_N] \subset K$ and $[y_1, \dots, y_N] \subset K$, then $[x_1 \otimes y_1, \dots, x_N \otimes y_N)] \subset K$ and*

$$\widehat{\varphi}_1(x_1, \dots, x_N) \otimes \widehat{\varphi}_2(y_1, \dots, y_N) \le C\widehat{\varphi}_0(x_1 \otimes y_1, \dots, x_N \otimes y_N),$$
$$\widehat{\psi}_1(x_1, \dots, x_N) \otimes \widehat{\psi}_2(y_1, \dots, y_N) \ge C\widehat{\psi}_0(x_1 \otimes y_1, \dots, x_N \otimes y_N).$$

$\lhd$ Put $u = \widehat{\varphi}_1(x_1, \dots, x_N)$ and $v = \widehat{\varphi}_2(y_1, \dots, y_N)$. Let E_0 and F_0 be the vector sublattices in E and F generated by $\{u, x_1, \dots, x_N\}$ and $\{v, y_1, \dots, y_N\}$, respectively. According to [11, Corollary 4.5] $G_0 := E_0 \bar{\otimes} F_0$ is the sublattice of $E \bar{\otimes} F$ generated by $E_0 \otimes F_0$. Let G stands for the G_0-closure of G_0 in $E\widetilde{\otimes}F$. Then G is a uniformly complete sublattice of $E\widetilde{\otimes}F$ and any real valued lattice homomorphism on G_0 extends uniquely to a real valued lattice homomorphism on G, see [8, Lemma 1.1]. Therefore, the set H of all lattice homomorphisms $\rho : G \to \mathbb{R}$ with $\rho\otimes = \sigma \otimes \tau$ for some $\sigma \in \mathrm{Hom}(E_0)$ and $\tau \in \mathrm{Hom}(F_0)$ separates the points of G. The relations $[x_1, \dots, x_N] \subset K$ and $[y_1, \dots, y_N] \subset K$ imply $[x_1 \otimes y_1, \dots, x_N \otimes y_N)] \subset K$, since K is multiplicative. Thus we can conclude that $\widehat{\varphi}_0((x_1 \otimes y_1, \dots, x_N \otimes y_N))$ exists in G and

$$\rho(\widehat{\varphi}_0(x_1 \otimes y_1, \dots, x_N \otimes y_N)) = \varphi_0(\rho(x_1 \otimes y_1), \dots, \rho(x_N \otimes y_N))$$

for all $\rho \in H$. Now, making use of Proposition 2.6 and C-supermultiplicativity of the triple $(\varphi_0, \varphi_1, \varphi_2)$ we deduce

$$\begin{aligned}\rho(u \otimes v) = \sigma(u)\tau(v) &= \varphi_1(\sigma(x_1), \dots, \sigma(x_N))\varphi_2(\tau(y_1), \dots, \tau(y_N))\\ &\le C\varphi_0(\sigma(x_1)\tau(y_1), \dots, \sigma(x_N)\tau(y_N))\\ &= C\varphi_0((\sigma \otimes \tau)(x_1, y_1), \dots, (\sigma \otimes \tau)(x_N, y_N))\\ &= \rho(C\widehat{\varphi}_0(x_1 \otimes y_1, \dots, x_N \otimes y_N)).\end{aligned}$$

Since H separates the points of G, we have $u\otimes v \le C\widehat{\varphi}_0(x_1\otimes y_1, \dots, x_N\otimes y_N))$. The second inequality is derived in a similar way. $\rhd$

6.5. Theorem. *Let E, F, and G be uniformly complete vector lattices, $b : E \times F \to G$ a positive bilinear operator, $x_1, \dots, x_N \in E$ and $y_1, \dots, y_N \in E$. Suppose that $\psi_0, \psi_1, \psi_2 \in \mathscr{G}_\vee(\mathbb{R}^N, K)$, $\varphi_0, \varphi_1, \varphi_2 \in \mathscr{G}_\wedge(\mathbb{R}^N, K)$ with a multiplicative closed conic set $K \subset \mathbb{R}^N$, the triple $(\varphi_0, \varphi_1, \varphi_2)$ is C-supermultiplicative on K and the* *triple* (ψ_0, ψ_1, ψ_2) *is* C*-submultiplicative on* K*. If* $[x_1, \dots, x_N] \subset K$*,* $[y_1, \dots, y_N] \subset K$*, and* $[b(x_1, y_1), \dots, b(x_N, y_N)] \subset K$*, then*

$$b(\widehat{\varphi}_1(x_1, \dots, x_N), \widehat{\varphi}_2(y_1, \dots, y_N)) \leq C\widehat{\varphi}_0(b(x_1, y_1), \dots, b(x_N, y_N)),$$
$$b(\widehat{\psi}_1(x_1, \dots, x_N), \widehat{\psi}_2(y_1, \dots, y_N)) \geq C\widehat{\psi}_0(b(x_1, y_1), \dots, b(x_N, y_N)).$$

$\lhd$ Let a positive linear operator $\Phi_b : E\widetilde{\otimes}F \to G$ be the linearization of b via tensor product, so that $b = \Phi_b \otimes$. Observe that all hypothesis of Lemma 6.4 are fulfilled. Applying Φ_b to the inequalities from Lemma 6.4 we get

$$b(\widehat{\varphi}_1(x_1, \dots, x_N), \widehat{\varphi}_2(y_1, \dots, y_N)) \leq C\Phi_b(\widehat{\varphi}_0(b(x_1, y_1), \dots, b(x_N, y_N))),$$
$$b(\widehat{\psi}_1(x_1, \dots, x_N), \widehat{\psi}_2(y_1, \dots, y_N)) \geq C\Phi_b(\widehat{\psi}_0(b(x_1, y_1), \dots, b(x_N, y_N))).$$

It remains to apply Lemma 6.3. $\rhd$

6.6. Corollary. *Let* E*,* F*, and* G *be relatively uniformly complete vector lattices,* $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$*, and* $\mathfrak{y} := (y_1, \dots, y_N) \in F^N$*. Let* $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N, K)$*,* $\psi \in \mathscr{G}_\wedge(\mathbb{R}^N, K)$ *with a multiplicative closed conic set* $K \subset \mathbb{R}^N$ *and* $[\mathfrak{x}] \cup [\mathfrak{y}] \subset K$*. Then for any positive bilinear operator* $b : E \times F \to G$ *we have*

$$\sum_{k=1}^{N} b(x_k, y_k) \leq b(\widehat{\varphi}(x_1, \dots, x_N), \widehat{\varphi^\circ}(y_1, \dots, y_N)),$$
$$\sum_{k=1}^{N} b(x_k, y_k) \geq b(\widehat{\psi}(x_1, \dots, x_N), \widehat{\psi_\circ}(y_1, \dots, y_N)).$$

$\lhd$ Put $\lambda(s) := s_1 + \dots + s_N$ $(s = (s_1, \dots, s_N))$. The triples $(\lambda, \varphi, \varphi^\circ)$ and $(\lambda, \psi, \psi_\circ)$ are 1-submultiplicative and 1-supermultiplicative, respectively, see 4.8. Since $\widehat{\lambda}(u_1, \dots, u_N) = u_1 + \dots + u_N$, we need only to apply Theorem 6.5. $\rhd$

6.7. Corollary. *Let* E *be a uniformly complete vector lattices,* F *be a Dedekind complete vector lattice,* $x_1, \dots, x_N \in E$*, and* $T_1, \dots, T_N \in L^\sim(E, F)$*. Suppose that* $\psi_0, \psi_1, \psi_2 \in \mathscr{G}_\vee(\mathbb{R}^N, K)$ *and* $\varphi_0, \varphi_1, \varphi_2 \in \mathscr{G}_\wedge(\mathbb{R}^N, K)$*, the triple* $(\varphi_0, \varphi_1, \varphi_2)$ *is* C*-supermultiplicative on* K *and the triple* (ψ_0, ψ_1, ψ_2) *is* C*-submultiplicative on* K*. If* $[x_1, \dots, x_N] \subset K$*,* $[y_1, \dots, y_N] \subset K$*, and* $[T_1x_1, \dots, T_Nx_N)] \subset K$*, then*

$$\widehat{\varphi}_2(T_1, \dots, T_N)\big(\widehat{\varphi}_1(x_1, \dots, x_N)\big) \leq C\widehat{\varphi}_0(T_1x_1, \dots, T_Nx_N),$$
$$\widehat{\psi}_2(T_1, \dots, T_N)\big(\widehat{\psi}_1(x_1, \dots, x_N)\big) \geq C\widehat{\psi}_0(T_1x_1, \dots, T_Nx_N).$$

$\lhd$ Apply Theorem 6.5 to positive bilinear operator b from $E \times L^\sim(E, F)$ to F defined by $b(x, T) := Tx$. $\rhd$

6.8. Corollary. *Let* E *and* F *be relatively uniformly complete vector lattices,* $\mathfrak{x} := (x_1, \dots, x_N) \in E^N$*, and* $\mathfrak{T} := (T_1, \dots, T_N) \in L^\sim(E, F)^N$*. Let* $\varphi \in \mathscr{G}_\vee(\mathbb{R}^N, K)$*,* $\psi \in \mathscr{G}_\wedge(\mathbb{R}^N, K)$*, and* $[\mathfrak{x}] \cup [\mathfrak{T}] \subset K$*. Then we have*

$$\sum_{k=1}^{N} T_kx_k \leq \widehat{\varphi^\circ}(T_1, \dots, T_N)(\widehat{\varphi}(x_1, \dots, x_N)),$$
$$\sum_{k=1}^{N} T_kx_k \geq \widehat{\psi_\circ}(T_1, \dots, T_N)(\widehat{\psi}(x_1, \dots, x_N)).$$

$\lhd$ Apply 6.6 to positive bilinear operator b from $E \times L^{\sim}(E,F)$ to F defined by $b(x,T) := Tx$. $\rhd$

7. Functions of Bilinear Operators

In this section we compute $\varphi(b_1,\dots,b_N)$ for regular bilinear operators $b_1,\dots,b_N$. A *partition* of $x \in E_+$ is any finite sequence $(x_1,\dots,x_n)$, $n \in \mathbb{N}$, of elements of E_+ whose sum equals x. Denote by $\mathrm{Prt}(x)$ and $\mathrm{DPrt}(x)$ the sets of all partitions of x and all partitions with pairwise disjoint terms, respectively.

