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Цель статьи — исследование инъективных банаховых решеток на основе булевозначного принцип переноса из теории AL-пространств в теорию инъективных банаховых решеток. Устанавливается, что инъективная банахова решетка при погружении в подходящий булевозначный универсум превращается в AL-пространство. Тем самым, каждая теорема об AL-пространстве, доказанная в рамках теории множеств Цермело — Френкеля, имеет свой аналог для исходной инъективной банаховой решетки, интерпретируемой как булевозначное AL-пространство. Перевод теорем об AL-пространствах в теоремы об инъективных банаховых решетках осуществляется с помощью общих процедур булевозначного анализа.

Ключевые слова: AL-пространство, AM-пространство, инъективная банахова решетка, атомическая банахова решетка, булевозначное представление, оператор Магарам, регулярный оператор, положительно суммирующий оператор.

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The aim of this paper is to give a Boolean valued analysis approach to the theory of injective Banach lattices and establish a Boolean valued transfer principle from AL-spaces to injective Banach lattices. We prove that every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space. According to this fact and fundamental principles of Boolean valued models, each theorem about the AL-space within Zermelo–Fraenkel set theory has an analog for the original injective Banach lattice interpreted as the Boolean-valued AL-space. Translation of theorems from AL-spaces to injective Banach lattices is carried out by appropriate general operations of Boolean-valued analysis.

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Keywords: AL-space, AM-space, injective Banach lattice, atomic Banach lattice, Boolean valued representation, Maharam operator, regular operator, cone absolutely summing operator.

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BOOLEAN VALUED ANALYSIS APPROACH TO INJECTIVE BANACH LATTICES

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1. Introduction

A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ with $\|T_0\| = \|T\|$. Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a positive contractive projection from Y onto X. Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [7, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms.

Lotz [27] was the first who introduced this concept and proved among other things that a Dedekind complete AM-space with unit and an AL-space are injective Banach lattices ([27, Propositions 2.1 and 3.2]). Cartwright [13, Chapter 2] was able to characterize injective Banach lattices in terms of order intersection property. Haydon [19] discovered that a general injective Banach lattice has a mixed AM-AL-structure.

The aim of this paper is to give a Boolean valued analysis approach to the theory of injective Banach lattices and establish a *Boolean valued transfer principle* from AL-spaces to injective Banach lattices. We prove that every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space. According to this fact and fundamental principles of Boolean-valued models, each theorem about the AL-space within Zermelo–Fraenkel set theory has an analog for the original injective Banach lattice interpreted as the Boolean-valued AL-space. Translation of theorems from AL-spaces to injective Banach lattices is carried out by appropriate general operations of Boolean-valued analysis.

In Section 2 we recall some basic notions and facts about injective Banach lattices, Banach spaces with mixed norm and Banach spaces with Boolean algebra of projections. Section 3 collects some Boolean-valued requisits. Section 4 is devoted to Boolean valued interpretation of Banach lattice theory. Section 5 presents some representation results and examples of injective Banach lattices. Section 6 contains some instances of transferring of several well known AL-results to the results on injective Banach lattices by means of Boolean valued interpretation. Section 7 deals with the operators admitting factorization through injective Banach lattice and represents Boolean valued interpretation of a portion of the theory of cone absolutely summing operators. Section 8 presents a Boolean valued representation of the notion of atomic Banach lattice and complete description of the corresponding class of injective Banach lattices. In this work we deal only with the isometric theory, i.e. with 1-injective Banach lattices. For λ-injective Banach lattices ($\lambda > 1$) see [26, 28].

In what follows X and Y denote Banach lattices, while $\mathscr{L}(X, Y)$ and $\mathscr{L}^r(X, Y)$ stand respectively for bounded and regular operators from X into Y and $X^* := \mathscr{L}(X, \mathbb{R})$. For the theory of Banach lattices and positive operators we refer to the books [4, 6, 29]. The needed information on the theory of Boolean-valued models is briefly presented in [22, Chapter 9]; details may be found in [8, 23, 36].