7.1. Lemma. *Let E, F, and G be vector lattices, $b_1,\dots,b_N \in BL^r(E,F;G)$, and $\mathfrak{b} := (b_1,\dots,b_N)$. Let $\varphi \in \mathscr{H}_\vee(\mathbb{R}^N)$, $\psi \in \mathscr{H}_\wedge(\mathbb{R}^N)$, $\widehat{\varphi}(b_1(x_0,y_0),\dots,b_N(x_0,y_0))$ and $\widehat{\psi}(b_1(x_0,y_0),\dots,b_N(x_0,y_0))$ are well defined in G for all $0 \le x_0 \le x$ and $0 \le y_0 \le y$. Denote $\mathfrak{x} := (x_1,\dots,x_n) \in E^n$ and $\mathfrak{y} := (y_1,\dots,y_m) \in F^m$. Then the sets*

$$\varphi(\mathfrak{b};x,y) :=$$
$$\left\{ \sum_{i=1}^{n}\sum_{j=1}^{m} \widehat{\varphi}(b_1(x_i,y_j),\dots,b_N(x_i,y_j)) :\ n,m \in \mathbb{N}, \mathfrak{x} \in \mathrm{Prt}(x),\ \mathfrak{y} \in \mathrm{Prt}(y) \right\},$$
$$\psi(\mathfrak{b};x,y) :=$$
$$\left\{ \sum_{i=1}^{n}\sum_{j=1}^{m} \widehat{\psi}(b_1(x_i,y_j),\dots,b_N(x_i,y_j)) :\ n,m \in \mathbb{N}, \mathfrak{x} \in \mathrm{Prt}(x),\ \mathfrak{y} \in \mathrm{Prt}(y) \right\},$$

are upward directed and downward directed, respectively.

$\lhd$ Assume that $(x_1,\dots,x_n)$ and $(x'_1,\dots,x'_{n'})$ are partitions of x while $(y_1,\dots,y_m)$ and $(y'_1,\dots,y'_{m'})$ are partitions of y. By the Riesz Decomposition Property of vector lattices there exist finite double sequences $(u_{i,j})_{i\le n,\, j\le n'}$ in E_+ and $(v_{i,j})_{i\le m,\, l\le m'}$ in F_+ such that

$$\textstyle\sum_{k=1}^{n'} u_{i,k} = x_i, \quad \sum_{i=1}^{n} u_{i,k} = x'_k \quad \big(i := 1,\dots,n,\ k := 1,\dots,n'\big).$$
$$\textstyle\sum_{l=1}^{m'} v_{j,l} = y_j, \quad \sum_{i=1}^{m} v_{j,l} = y'_l \quad \big(j := 1,\dots,m,\ l := 1,\dots,m'\big).$$

In particular, $(u_{i,k})_{i\le n,\, k\le n'}$ and $(v_{i,l})_{i\le m,\, j\le m'}$ are partition of x and y, respectively. Taking subadditivity of φ into consideration we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{n,m} \varphi(b_1(x_i,y_j),\dots,b_N(x_i,y_j))\\
&\qquad = \sum_{i,j=1}^{n,m} \varphi\bigg(\sum_{k,l=1}^{n',m'} b_1(u_{i,k},v_{j,l}),\dots,\sum_{k,l=1}^{n',m'} b_N(u_{i,k},v_{j,l}) \bigg)\\
&\qquad = \sum_{i,j=1}^{n,m} \varphi\bigg(\sum_{k,l=1}^{n',m'} \Big(b_1(u_{i,k},v(j,l)),\dots,b_N(u_{i,k},v_{j,l}) \Big) \bigg)\\
&\qquad \le \sum_{i,j=1}^{n,m}\sum_{k,l=1}^{n',m'} \varphi(b_1(u_{i,k},v_{j,l}),\dots,b_N(u_{i,k},v_{j,l})).
\end{aligned}$$

In a similar way we get

$$\sum_{i,j=1}^{n',m'} \varphi(b_1(x'_i,y'_j),\dots,b_N(x'_i,y'_j) \le \sum_{i,j=1}^{n',m'}\sum_{k,l=1}^{n,m} \varphi(b_1(u_{i,k},v_{j,l}),\dots,b_N(u_{i,k},v_{j,l})),$$

so that the first set is upward directed. Similarly, the second set is downward directed. ▷

7.2. Lemma. *Let Let E, F, and G be vector lattices with G Dedekind complete and $\mathscr{B}$ be an order bounded set of regular bilinear operators from $E\times F$ to G. Then for every $x\in E_+$ and $y\in F_+$ we have:*

$$(\sup\mathscr{B})(x,y) = \sup\left\{\sum_{i=1}^{n}\sum_{j=1}^{m} b_{k(i,j)}(x_i,y_j)\right\},$$

$$(\inf\mathscr{B})(x,y) = \inf\left\{\sum_{i=1}^{n}\sum_{j=1}^{m} b_{k(i,j)}(x_i,y_j)\right\},$$

where supremum and infimum are taken over all naturals $n,m,l\in\mathbb{N}$, functions $k:\{1,\dots,n\}\times\{1,\dots,m\}\to\{1,\dots,l\}$, partitions $(x_1,\dots,x_n)\in\mathrm{Prt}(x)$ and $(y_1,\dots,y_m)\in\mathrm{Prt}(y)$, and arbitrary finite collections $b_1\dots,b_l\in\mathscr{B}$.

◁ See [23, Proposition 2.6]. ▷

7.3. Theorem. *Let E, F, and G be vector lattices with G Dedekind complete, $b_1,\dots,b_N\in BL^\sim(E,F;G)$, and $\mathfrak{b}:=(b_1,\dots,b_N)$. Assume that $\varphi\in\mathscr{H}_\vee(\mathbb{R}^N)$, $\psi\in\mathscr{H}_\wedge(\mathbb{R}^N)$, $\widehat{\varphi}(b_1(x_0,y_0),\dots,b_N(x_0,y_0))$ and $\widehat{\psi}(b_1(x_0,y_0),\dots,b_N(x_0,y_0))$ are well defined in G for all $0\le x_0\le x$ and $0\le y_0\le y$, $\varphi(\mathfrak{b};x,y)$ is order bonded above, and $\psi(\mathfrak{b};x,y)$ is order bounded below for all $x\in E_+$ and $y\in F_+$. Then $\widehat{\varphi}(b_1,\dots,b_N)$ and $\widehat{\psi}(b_1,\dots,b_N)$ are well defined in $BL^\sim(E,F;G)$ and for every $x\in E_+$ and $y\in F_+$ the representations*

$$\widehat{\varphi}(b_1,\dots,b_N)(x,y) = \sup\varphi(\mathfrak{b};x,y),$$

$$\widehat{\psi}(b_1,\dots,b_N)(x,y) = \inf\psi(\mathfrak{b};x,y)$$

hold with supremum over upward directed set and infimum over downward directed set. If E and F have the strong Freudenthal property (or principal projection property) then $\mathrm{Prt}(x)$ and $\mathrm{Prt}(y)$ may be replaced by $\mathrm{DPrt}(x)$ and $\mathrm{DPrt}(y)$, respectively.

◁ Denote $b_\lambda:=\lambda_1b_1+\dots+\lambda_Nb_N$ for $\lambda:=(\lambda_1,\dots,\lambda_N)\in\mathbb{R}^N$ and observe that if the set $\{b_\lambda:\lambda\in\underline{\partial}\varphi\}$ is order bounded in $BL^r(E,F;G)$, then by 4.4 $\widehat{\varphi}(b_1,\dots,b_N)$ exists in $BL^r(E,F;G)$ and $\widehat{\varphi}(b_1,\dots,b_N)=\sup\{b_\lambda:\lambda\in\underline{\partial}\varphi\}$. Take arbitrary $\lambda^r:=(\lambda^r_1,\dots,\lambda^r_N)\in\underline{\partial}\varphi$ $(r:=1,\dots,l)$, $k:\{1,\dots,n\}\times\{1,\dots,m\}\to\{1,\dots,l\}$, $\mathfrak{x}:=(x_1,\dots,x_n)\in\mathrm{Prt}(x)$, and $\mathfrak{y}:=(y_1,\dots,y_m)\in\mathrm{Prt}(y)$. Making use of Lemma 7.2 and Theorem 4.4 we deduce:

$$\sum_{i,j=1}^{n,m} b_{\lambda^{k(i,j)}}(x_i,y_j) = \sum_{i,j=1}^{n,m}\sum_{s=1}^{N}\lambda_s^{k(i,j)}b_s(x_i,y_j) \le \sum_{i,j=1}^{n,m}\widehat{\varphi}(b_1(x_i,y_j),\dots,b_N(x_i,y_j)) \le a,$$

where a is an upper bound of $\varphi(\mathfrak{b};x,y)$. Passing to supremum over all $(\lambda^1\dots,\lambda^l)$, k, $\mathfrak{x}$, and $\mathfrak{y}$ and taking Theorem 4.4 into account we get that $\widehat{\varphi}(b_1,\dots,b_N)$ is well defined and $\widehat{\varphi}(b_1,\dots,b_N)(x,y)\le\varphi(\mathfrak{b};x,y)$. Surely, in above reasoning we could take $(x_1,\dots,x_n)\in\mathrm{DPrt}(x)$ provided that E has the principal projection property.

Conversely, let $f(x,y)$ stands for the right-hand side of the first equality. Observe that if $(\lambda_1,\dots,\lambda_n)\in\underline{\partial}\varphi$ and $u\in E_+$, $v\in F_+$, then by 4.4 we have

$$\sum_{k=1}^{N}\lambda_k b_k(u,v)=\Big(\sum_{k=1}^{N}\lambda_k b_k\Big)(u,v)\le\widehat{\varphi}(b_1,\dots,b_N)(u,v)$$

and again $\widehat{\varphi}(b_1(u,v),\dots,b_N(u,v))\le\widehat{\varphi}(b_1,\dots,b_N)(u,v)$ by Theorem 4.4. Now, given $(x_1,\dots,x_n)$ in $\mathrm{Prt}(x)$ or $\mathrm{DPrt}(x)$ and $(y_1,\dots,y_n)$ in $\mathrm{Prt}(y)$ or $\mathrm{DPrt}(y)$, we can estimate

$$\sum_{i,j=1}^{n,m}\widehat{\varphi}(b_1(x_i,y_j),\dots,b_N(x_i,y_j))\le\sum_{i,j=1}^{n,m}\widehat{\varphi}(b_1,\dots,b_N)(x_i,y_j)=\widehat{\varphi}(b_1,\dots,b_N)(x,y)$$

and thus $f(x,y)\le\widehat{\varphi}(b_1,\dots,b_N)(x,y)$. Thus the first equality is hold true. By Lemma 7.1 the supremum on the right-hand side of the required formula is taken over upward directed set.

The second representation is proved in a similar way. ▷

7.4. Corollary. *Let E, F, G, φ, ψ, $b_1,\dots,b_N$ be the same as in 7.1, $\overline{b}:=\widehat{\varphi}(b_1,\dots,b_N)$ and $\underline{b}:=\widehat{\psi}(b_1,\dots,b_N)$. Assume that, in addition, $E=F$ has the strong Freudenthal property and $b_1,\dots,b_N$ are orthosymmetric. Then for every $x\in E$ the representations*

$$\overline{b}(x,x)=\sup\Big\{\sum_{i=1}^{n}\varphi(b_1(x_i,|x|),\dots,b_N(x_i,|x|)):\ (x_1,\dots,x_n)\in\mathrm{DPrt}(|x|)\Big\},$$
$$\underline{b}(x,x)=\inf\Big\{\sum_{i=1}^{n}\psi(b_1(x_i,|x|),\dots,b_N(x_i,|x|)):\ (x_1,\dots,x_n)\in\mathrm{DPrt}(|x|)\Big\},$$

hold with supremum and infimum over upward and downward directed sets, respectively.

◁ It is sufficient to check the first formula. We can assume $x\in E_+$. Denote by $g(x)$ the right-hand side of the desired equality. From 7.3 we have $g(x)\le\widehat{\varphi}(b_1,\dots,b_N)(x,x)$. To prove the reverse inequality take two disjoint partitions of x, say $\mathfrak{x}':=(x'_1,\dots,x'_l)$ and $\mathfrak{x}'':=(x''_1,\dots,x''_m)$, and let $(x_1,\dots,x_n)\in\mathrm{DPrt}(x)$ be their common refinement. Since $b_1,\dots,b_N$ are orthosymmetric we deduce

$$\sum_{r,s=1}^{l,m}\widehat{\varphi}(b_1(x'_r,x''_s),\dots,b_N(x'_r,x''_s))$$
$$=\sum_{i=1}^{n}\widehat{\varphi}(b_1(x_i,x_i),\dots,b_N(x_i,x_i))=\sum_{i=1}^{n}\widehat{\varphi}(b_1(x_i,x),\dots,b_N(x_i,x)).$$

Passing to supremum over all $\mathfrak{x}'$ and $\mathfrak{x}''$ we get the desired inequality. ▷

8. Functions of Linear Operators

The above machinery is applicable to the calculus of order bounded operators. By way of illustration compute and estimate $\varphi(T_1,\dots,T_N)$ for order bounded linear operators $T_1,\dots,T_N$. We use the above symbols $\mathrm{Prt}(x)$ and $\mathrm{DPrt}(x)$ for the sets of partitions and disjoint partitions of $x\in E_+$, respectively.

8.1. Theorem. *Let E and F be vector lattices with F Dedekind complete, $T_1,\dots,T_N \in L^{\sim}(E,F)$, and $\mathfrak{T} := (T_1,\dots,T_N)$. Let $\varphi \in \mathscr{H}_{\vee}(\mathbb{R}^N)$, $\psi \in \mathscr{H}_{\wedge}(\mathbb{R}^N)$, and $[T_1,\dots,T_N]$ is contained in $\operatorname{dom}(\varphi)\cap\operatorname{dom}(\psi)$. If for every $x \in E_+$ the sets*

$$\varphi(\mathfrak{T};x) = \left\{ \sum_{k=1}^{n} \widehat{\varphi}(T_1x_k,\dots,T_Nx_k) :\ (x_1,\dots,x_n) \in \operatorname{Prt}(x) \right\},$$
$$\psi(\mathfrak{T};x) = \left\{ \sum_{k=1}^{n} \widehat{\psi}(T_1x_k,\dots,T_Nx_k) :\ (x_1,\dots,x_n) \in \operatorname{Prt}(x) \right\}$$

are order bounded from above and from below respectively, then $\widehat{\varphi}(T_1,\dots,T_N)$ and $\widehat{\psi}(T_1,\dots,T_N)$ exist in $L^{\sim}(E,F)$, and the representations

$$\widehat{\varphi}(T_1,\dots,T_N)x = \sup \varphi(\mathfrak{T};x),$$
$$\widehat{\psi}(T_1,\dots,T_N)x = \inf \psi(\mathfrak{T};y)$$

hold with supremum over upward directed set and infimum over downward directed set. If E has the principal projection property then $\operatorname{Prt}(x)$ may be replaced by $\operatorname{DPrt}(x)$.

$\lhd$ Follows immediately from 7.1. $\rhd$

8.2. Remark. (1) Assume that E, F, $T_1,\dots,T_N$, φ, and ψ are the same as in 5.1. Then $\varphi(T_1,\dots,T_N)x \geq \varphi(T_1x,\dots,T_Nx)$ and $\psi(T_1,\dots,T_N)x \leq \psi(T_1x,\dots,T_Nx)$ for all $x \in E_+$. In particular, if $\mathbb{R}^N_+ \subset \operatorname{dom}(\varphi)\cap\operatorname{dom}(\psi)$ and $\varphi(T_1x,\dots,T_Nx) \geq \psi(T_1x,\dots,T_Nx)$ for all $x \in E_+$, then $\varphi(T_1,\dots,T_N) \geq \psi(T_1,\dots,T_N)$.

(2) If the sets in braces at the right-hand sides of 8.1 are order bounded below and above respectively, then $\widehat{\varphi}(T_1,\dots,T_N)$ and $\widehat{\psi}(T_1,\dots,T_N)$ are well defined.

(3) Assume that $\varphi \in \mathscr{H}(C;[\mathfrak{x}])$ and $\varphi(0,t_2,\dots,t_N) = 0$ for all $(t_1,\dots,t_N) \in \operatorname{dom}(\varphi)$. Then evidently $\widehat{\varphi}(x_1,\dots,x_N) \in \{x_1\}^{\perp\perp}$ provided that $[\mathfrak{x}] \subset \operatorname{dom}(\varphi)$. This simple observation together with 5.1 enables one to attack the nonlinear majorization problem for wider variety of majorants $\varphi(T_1,\dots,T_N)$, cp. [5].

8.3. *Let E and F be vector lattices with E relatively uniformly complete and F Dedekind complete. Then for $T_1,\dots,T_N \in L^{\sim}_{+}(E,F)$, $x_1,\dots,x_N \in E_+$, and $\alpha_1,\dots,\alpha_N \in \mathbb{R}_+$ with $\alpha_1+\dots+\alpha_N = 1$ we have*

$$(T_1^{\alpha_1}\dots T_N^{\alpha_N})(x_1^{\alpha_1}\dots x_N^{\alpha_N}) \leq (T_1x_1)^{\alpha_1}\dots(T_Nx_N)^{\alpha_N}.$$

The reverse inequality holds provided that $\alpha_1+\dots+\alpha_N = 1$, $(-1)^k(1-\alpha_1-\dots-\alpha_k)\alpha_1\cdot\ldots\cdot\alpha_k \geq 0$ $(k := 1,\dots,N-1)$, and $x_i \gg 0$, $f(x_i) \gg 0$ for all i with $\alpha_i < 0$.

$\lhd$ Apply 6.7 with $K = \mathbb{R}^N_+$, $C = 1$, $\varphi_0(t) = \varphi_1(t) = \varphi_2(t) = t_1^{\alpha_1}\dots t_N^{\alpha_N}$. $\rhd$

8.4. Theorem. *Let E and F be vector lattices with F Dedekind complete and $T_1,\dots,T_N \in L^{\sim}(E,F)$. Suppose that $\varphi \in \mathscr{G}_{\vee}(\mathbb{R}^N)$ and $\psi \in \mathscr{G}_{\wedge}(\mathbb{R}^N)$ are increasing and $[T_1,\dots,T_N] \subset \operatorname{dom}(\varphi)\cap\operatorname{dom}(\psi)$. Then for every $x \in E_+$ the representations hold*

$$\varphi(T_1,\dots,T_N)x = \sup\left\{ \sum_{k=1}^{N} T_kx_k :\ \varphi^{\circ}(x_1,\dots,x_N) \leq x \right\},$$
$$\psi(T_1,\dots,T_N)x = \inf\left\{ \sum_{k=1}^{N} T_kx_k :\ \psi_{\circ}(x_1,\dots,x_N) \geq x \right\},$$

with supremum over upward directed set and infimum over downward directed set.