We denote by $\mathbb{P}(X)$ the Boolean algebra of all band projections in a vector lattice X. Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$, while $\Lambda := \Lambda(\mathbb{B})$ is a Dedekind complete AM-space with unit such that $\mathbb{B} = \mathbb{P}(\Lambda)$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$.

We let := denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ symbolize the naturals, the rationals, and the reals.

2. Preliminaries

Injective Banach lattices. A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ for all $x \in X$, where $\pi^\perp := I_X - \pi$. The collection of all M-projections forms a subalgebra $\mathbb{M}(X)$ of the Boolean algebra of all band projections $\mathbb{P}(X)$ in X. The notion of an M-projection plays a crucial role in the theory of injective Banach lattices. In a wider context of a general Banach space theory the concept is presented in [9] and [18].

Consider a complete Boolean algebra $\mathbb{B}$. A Banach lattice X is said to be a $\mathbb{B}$-cyclic if $\mathbb{B}$ is a subalgebra of $\mathbb{M}(X)$ and for every partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and every norm bounded net $(x_\xi)_{\xi\in\Xi}$ there exists a (unique) $x \in X$ such that $\pi_\xi x_\xi = \pi_\xi x$ for all $\xi \in \Xi$. If X has the Fatou and Levi properties, then $\mathbb{M}(X)$ is an order closed subalgebra of the complete Boolean algebra $\mathbb{P}(X)$. In this event X is a $\mathbb{B}$-cyclic Banach lattice for every order closed subalgebra $\mathbb{B} \subset \mathbb{M}(X)$. A geometric property which enables us to characterize injectivity was discovered by Cartright [13].

Definition 2.1. (1) A Banach lattice X has the *finite order intersection property* if, given $z \in X_+$, finite collections $x_1, \dots, x_n \in X_+$, $y_1, \dots, y_m \in X_+$, and strictly positive reals $r_1, \dots, r_n \in \mathbb{R}_+$, $s_1, \dots, s_m \in \mathbb{R}_+$ such that $\|x_i\| \leqslant r_i$, $\|y_j\| \leqslant s_j$, and $\|x_i + y_j + z\| \leqslant r_i + s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$, there exist $u, v \in X_+$ with $z = u+v$, $\|x_i+u\| \leqslant r_i$, and $\|y_j+v\| \leqslant s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$.

(2) A Banach lattice X has the *Cartwright property* if, given $x_1, x_2, y \in X_+$ and $0 < r_1, r_2 \in \mathbb{R}$ with $\|x_1\| \leqslant r_1$, $\|x_2\| \leqslant r_2$, and $\|x_1 + x_2 + y\| \leqslant r_1 + r_2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant r_1$, and $\|x_2 + y_2\| \leqslant r_2$.

(3) A Banach lattice X has the *splitting property* if, given $x_1, x_2, y \in X_+$ with $\|x_1\| \leqslant 1$, $\|x_2\| \leqslant 1$, and $\|x_1 + x_2 + y\| \leqslant 2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant 1$, and $\|x_2 + y_2\| \leqslant 1$.

The above three properties are equivalent and characterize Banach lattices with the injective second dual as shown by Cartwright [13, Theorems 2.9 and 3.6]

Theorem 2.2. *A Banach lattice has the finite order intersection property if and only if it has the Cartwright property if and only if it has the splitting property.*

Theorem 2.3. *A Banach lattice has the splitting property if and only if its second dual is injective.*

The results from Haydon [19, Theorem 3F(ii) and Corollary 5D] can be summarized as follows. Concerning the terms Levi property and Fatou property see [4, Definition 7] and [15].

Theorem 2.4. *A Banach lattice is injective if and only if it has Cartwright, Fatou, and Levi properties. An injective Banach lattice is an AL-space if and only if there is no M-projection in it other than* 0 *and* I_X.