$\lhd$ Suppose that $\varphi(T_1,\dots,T_N)$ exists and $x \in E_+$. If $x_1,\dots,x_N \in E_+$ and $\varphi^\circ(x_1,\dots,x_N) \le x$, then making use of the Bipolar Theorem, positivity of $\varphi(T_1,\dots,T_N)$, and 6.8 we deduce

$$\sum_{k=1}^{N} T_k x_k \le \varphi(T_1,\dots,T_N)(\varphi^\circ(x_1,\dots,x_N)) \le \varphi(T_1,\dots,T_N)x.$$

To prove the reverse inequality take $(x_1,\dots,x_n) \in \mathrm{Prt}(x)$, $\lambda^k = (\lambda_1^k,\dots,\lambda_N^k) \in \underline{\partial}\varphi = \{\varphi^\circ \le 1\}$ $(k := 1,\dots,n)$, and put $u_i := \sum_{k=1}^n \lambda_i^k x_k$. If $\alpha := (\alpha_1,\dots,\alpha_N) \in \underline{\partial}\varphi^\circ = \{\varphi \le 1\}$, then $\langle\alpha,\lambda^k\rangle \le \varphi(\alpha)\varphi^\circ(\lambda^k) \le 1$ and thus

$$\sum_{i=1}^{N} \alpha_i u_i = \sum_{i=1}^{N} \alpha_i \sum_{k=1}^{n} \lambda_i^k x_k = \sum_{k=1}^{n} \langle\alpha,\lambda^k\rangle x_k \le x.$$

It follows from 4.4 that $\varphi^\circ(u_1,\dots,u_N) \le x$.

Denote $S(\lambda) := \lambda_1 T_1 + \dots + \lambda_N T_N$ with $\lambda := (\lambda_1,\dots,\lambda_N)$. Let $f(x)$ is the right-hand side of the first equality. Then

$$\sum_{k=1}^{n} S(\lambda^k)(x_k) = \sum_{i=1}^{N} T_i u_i \le f(x).$$

It remains to observe that $\varphi(T_1,\dots,T_N) = \sup\{S(\lambda) : \lambda \in \underline{\partial}\varphi\}$ by 4.4. $\rhd$

8.5. Proposition. *Let E, F, and G be vector lattices with F Dedekind complete, $R : E \to G$ an order interval preserving operator, $T : G \to F$ an order continuous lattice homomorphism, and $\varphi \in \mathscr{H}(C,K)$. Assume that $S_1,\dots,S_N \in L^\sim(E,F)$ and $[S_1,\dots,S_N] \subset K$. Then $[S_1 \circ R,\dots,S_N \circ R] \subset K$ and*

$$\widehat{\varphi}(S_1,\dots,S_N) \circ R = \widehat{\varphi}(S_1 \circ R,\dots,S_N \circ R).$$

If, in addition, G is Dedekind complete, then $[T \circ S_1,\dots,T \circ S_N] \subset K$ and

$$T \circ \widehat{\varphi}(S_1,\dots,S_N) = \widehat{\varphi}(T \circ S_1,\dots,T \circ S_N).$$

$\lhd$ Under the indicated hypotheses the operators $S \mapsto S \circ R$ from $L^\sim(G,F)$ to $L^\sim(E,F)$ and $S \mapsto T \circ S$ from $L^\sim(E,G)$ to $L^\sim(E,F)$ are lattice homomorphisms, see [1, Theorem 7.4 and 7.5]. Therefore, it is sufficient to apply Proposition 2.6. $\rhd$

8.6. Proposition. *Let E and F be vector lattices with F Dedekind complete. Assume that $S_1,\dots,S_N \in L^\sim(E,F)$ and $[S_1,\dots,S_N] \subset K$. If S^* denotes the restriction of the order dual S' to $F_n^\sim$, the order continuous dual of F, then $[S_1^*,\dots,S_N^*] \subset K$ and*

$$\widehat{\varphi}(S_1,\dots,S_N)^* = \widehat{\varphi}(S_1^*,\dots,S_N^*).$$

$\lhd$ By Krengel–Synnatschke Theorem [1, Theorem 5.11] the map $S \mapsto S^*$ is a lattice homomorphism from $L^\sim(E,F)$ into $L^\sim(F_n^\sim,E^\sim)$, see [1, Theorem 7.6]. Thus, we need only to apply Proposition 2.6. $\rhd$

8.7. Let E denotes an ideal spaces on (Ω,Σ,μ). Consider another measure space (Ω',Σ',μ') and let F be an ideal spaces on (Ω',Σ',μ'). A linear operator $S : E \to F$ is called a *kernel operator* with kernel $k \in \mathscr{L}^0(\mu'\otimes\mu)$ if it admits the representation

$$(S\tilde{u})(s) = \int_{\Omega} k(s,t)\,u(t)\,d\mu(t) \quad (\tilde{u} \in E).$$

More precisely, there exists a $\mu' \otimes \mu$-measurable function $k : \Omega' \times \Omega \to \bar{\mathbb{R}}$ such that for every $\tilde{u} \in E$ the value $\tilde{v} = S\tilde{u}$ is the equivalence class of the function $v(s) = \int_\Omega k(s,t)\, u(t)\, d\mu(t)$ $(s \in \Omega')$. The integral is understood to be the usual Lebesgue integral.

8.8. Proposition. *Let E and F be ideal spaces over σ-finite measure spaces (Ω, Σ, μ) and (Ω', Σ', μ'), respectively. Suppose that $S_1, \dots, S_N$ are order bounded kernel operators from E to F with respective kernels $k_1, \dots, k_N$ and $[S_1, \dots, S_N] \subset K$. Then $(k_1(s,t), \dots, k_N(s,t)) \in K$ for $\mu' \otimes \mu$-almost all $(s,t) \in \Omega' \times \Omega$ and $\widehat{\varphi}(S_1, \dots, S_N)$ is also a kernel operator from E to F with kernel $\varphi \circ (k_1(\cdot,\cdot), \dots, k_N(\cdot,\cdot))$; in symbols,*

$$\big(\widehat{\varphi}(S_1, \dots, S_N)u\big)(s) = \int\limits_\Omega \varphi(k_1(s,t), \dots, k_N(s,t))\, u(t)\, d\mu(t) \quad (u \in E).$$

$\lhd$ The set $\mathscr{I}^\sim(E,F)$ of order bounded kernel operators from E into F is a band in $L_n^\sim(E,F)$. The map σ sending every operator from $\mathscr{I}^\sim(E,F)$ to the equivalence class of its kernel is a lattice isomorphism of $\mathscr{I}^\sim(E,F)$ onto some order ideal in $L^0(\mu \otimes \mu')$. Thus, $[\tilde{k}_1, \dots, \tilde{k}_N] \subset K$ and

$$\sigma\widehat{\varphi}(S_1, \dots, S_N) = \widehat{\varphi}(\tilde{k}_1, \dots, \tilde{k}_N)$$

by Proposition 2.6. According to Proposition 3.5 there exists a measurable set $\Omega_0 \subset \Omega' \times \Omega$ such that $\mu' \otimes \mu(\Omega' \times \Omega \setminus \Omega_0) = 0$, $[k_1(s,t), \dots, k_N(s,t)] \subset K$ for all $(s,t) \in \Omega_0$, and $\widehat{\varphi}(\tilde{k}_1, \dots, \tilde{k}_N)$ is the equivalence class of the measurable function $(s,t) \mapsto \varphi(k_1(s,t), \dots, u_N(s,t))$ $\big((s,t) \in \Omega_0\big)$. $\rhd$

9. Continuous and measurable bundles of Banach lattices

Now, we consider an instance of homogeneous functional calculus on vector lattices of continuous and measurable sections of bundles of Banach lattices is also considered. All necessary information on continuous and measurable Banach bundles can be found in [14, 15] and [19].

9.1. Let Q be a topological space. A *bundle of Banach lattices over* Q is a mapping $\mathscr{X}$ defined on Q and associating a Banach lattice $\mathscr{X}_q := \mathscr{X}(q)$ with every point $q \in Q$. The value $\mathscr{X}_q$ of a bundle $\mathscr{X}$ is called its *stalk* over q. A mapping s defined on a nonempty set $\operatorname{dom}(s) \subset Q$ is called a *section* over $\operatorname{dom}(s)$ if $s(q) \in \mathscr{X}_q$ for each $q \in \operatorname{dom}(s)$. A section s is called *almost global,* or *global,* whenever its domain $\operatorname{dom}(s)$ is respectively a comeager subset of Q or the whole of Q.

Let $S(Q, \mathscr{X})$ stands for the set of all global sections of $\mathscr{X}$ endowed with the structure of a vector lattice by letting $u \le v \Leftrightarrow (\forall q \in Q)\, u(q) \le v(q)$ and $(\alpha u + \beta v)(q) = \alpha u(q) + \beta v(q)$ $(q \in Q)$, where $\alpha, \beta \in \mathbb{R}$ and $u, v \in S(Q, \mathscr{X})$. For each section $s \in S(Q, \mathscr{X})$ we consider its point-wise norm $|\!|\!| s |\!|\!| : q \mapsto \|s(q)\|_q$ $(q \in Q)$.

A set $\mathscr{C} \subset S(Q, \mathscr{X})$ of global sections of a bundle of Banach lattices $\mathscr{X}$ over Q is called a *continuity structure* in $\mathscr{X}$ if the following conditions are met:

(1) $\mathscr{C}$ is a vector sublattice of $S(Q, \mathscr{X})$;

(2) for each $s \in \mathscr{C}$, the function $|\!|\!| s |\!|\!|$ is continuous;

(3) $\mathscr{C}$ is stalkwise dense in $\mathscr{X}$, i.e. for each $q \in Q$, the set $\big\{s(q) : s \in \mathscr{C}\big\}$ is dense in the stalk $\mathscr{X}_q$.

A *continuous bundle of Banach lattices over* Q is a pair $(\mathscr{X},\mathscr{C})$, where $\mathscr{X}$ is a bundle of Banach lattices over Q and $\mathscr{C}$ is a fixed continuity structure in $\mathscr{X}$. In the sequel we shall write simply $\mathscr{X}$ instead of $(\mathscr{X},\mathscr{C})$.

A section $u \in S(Q,\mathscr{X})$ is called *continuous* at $q_0 \in Q$ if the function $|||u-c||| : q \mapsto \|u(q)-s(q)\|_q$ $(q \in Q)$ is continuous at q_0 for every section $s \in \mathscr{C}$. If u is continuous at each $q_0 \in \operatorname{dom}(u)$, then u is said to be *continuous section* (see [14]) and [19] for more details).

9.2. Suppose that $\mathscr{X}$ is a continuous bundle of Banach lattices over an extremally disconnected compact space Q. Let u be a continuous section of $\mathscr{X}$ defined on a dense subset $D \subset Q$. Just as in 3.2 denote by $\bar{D}$ the totality of all points in Q at which u has limit and put $\bar{u}(q) := \lim_{p\to q} u(p)$ for all $q \in \bar{D}$. Then the set $\bar{D}$ is comeager in Q and the section $\bar{u}$ is continuous. The section $\bar{u}$ is called the *maximal extension* of u and denoted by $\operatorname{ext}(u)$. A continuous section u defined on a dense subset of Q is said to be *extended*, if $\operatorname{ext}(u)=u$. Denote by $C_\infty(Q,\mathscr{X})$ the space of all extended almost global sections of the bundle $\mathscr{X}$.

The set $C_\infty(Q,\mathscr{X})$ is endowed by the structure of a lattice normed vector lattice over $C_\infty(Q)$ in the following way. If $\lambda,\mu \in \mathbb{R}$ and $u,v \in C_\infty(Q,\mathscr{X})$, then the sum $\lambda u+\mu v$ is defined to be $\operatorname{ext}(\lambda u|_D+\mu v|_D)$, and $u \le v$ means hat $u(t) \le v(t)$ for all $q \in D$, where $D=\operatorname{dom}(u)\cap\operatorname{dom}(v)$. The maximal extension $\operatorname{ext}(|||u|||) \in C_\infty(Q)$ of the continuous function $|||u|||$ is taken as the norm $\lvert u\rvert$ of a section $u \in C_\infty(Q,\mathscr{X})$. The notation $\lvert u\rvert$ for the function $\operatorname{ext}(u)$ is also used if the continuous section u is defined on an arbitrary dense subset of Q. The space $C_\infty(Q,\mathscr{X})$ is a module over $C_\infty(Q)$, where $eu := \operatorname{ext}(e|_{\operatorname{dom}(u)}\cdot u|_{\operatorname{dom}(e)}$ for $e \in C_\infty(Q)$ and $u \in C_\infty(Q,\mathscr{X})$. If E is an order ideal in $C_\infty(Q)$ then we assign

$$E(\mathscr{X}) := \big\{u \in C_\infty(Q,\mathscr{X}) : \lvert u\rvert \in E\big\}.$$

It can be easily checked that $E(\mathscr{X})$ is a uniformly complete vector lattice.

9.3. Theorem. *Let $\mathscr{X}$ be a $u_1,\dots,u_N \in C_\infty(Q,\mathscr{X})$ and $[u_1,\dots,u_N] \subset K$. Then there exists a comeager subset $Q_0 \subset Q$ such that $Q_0 \subset \operatorname{dom}(u_1)\cap\dots\cap\operatorname{dom}(u_N)$, $[u_1(q),\dots,u_N(q)] \subset K$ for every $q \in Q_0$, and $\widehat{\varphi}(u_1,\dots,u_N) \in C_\infty(Q,\mathscr{X})$ is the maximal extension of the continuous section $q \mapsto \widehat{\varphi}(u_1(q),\dots,u_N(q))$ $(q \in Q_0)$, i.e.*

$$\widehat{\varphi}(u_1,\dots,u_N)(q)=\widehat{\varphi}(u_1(q),\dots,u_N(q)) \quad (q \in Q_0).$$

$\lhd$ The proof is a dully modification of the reasoning in 3.3. $\rhd$

9.4. A continuous Banach bundle $\mathscr{X}$ over an extremal compact space Q is called *ample* (or *complete* if every bounded almost global continuous section of it can be extended to a global continuous section. Put

$$C_\#(Q,\mathscr{X}) := \big\{u \in C_\infty(Q,\mathscr{X}) : \lvert u\rvert \in C(Q)\big\}.$$

Let $C(Q,\mathscr{X})$ denote the set of all global continuous sections of $\mathscr{X}$. Then a bundle $\mathscr{X}$ is ample if and only if $C_\#(Q,\mathscr{X})=C(Q,\mathscr{X})$.

9.5. Let G be a universally complete vector lattice with a fixed order unit and the corresponding structure of a semiprime f-algebra. A *duality pair* in G is a pair (E,D) of order dense ideals E and D in G such that the ideal $E^* := \{e^* \in G : (\forall e \in E)\, ee^* \in D\}$ is also order dense in G.

Take a lattice-normed space X with $\lvert X\rvert^{\perp\perp}=E$. The *operator-dual space* X^* is defined as follows. An operator $x^* : X \to D$ belongs to X^* if and only if there exists an element $0 \le c \in E^*$ such that

$$\langle x, x^* \rangle := x^*(x) \le c \ |x| \quad (x \in X).$$

The least element $0 \le c \in E^*$ satisfying the indicated relation exists. This element is denoted by $|x^*|$. The mapping $x^* \mapsto |x^*|$ is an E^*-valued norm in X^* and the following inequality holds:

$$\langle x, x^* \rangle \le |x| \ |x^*| \quad (x \in X).$$

Two lattice normed lattices X and Y over E are said to *isometrically isomorphic* if there exists a lattice isomorphism i of X into Y such that $\big|i(x)\big| = |x|$ for all $x \in X$.

9.6. Theorem. *Let $\mathscr{X}$ be an ample continuous bundle of Banach lattices, $S_1, \dots, S_N \in E(\mathscr{X})^*$, and $[S_1, \dots, S_N] \subset K$. Then there exist $v_1, \dots, v_N \in E^*(\mathscr{X}')$ and a comeager subset $Q_0 \subset Q$ such that*

(1) $Q_0 \subset \mathrm{dom}(u_1) \cap \dots \cap \mathrm{dom}(u_N)$;

(2) $[u_1(q), \dots, u_N(q)] \subset K$ *for every* $q \in Q_0$;

(3) *for every $\varphi \in \mathscr{H}(C, K)$ the map $q \mapsto \widehat{\varphi}(v_1(q), \dots, v_N(q))$ $(q \in Q_0)$ is a continuous section of $\mathscr{X}'$ over Q_0, and for all $u \in E(\mathscr{X})$ the representation holds:*

$$\big(\widehat{\varphi}(S_1, \dots, S_N)(u)\big)(q) = \langle u(q), \widehat{\varphi}(v_1(q), \dots, v_N(q)) \rangle \quad (q \in Q_0);$$

(4) $\big|\widehat{\varphi}(S_1, \dots, S_N)\big|(q) = \|\widehat{\varphi}(v_1(q), \dots, v_N(q))\|_{\mathscr{X}'(q)} \quad (q \in Q_0)$.

In particular, the lattice normed lattice $E(\mathscr{X})^*$ is isometrically isomorphic to $E^*(\mathscr{X}')$, where the isometric isomorphism is performed by associating with each section $v \in E^*(\mathscr{X}')$ the operator $u \mapsto \langle u, v \rangle$ from $E(\mathscr{X})$ to D, see [15]. (Here $\langle u, v \rangle$ denotes the coset of the function $\langle u_0(\cdot), v_0(\cdot) \rangle$ with $u_0 \in u$ and $v_0 \in v$.)

9.7. Now consider a nonzero measure space (Ω, Σ, μ) with the direct sum property. Let $\mathscr{X}$ be a bundle of Banach lattices over Ω. Denote by $S_\sim(\Omega, \mathscr{X})$ the set of all sections of $\mathscr{X}$ defined almost everywhere on Ω. A set of sections $\mathscr{C} \subset S_\sim(\Omega, \mathscr{X})$ is called a *measurability structure* on $\mathscr{C}$, if it satisfies the following conditions:

(a) $\lambda_1 c_1 + \lambda_2 c_2 \in \mathscr{C}$ and $|c| \in \mathscr{C}$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $c, c_1, c_2 \in \mathscr{C}$;

(b) the point-wise norm $|||c||| : \Omega \to \mathbb{R}$ of every element $c \in \mathscr{C}$ is measurable;

(c) the set $\mathscr{C}$ is stalkwise dense in $\mathscr{X}$.

If $\mathscr{C}$ is a measurability structure in $\mathscr{X}$ then we call the pair $(\mathscr{X}, \mathscr{C})$ a *measurable bundle of Banach lattices* over Ω. We shall usually write simply $\mathscr{X}$ instead of $(\mathscr{X}, \mathscr{C})$.