Banach–Kantorovich spaces. Recall that a *Banach–Kantorovich space* over a Dedekind complete vector lattice E is a vector spaces X with a decomposable norm $|\!\cdot\!| : X \to E$ which is norm complete with respect to order convergence in E. Decomposability means that for all $e_1, e_2 \in E_+$ and $x \in X$, with $|x| = e_1 + e_2$ there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = e_k$ $(k := 1, 2)$. If a Banach–Kantorovich space is in addition a vector lattice and the norm is monotone ($|x| \leqslant |y| \Rightarrow |x| \leqslant |y|$) then it is called a *Banach–Kantorovich lattice.* Any Banach–Kantorovich lattice is a lattice ordered module over the f-algebra $\mathrm{Orth}(|X|^{\perp\perp})$, where $|X| = \{|x| : x \in X\}$. A detailed presentation see in [22].

Definition 2.5. A positive operator $T : X \to F$ is said to have the *Levi property* if $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in F. A *Maharam operator* is an order continuous order intervals preserving ($\equiv T([0, x] = [0, Tx])$ for all $x \in X_+$) operator.

Consider vector lattices X and F, with F order complete, and an operator $\Phi \in L_+(X, F)$. Suppose that Φ is strictly positive ($\equiv x > 0$ implies $\Phi(x) > 0$) and put $|x| := \Phi(|x|)$ $(x \in X)$. Then $(X, |\!\cdot\!|)$ is a lattice-normed space. The *bo*-completion of X denoted by $L^1(\Phi)$ is a Banach–Kantorovich lattice, see [22, Theorems 2.2.8 and 2.2.11]. It is easy to see that $L^1(\Phi) = X$ if and only if Φ is a strictly positive Maharam operator with the Levi property.

Assume that X is a vector lattice endowed with an E-valued norm $x \mapsto |x| \in E$ $(x \in X)$ which is monotone in the sense that $x \leqslant y$ implies $|x| \leqslant |y|$. Then we are able to introduce a *mixed norm* in X by

$$|\!|\!| x |\!|\!| := \big\| |x| \big\| \quad (x \in X).$$

In this event $(X, |\!|\!|\cdot|\!|\!|)$ is called a *normed lattice with mixed norm* and a *Banach lattice with mixed norm* if, in addition, it is norm complete. It can be easily seen that $(X, |\!|\!|\cdot|\!|\!|)$ is a Banach lattice if and only if $(X, |\!\cdot\!|)$ is *br*-complete, i .e. norm complete with respect to uniform convergence in E, [22, Theorem 7.1.2].

Banach spaces with Boolean algebra of projections. Let X be a Banach space and let $\mathscr{L}(X)$ be a set of all bounded linear operators in it. Assume that the mapping $\varphi := \varphi_X : \mathbb{B} \to \mathscr{L}(X)$ is injective and satisfies the following conditions:

(1) $\varphi(b)$ is a norm one projection for each $b \in \mathbb{B}$;

(2) $\varphi(\mathbb{1})$ and $\varphi(\mathbb{O})$ coincide with an identical and a zero operator, respectively;

(3) the projections $\varphi(b)$ and $\varphi(b')$ commute for all $b, b' \in \mathbb{B}$;

(4) $\varphi(b \vee b') = \varphi(b) \circ \varphi(b')$ and $\varphi(b^*) = I_X - \varphi(b)$ hold for all b and b'.

Then the set $\mathscr{B} := \varphi(\mathbb{B})$ is said to be a complete Boolean algebra of projections in the space X. In the sequel we identify Boolean algebras $\mathbb{B}$ and $\mathscr{B}$ and speak about the Boolean algebra of projections $\mathbb{B} \subset \mathscr{L}(X)$.

If $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$ and $(x_\xi)_{\xi\in\Xi}$ is a family in X, then the element $x \in X$, for which $b_\xi x_\xi = b_\xi x$ whatever be $\xi \in \Xi$, is called a *mixing* of (x_ξ) with respect to (b_ξ). A Banach space X is said to be $\mathbb{B}$-*cyclic* if $\mathbb{B} \subset \mathscr{L}(X)$ and the following conditions hold:

(5) a mixing of any bounded family in X with respect to any partition of unity in $\mathbb{B}$ (with the same index set) exists and is unique;

(6) the unit ball of X is closed with respect to any mixings.