Let $(\mathscr{X}, \mathscr{C})$ be a measurable bundle of Banach lattices over Ω. We say that $s \in S_\sim(\Omega, \mathscr{X})$ is a *step-section*, if $s = \sum_{k=1}^n [A_k] c_k$ for some $n \in \mathbb{N}$, $A_1, \dots, A_n \in \Sigma$ and $c_1, \dots, c_n \in \mathscr{C}$. A section $u \in S_\sim(\Omega, \mathscr{X})$ is called *measurable* if, for every $L \in \Sigma$ with $\nu(L) < +\infty$, there is a sequence $(s_n)_{n \in \mathbb{N}}$ of step-sections such that $s_n(\omega) \to u(\omega)$ for almost all $\omega \in L$. The set of all measurable sections of $\mathscr{X}$ is denoted by $\mathscr{L}^0(\Omega, \Sigma, \mu, \mathscr{X})$ or $\mathscr{L}^0(\mu, \mathscr{X})$ for brevity.

9.8. Suppose that $\mathscr{X}$ is a measurable Banach bundle over Ω. Consider the equivalence relation $\sim$ in the set $\mathscr{L}^0(\mu, \mathscr{X})$: $u \sim v$ means that $u(\omega) = v(\omega)$ for almost all $\omega \in \Omega$. The coset containing $u \in \mathscr{L}^0(\mu, \mathscr{X})$ is denoted by $\tilde{u}$. The quotient set $L^0(\mu, \mathscr{X}) := L^0(\Omega, \Sigma, \mu, \mathscr{X}) := \mathscr{L}^0(\mu, \mathscr{X})/\sim$ is a vector lattice: $(s\tilde{u} + t\tilde{v}) = (su + tv)^\sim$ and $\tilde{u} \le \tilde{v} \Leftrightarrow u(\omega) \le v(\omega)$ for almost all $\omega \in \Omega$, where $s, t \in \mathbb{R}$ and $u, v \in \mathscr{L}^0(\mu, \mathscr{X})$. For every $u \in \mathscr{L}^0(\mu, \mathscr{X})/\sim$ we may define its (vector) *norm* $|u| := |\tilde{u}| := |||u|||^\sim \in L^0(\mu)$. It is clear that the vector lattice $L^0(\mu, \mathscr{X})$ is uniformly complete.

9.9. Let $\mathscr{X}$ be a measurable bundle of Banach lattices over Ω. Consider a lifting $\rho : L^\infty(\Omega) \to \mathscr{L}^\infty(\Omega)$. We call a mapping $\rho_{\mathscr{X}} : L^\infty(\mu, \mathscr{X}) \to \mathscr{L}^\infty(\mu, \mathscr{X})$ a *lifting* of $L^\infty(\mu, \mathscr{X})$ *associated with* ρ if, for all $u, v \in L^\infty(\Omega, \mathscr{X})$ and $e \in L^\infty(\Omega)$ the following relations hold:

(1) $\rho_{\mathscr{X}}(u) \in u$ and $\operatorname{dom}(\rho_{\mathscr{X}}(u)) = \Omega$;

(2) $\lvert\rho_{\mathscr{X}}(u)\rvert = \rho(\lvert u\rvert)$;

(3) $\rho_{\mathscr{X}}(u + v) = \rho_{\mathscr{X}}(u) + \rho_{\mathscr{X}}(v)$;

(4) $|\rho_{\mathscr{X}}(u)| = \rho_{\mathscr{X}}(|u|)$;

(5) $\rho_{\mathscr{X}}(eu) = \rho(e)\rho_{\mathscr{X}}(u)$;

(6) the set $\{\rho_{\mathscr{X}}(u) : u \in L^\infty(\Omega, \mathscr{X})\}$ is stalkwise dense in $\mathscr{X}$.

We say that $\mathscr{X}$ is a *liftable measurable bundle of Banach lattices* provided that there exists a lifting of $L^\infty(\Omega)$ and a lifting of $L^\infty(\Omega, \mathscr{X})$ associated with it. The following result is due to A. E. Gutman, see [15].

9.10. Theorem. *Let $\mathscr{X}$ be a liftable measurable bundle of Banach lattices over Ω. Then there exists (a unique) liftable measurable bundle of Banach lattices $\mathscr{X}'$ such that*

(1) *at each point $\omega \in \Omega$, the stalk $\mathscr{X}'(\omega)$ is a Banach sublattice of $\mathscr{X}(\omega)'$;*

(2) *if $u \in \mathscr{L}^0(\mu, \mathscr{X})$ and $u' \in \mathscr{L}^0(\mu, \mathscr{X}')$, then $\langle u, u'\rangle \in \mathscr{L}^0(\mu)$;*

(3) *for all $u \in \mathscr{L}^\infty(\mu, \mathscr{X})$ and $u' \in \mathscr{L}^\infty(\mu, \mathscr{X}')$, we have $\rho(\langle u, u'\rangle) = \langle \rho_{\mathscr{X}}(u), \rho_{\mathscr{X}'}(u')\rangle$, where $\rho_{\mathscr{X}}$ and $\rho_{\mathscr{X}'}$ are respective liftings of $\mathscr{X}$ and $\mathscr{X}'$ associated with ρ;*

(4) *if a bounded mapping $u' : \omega \mapsto u'(\omega)$ is such that, for every $u \in \mathscr{L}^\infty(\mu, \mathscr{X})$ the function $\langle u, u'\rangle$ is measurable and $\rho(\langle u, u'\rangle) = \langle \rho_{\mathscr{X}}(u), u'\rangle$, then $u' \in \mathscr{L}^\infty(\mu, \mathscr{X}')$.*

9.11. Theorem. *Let $S_1, \dots, S_N \in E(\mathscr{X})^*$, $e := \lvert S_1\rvert + \dots + \lvert S_N\rvert$, and $[S_1, \dots, S_N] \subset K$. Then there exist measurable sections $v_1, \dots, v_N \in \mathscr{L}^0(\mathscr{X}')$ such that*

(1) *$\tilde{v}_1, \dots, \tilde{v}_N \in E^*(\mathscr{X}')$;*

(2) *$[v_1(\omega), \dots, v_N(\omega)] \subset K$ for every $\omega \in \Omega$;*

(3) *for every $\varphi \in \mathscr{H}(C, K)$ the map $\omega \mapsto \widehat{\varphi}(v_1(\omega), \dots, v_N(\omega))$ $(\omega \in \Omega)$ is a measurable section of $\mathscr{X}'$ and for all $u \in E(\mathscr{X})$ and $\omega \in \Omega$ the representation holds:*

$$\rho_e\big(\widehat{\varphi}(S_1, \dots, S_N)(u)\big)(\omega) = \langle u(\omega), \widehat{\varphi}(v_1(\omega), \dots, v_N(\omega))\rangle \quad (\omega \in \Omega);$$

(4) *for every $\varphi \in \mathscr{H}(C, K)$ the map $\omega \mapsto \|\widehat{\varphi}(v_1(\omega), \dots, v_N(\omega))\|$ $(\omega \in \Omega)$ is measurable and the corresponding coset coincide with $\lvert\widehat{\varphi}(S_1, \dots, S_N)\rvert$.*

10. Functions of Dominated Operators

In this section we prove two representation theorems for $\widehat{\varphi}(T_1, \dots, T_N)$ with dominate operators $T_1, \dots, T_N$.

10.1. At first, we introduce a vector lattice $E_w(X')$ of weakly measurable vector-valued functions. Let (Ω, Σ, μ) be a measure space with the direct sum property, E an order dense ideal in $L^0(\Omega, \Sigma, \mu)$, and X a Banach lattice. An X'-valued function u defined almost everywhere on Ω is called $\sigma(X', X)$-*measurable* or simply X-*measurable* if, for each $x \in X$, the function $t \mapsto \langle x, u(t)\rangle$ $(t \in \Omega)$ is measurable. Denote the coset of the last function by $\langle x, u\rangle$, so that $\langle u, z\rangle \in L^0(\mu)$. Let $\mathscr{L}^0_w(\Omega, X')$ be the set of X-measurable vector-valued functions $u : \Omega \to X'$. We say that X-measurable vector-functions u and v are X-equivalent and write $u \simeq v$ if, for each $x \in X$, the measurable functions $\langle x, u(\cdot)\rangle$ and $\langle x, v(\cdot)\rangle$ are equal almost everywhere.

Consider the quotient set $L^0_w(\mu, X') := L^0_w(\Omega, \Sigma, \mu, X') := \mathscr{L}^0_w(\Omega, X')/\simeq$ and define vector space structure in it by setting $\alpha\widetilde{u} + \beta\widetilde{v} := (\alpha u + \beta v)^{\sim}$. For a coset $\widetilde{u} \in L^0_w(\mu, X')$ with $u \in \mathscr{L}^0_w(\Omega, X')$ put $\langle x, \widetilde{u}\rangle := \langle x, u\rangle$. The set $R(\widetilde{u}) := \big\{\langle \widetilde{u}, z\rangle : z \in Z, \|z\| \leq 1\big\}$ is order-bounded in $L^0(\mu)$ and we can assign

$$|\!|\widetilde{u}|\!| := \sup\big\{\langle x, u\rangle : x \in X,\ \|x\| \leq 1\big\},$$

where the supremum is taken in $L^0(\Omega, \Sigma, \mu)$. Define now the set

$$E_w(X') := \big\{u \in L^0_w(\mu, X') : |\!|u|\!| \in E\big\}.$$

It is easy to verify that for every order ideal $E \subset L^0(\mu)$ the space $E_w(X')$ endowed with the operations and E-valued norm $|\!|\cdot|\!|$ induced from $L^0_w(\Omega, \Sigma, \mu, X')$ is a Banach–Kantorovich space over $L^0(\Omega, \Sigma, \mu)$ [19].

10.2. Let ρ is a lifting of $L^\infty(\mu)$ and $\rho(\widetilde{e}) = e$ for some $0 \leq e \in \mathscr{L}^0(\mu)$. Given $g \in \mathscr{L}^0(\mu)$, defined the function g/e by $(g/e)(\omega) = 0$ if $e(\omega) = 0$ and $(g/e)(\omega) = g(\omega)/e(\omega)$ if $e(\omega) > 0$. Put

$$\mathscr{L}^\infty_e(\mu) := \{g \in \mathscr{L}^0(\mu) : \ g/e \in \mathscr{L}^\infty(\mu)\},$$
$$L^\infty_e(\mu) := \mathscr{L}^\infty_e(\mu)/\sim,$$
$$\rho_e(\widetilde{g}) := e\rho(\widetilde{g/e}) \quad \big(g \in \mathscr{L}^\infty_e(\mu)\big)$$

Then $L^\infty_e(\mu)$ is an ideal space on (Ω, Σ, μ) and ρ_e is a lattice isomorphism of $L^\infty_e(\mu)$ into $\mathscr{L}^\infty_e(\mu)$. Moreover, $\rho_e(\widetilde{g}) \in \widetilde{g}$ for any $g \in L^\infty_e(\mu)$ and $\rho_e(\widetilde{e}) = e$.

Let (g_α) is an order bounded subset of $\mathscr{L}^\infty_e$ and let $\rho_e(\widetilde{g}_\alpha) = g_\alpha$ for every α. Then the point-wise supremum $g(t) = \sup_\alpha\{g_\alpha(t)\}$ is measurable and $\widetilde{g} = \sup \widetilde{g}_\alpha$ in $L^0(\mu)$.

10.3. We now recall two types of *dominated operators*, see [19]. Let X be a Banach space and E an ideal space. An operator $S : X \to E$ is dominated if the image of the unit ball in X is order bounded in E. The element $|\!|S|\!|$ defined as

$$|\!|S|\!| = \sup\{|Sx| : \ x \in X,\ \|x\| \leq 1\}$$

is called the *abstract norm* of S. The linear space of all dominated operators $M(X, E)$ is denoted also by $L_A(X, F)$ and is called the space of operators with abstract norm. If X is a Banach lattice then $M(X, E)$ is a Dedekind complete vector lattice. Actually, the exact dominant is presented by the mapping $t \mapsto t\,|\!|S|\!|$ $(t \in \mathbb{R})$.

An operator $S : E \to Y$ is dominated if there exists a positive functional e^* on E such that

$$\|Te\| \leq \langle |e|, e^*\rangle \quad (e \in E).$$

The exact dominant is calculated as follows:

$$|\!|T|\!|\, e = \sup\left\{\sum_{k=1}^{n} \|Te_k\| : \ e_1, \dots, e_n \in E_+,\ \sum_{k=1}^{n} e_k = e,\ n \in \mathbb{N}\right\} \quad (e \in E_+).$$

10.4. Theorem. *Let X be a Banach lattice, E an ideal space on (Ω, Σ, μ), and $S \in M(X, E)$ with $\widetilde{e} := |\!|S|\!|$ for some $e \in \mathscr{L}^0(\mu)$. Then there exists an X-measurable function $v : \Omega \to X'$ such that*

(1) *$\widetilde{v} \in E_w(X')$;*

(2) *$\rho_e(Sx)(\omega) = \langle x, v(\omega)\rangle$ for all $x \in X$ and $\omega \in \Omega$;*

(3) *the function* $\omega \mapsto \|v(\omega)\|$ $(\omega \in \Omega)$ *is measurable and the corresponding coset coincides with* $\boldsymbol{|}S\boldsymbol{|}$*;*

(4) *the function* $\omega \mapsto |v(\omega)|$ $(\omega \in \Omega)$ *is* X*-measurable and for every* $x \in X$ *we have* $\rho_e(|S|x)(\omega) = \langle x, |v(\omega)|\rangle$ *for almost all* $\omega \in \Omega$*.*

$\lhd$ Define $v : \Omega \to X'$ by $\langle x, u(\omega)\rangle = \rho_e(Sx)(\omega)$ $(x \in X,\, \omega \in \Omega)$. It is well known that v obeys (1)–(3) for any Banach space X, see [3, Theorem 2.1]. Let X be a Banach lattice and $0 \le x \in X$. Then $|S|x = \sup_{0 \le |a| \le x} S(a)$. According to 10.2 the point-wise supremum g_x of the family $(\langle a, v(\omega)\rangle)_{0 \le |a| \le x}$ is measurable and the coset of g_x is equal to $|S|x$. It remains to observe that $g_x(\omega) = \langle x, |v(\omega)|\rangle$ for every $\omega \in \Omega$. $\rhd$

10.5. Remark. If we deal with equivalence classes of measurable functions instead of measurable functions, then we have the following simple representation result: There exists a norm reserving lattice isomorphism $\iota : S \to \mathbf{v} := \iota(S)$ of $M(X, E)$ onto $E_w(X')$ such that $Sx = \langle x, v\rangle$ $(x \in X)$, see [3, Theorem 2.2]. As an easy corollary to this fact we get the representation $\widehat{\varphi}(S_1, \dots, S_N)x = \langle x, \widehat{\varphi}(\mathbf{v}_1, \dots, \mathbf{v}_N)\rangle$ $(x \in X)$ for any finite collection $S_1, \dots, S_N \in M(X, E)$, where $\mathbf{v}_i := \iota(S_i)$. However, we have to work with individual functions if we want to describe explicitly $\widehat{\varphi}(\mathbf{v}_1, \dots, \mathbf{v}_N)$ or at least $|\mathbf{v}| \in E_w(X)$. The choice of a representing function $v \in \mathbf{v}$ is not suitable for this purpose. Indeed, by 10.4 (2) there is an X-measurable function $v_0 : \Omega \to X$ such that $\rho_e(\langle x, \mathbf{v}\rangle)(\omega) = \langle x, v_0(\omega)\rangle$. At the same time for each $x \in X$ we have $\langle x, v_0(\omega)\rangle = \langle x, |v(\omega)|\rangle$ for almost every $\omega \in \Omega$ according to 10.4 (4). This problem disappear if X' have the Radon-Nikodým property, since in this event $E_w(X') = E(X')$ and, if v is a representing function for S, then $|v| : \omega \mapsto |v(\omega)|$ is a representing function for $|S|$, i.e v_0 and $|v|$ are equal almost everywhere. But for general X it is not true and another tool should be involved. Such tool was invented by A. E. Gutman in [15]: the spaces $E(X)$ and $E(X')$ are representable as the spaces of measurable sections of liftable measurable Banach bundles. An easy modification of Gutman's approach covers the case of vector lattices.

10.6. Theorem. *Let* X *be a Banach lattice and* (Ω, Σ, μ) *a measure space with the direct sum property. There exists a liftable measurable bundle of Banach lattices* $\mathscr{X} := (\mathscr{X}(\omega))_{\omega \in \Omega}$ *over* Ω*, unique to within a* ρ*-isometry, and such that if* $\mathscr{X}' := (\mathscr{X}'(\omega))_{\omega \in \Omega}$ *is the dual measurable Banach bundle, than*

(1) X *is a Banach sublattice of each stalk* $\mathscr{X}(\omega)$ *and* $\mathscr{X}'(\omega)$ *is a Banach sublattice of* $\mathscr{X}(\omega)'$ *for all* $\omega \in \Omega$*;*

(2) *the respective liftings* $\rho_{\mathscr{X}}$ *and* $\rho_{\mathscr{X}'}$ *of* $\mathscr{X}$ *and* $\mathscr{X}'$ *are module preserving, are associated with* ρ*, and* $\rho_{\mathscr{X}}(\widetilde{c}) = c$ *for all constant functions* $c : \Omega \to X$*;*

(3) *for every section* $u \in \mathscr{L}^0(\Omega, \mathscr{X})$ *the function* $\overline{u}$ *coinciding with* u *on* $u^{-1}(X)$ *and vanishing on* $\Omega \setminus u^{-1}(X)$ *is contained in* $\mathscr{L}^0(\mu, X)$*;*

(4) *for every section* $v \in \mathscr{L}^0(\mu, \mathscr{X}')$ *the function* $v_X : \omega \mapsto v(\omega)|_X$ *from* Ω *to* X' *is contained in* $\mathscr{L}^0_w(\mu, X')$*;*

(5) *the mapping sending the coset of* $u \in \mathscr{L}^0(\mu, \mathscr{X})$ *to the coset of* $\overline{u} \in \mathscr{L}^0(\mu, X)$ *is a lattice isomorphism and an isometry of* $L^0(\mu, \mathscr{X})$ *onto* $L^0(\mu, X)$*;*

(6) *the mapping sending the coset of* $v \in \mathscr{L}^0(\mu, \mathscr{X}')$ *to the coset of* $v_X \in \mathscr{L}^0_w(\mu, X')$ *is a lattice isomorphism and an isometry of* $L^0(\mu, \mathscr{X}')$ *onto* $L^0_w(\mu, X')$*.*

$\lhd$ For an arbitrary Banach space X this fact was established by A. E. Gutman [15]. Consider the trivial Banach bundle $\mathscr{X}_0 : \omega \mapsto X$ and let the totality of constant functions $c : \Omega \to X$ be taken as the measurability structure $\mathscr{C}$ of $\mathscr{X}_0$.

Then $\mathscr{X}_0$ can be densely embedded into a liftable measurable Banach bundle $\mathscr{X}$. I. G. Ganiev [13, Theorem 2.1] observed that if X is a Banach lattice then then (since the measurability structure is a vector lattice) $\mathscr{X}$ is a measurable bundle of Banach lattice with the lifting $\rho_{\mathscr{X}}$ associated with ρ; moreover, $\rho_{\mathscr{X}}$ is module preserving. According to Theorem 10.4 $E_w(X')$ is a vector lattice. At the same time $\mathscr{X}'$ is the representing measurable Banach bundle for the space $E_w(X')$ and thus is a measurable bundle of Banach lattices and $\rho_{\mathscr{X}'}$ is module preserving. It remains to observe that the linear isometries indicated in (5) and (6) are order isomorphisms if X is a Banach lattice. $\triangleright$

We say that $(\mathscr{X}, \mathscr{X}')$ is a *representing pair of measurable Banach bundles* for $(E(X), E_w(X'))$.

Now we are ready to prove our representation result for $\widehat{\varphi}(S_1, \dots, S_N)$ with $S_1, \dots, S_N \in M(X, E)$. In the sequel we put $\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega)) = 0$ whenever $u_1, \dots, u_N \in \mathscr{L}^0(\mu, X')$ but $\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))$ cannot be correctly defined in X', i.e. $[u_1, \dots, u_N]$ is not contained in K.

10.7. Theorem. *Let X be a Banach lattice, E an ideal space on (Ω, Σ, μ), and $(\mathscr{X}, \mathscr{X}')$ a representing pair of measurable Banach bundles for $(E(X), E_w(X'))$. Consider $\varphi \in \mathscr{H}(\mathbb{R}^N, K)$ and $S_1, \dots, S_N \in M(X, E)$ with $[S_1, \dots, S_N] \subset K$ and put $e := \lvert S_1\rvert + \dots + \lvert S_N\rvert$, $S := \widehat{\varphi}(S_1, \dots, S_N)$. Then there exist measurable sections $u_1, \dots, u_N \in \mathscr{L}^0(\Omega, \mathscr{X}')$ such that*

(1) *$\widetilde{u}_1, \dots, \widetilde{u}_N \in E(\mathscr{X}')$;*

(2) *$[u_1(\omega), \dots, u_N(\omega)] \subset K$ for all $\omega \in \Omega$;*

(3) *the function $\omega \mapsto \widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))$ $(\omega \in \Omega)$ is a measurable section of $\mathscr{X}'$ and for all $x \in X$ and $\omega \in \Omega$ we have*

$$\rho_e(Sx)(\omega) = \langle x, \widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\rangle;$$

(3) *the function $\omega \mapsto \|\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\|_{\mathscr{X}'(\omega)}$ $(\omega \in \Omega)$ is measurable and the corresponding coset coincides with $\lvert S\rvert$.*

$\triangleleft$ Theorems 10.4 and 10.5 imply that there exists a lattice isomorphism ι of $M(X, E)$ onto $E(\mathscr{X}')$ such that for every $S \in M(X, E)$ we have $Sx = \langle \widetilde{x}, \iota(S)\rangle$, where $\widetilde{x}$ stands for the coset of the constant function $\omega \mapsto x$ $(\omega \in \Omega)$. Put $L^e_w(\mu, X) := \{\mathbf{u} \in L^0_w(\mu, X) : \lvert\mathbf{u}\rvert \in L^e_w(\mu)\}$ and consider the corresponding space of X-measurable vector functions $\mathscr{L}^e_w(\mu, X') := \bigcup\{\mathbf{u} : \mathbf{u} \in L^e_w(\mu, X')\}$. If $\mathbf{u} \in L^e_w(\mu, X')$, then $|\rho_e(\langle x, \mathbf{u}\rangle)| \leq \|x\| e$ for all $x \in X$ and the vector function $\omega \mapsto \rho_e(\langle \cdot, \mathbf{u}\rangle) \in X'$ lie in $\mathscr{L}^e_w(\mu, X')$. Thus, we can define an operator $\overline{\rho}_e$ from $L^e_w(\mu, X')$ to $\mathscr{L}^e_w(\mu, X')$ by putting $\langle x, \overline{\rho}_e(\mathbf{u})\rangle = \rho_e(\langle x, \mathbf{u}\rangle)$ for all $x \in X$ and $\mathbf{u} \in L^e_w(X')$. In view of 10.4 $\overline{\rho}_e$ is a linear operator; moreover, $\overline{\rho}_e(\mathbf{u}) \in \mathbf{u}$ and $\overline{\rho}_e(g\mathbf{u}) = \rho(g)\overline{\rho}_e(\mathbf{u})$ for all $\mathbf{u} \in L^e_w(\mu, X)$ and $g \in L^\infty(\mu)$.

It follows from 10.4 (4) that $\overline{\rho}_e(|\mathbf{u}|)(\omega) = |\overline{\rho}_e(\mathbf{u})(\omega)|$ for all $\mathbf{u} \in L^e_w(\mu, X)$ and $\omega \in \Omega$. Therefore, $\mathbb{L} := \overline{\rho}_e\big(L^e_w(\mu, X)\big)$ is a vector sublattice of the vector lattice $\prod_{\omega\in\Omega} \mathscr{X}'(\omega)$ with the point-wise ordering and $\overline{\rho}_e$ is a lattice isomorphism of $L^e_w(\mu, X)$ onto $\mathbb{L}$. In particular, $\mathbb{L}$ is a uniformly complete vector lattice. Put $h := \overline{\rho}_e \circ \iota$ and $u_i := h(S_i)$ $(i := 1, \dots, N)$. Clearly, $u_1, \dots, u_N \in \mathscr{L}^e_w(X')$ and thus $\widetilde{u}_1, \dots, \widetilde{u}_N \in L^e_w(X') \subset E_w(X')$. By Proposition 2.6 $[u_1, \dots, u_N] \subset K$ and

$$\begin{aligned}\rho_e\big(\widehat{\varphi}(S_1, \dots, S_N)x\big) &= \rho_e\big(\langle x, \iota(\widehat{\varphi}(S_1, \dots, S_N))\rangle\big)\\ &= \langle x, h(\widehat{\varphi}(S_1, \dots, S_N))\rangle = \langle x, \widehat{\varphi}(u_1, \dots, u_N)\rangle.\end{aligned}$$

For any $\omega \in \Omega$ define a lattice homomorphism $\widehat{\omega} : \mathbb{L} \to \mathscr{X}'(\omega)$ by $\widehat{\omega}(v) := v(\omega)$. Again by Proposition 2.6 we have $[u_1(\omega), \dots, u_N(\omega)] \subset K$ and $\widehat{\omega}(\widehat{\varphi}(u_1, \dots, u_N)) = \widehat{\varphi}(v_1(\omega), \dots, v_N(\omega))$ from which we have

$$\rho_e\big(\widehat{\varphi}(S_1, \dots, S_N)x\big)(\omega) = \langle x, \widehat{\varphi}(v_1(\omega), \dots, v_N(\omega))\rangle.$$

Now it is clear that the function $\widehat{\varphi}(u_1(\cdot), \dots, u_N(\cdot))\rangle$ is X-measurable, the function $\|\widehat{\varphi}(u_1(\cdot), \dots, u_N(\cdot))\|$ is measurable, and $\lvert S \rvert$ is the coset of $\|\widehat{\varphi}(u_1(\cdot), \dots, u_N(\cdot))\|$. $\rhd$

10.8. Theorem. *Let X be a Banach lattice, E an ideal space on (Ω, Σ, μ) with point separating dual $E_n^\sim$, F an order dense ideal in $E_n^\sim$, and $(\mathscr{X}, \mathscr{X}')$ a representing pair of measurable Banach bundles for $(E(X), E_w(X'))$. Let the dominated operators $S_1, \dots, S_N \in M_F(E, X')$ with $[S_1, \dots, S_N] \subset K$ are given, and $S := \widehat{\varphi}(S_1, \dots, S_N)$. Then there exist measurable sections $u_1, \dots, u_N \in \mathscr{L}^0(\Omega, \mathscr{X}')$ such that*

(1) *$\widetilde{u}_1, \dots, \widetilde{u}_N \in F(\mathscr{X}')$;*

(2) *$[u_1(\omega), \dots, u_N(\omega)] \subset K$ for all $\omega \in \Omega$;*

(3) *for every $\varphi \in \mathscr{H}(\mathbb{R}^N, K)$, the function $\omega \mapsto \widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))$ $(\omega \in \Omega)$ is a measurable section of $\mathscr{X}'$ and the representation holds*

$$\langle x, S(e)\rangle = \int_\Omega e(\omega)\langle x, \widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\rangle \, d\mu(\omega) \quad (e \in E,\ x \in X);$$

(4) *the function $\omega \mapsto \|\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\|$ $(\omega \in \Omega)$ is measurable and*

$$\lvert S \rvert (e) = \int_\Omega e(\omega)\|\widehat{\varphi}(u_1(\omega), \dots, u_N(\omega))\| \, d\mu(\omega) \quad (e \in E).$$

$\lhd$ For any Banach space X the mapping which sends a dominated operator $S \in M_F(E, X')$ to the restriction $h(S) := S'|_X$ of its adjoint $S' : X'' \to F$ to X is an isomorphism of $M(E, X')$ onto $M(X, F)$; moreover, $\lvert S \rvert = \big\lvert h(S) \big\rvert$ for all S, see [4, Theorem 3.3]. It remains to observe that if X is a Banach lattice, then h is also a lattice isomorphism and apply Theorem 10.5. $\rhd$

Acknowledgements

The author would like to thank A. E. Gutman and S. S. Kutateladze for reding the text and making valuable remarks, R. Drnovšek and A. Peperko for attracting his attention to the paper [17] and to S. M. Sitnik for useful discussions on inequalities.

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Кусраев Анатолий Георгиевич

Однородное функциональное исчисление
в векторных решетках

Ответственный за выпуск
Биченова М. С.

Подписано в печать 9.09.2008.
Формат $60 \times 84\,{}^{1}/_{8}$. Усл. п. л. 3,95. Тираж 50 экз.