Let X and Y be Banach spaces; moreover, $\mathbb{B} \subset \mathscr{L}(X)$ and $\mathbb{B} \subset \mathscr{L}(Y)$. An operator $T : X \to Y$ is called $\mathbb{B}$-*linear*, if it is linear and commutes with all projections from $\mathbb{B}$, i.e., if $b\circ T = T \circ b$. (Here we mean $\varphi_Y(b)\circ T = T\circ\varphi_X(b)$). Denote the set of all bounded $\mathbb{B}$-linear operators from X into Y by $\mathscr{L}_{\mathbb{B}}(X, Y)$. Then $Z := \mathscr{L}_{\mathbb{B}}(X, Y)$ is a Banach space and, clearly, $\mathbb{B} \subset Z$. In fact, the projection $\varphi_Z(b)$ can be defined by the formula $T \mapsto b\circ T$ $(T \in Z)$. It is easily verified that the space Z is $\mathbb{B}$-cyclic if Y is $\mathbb{B}$-cyclic. The reverse statement is also true if $X \neq \{0\}$. Call a bijective $\mathbb{B}$-linear operator a $\mathbb{B}$-*isomorphism* and if, in addition, it is norm-preserving, we shall speak about an *isometric* $\mathbb{B}$-*isomorphism.* The space $X^\# := \mathscr{L}_{\mathbb{B}}(X, \Lambda)$ is called $\mathbb{B}$-dual to X.

Theorem 2.6. *A Banach space is linearly isometric to a bo-complete lattice with mixed norm, whose norm lattice is a Dedekind complete AM-space with unit, if and only if it is $\mathbb{B}$-cyclic with respect to some complete Boolean algebra $\mathbb{B}$ of projections.*

In particular, a $\mathbb{B}$-cyclic Banach lattice can be equipped with the structure of Λ-module with $\Lambda = \Lambda(\mathbb{B})$ being a Dedekind complete f-algebra with unit whose Boolean algebra of projections is isomorphic to $\mathbb{B}$.

3. Boolean Valued Requisites

Boolean valued models. Given a complete Boolean algebra $\mathbb{B}$, one can define the universe $\mathrm{V}^{(\mathbb{B})}$ of $\mathbb{B}$-valued sets. For making statements about $\mathrm{V}^{(\mathbb{B})}$ take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of set theory and replace the variables $u_1, \dots, u_n$ by elements $x_1, \dots, x_n \in \mathrm{V}^{(\mathbb{B})}$. Then we obtain some statement about the objects $x_1, \dots, x_n$. There is a natural way of assigning to each such statement an element $[\![\varphi(x_1, \dots, x_n)]\!] \in \mathbb{B}$ which acts as the *'Boolean truth-value'* of $\varphi(u_1, \dots, u_n)$ in the universe $\mathrm{V}^{(\mathbb{B})}$ and is defined by induction, taking into consideration the way in which formulas are built up from atomic formulas $x \in y$ and $x = y$ and assigning truth-values $[\![x \in y]\!] \in \mathbb{B}$ and $[\![x = y]\!] \in \mathbb{B}$, where $x, y \in \mathrm{V}^{(\mathbb{B})}$.

We say that the *statement* $\varphi(x_1, \dots, x_n)$ *is valid inside* $\mathrm{V}^{(\mathbb{B})}$ or the *elements* $x_1, \dots, x_n$ *possess the property* φ if $[\![\varphi(x_1, \dots, x_n)]\!] = \mathbb{1}$. In this event, we write $\mathrm{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n)$.

The pair $(\mathrm{V}^{(\mathbb{B})}, [\![\cdot]\!])$ is called a *Boolean-valued model* of ZFC, Zermelo–Fraenkel set theory with the axiom of choice. The most important properties of a Boolean-valued model are stated in the following four principles (see [8, 23, 36]):

Theorem 3.1 (*The Transfer Principle*)**.** *All the theorems of ZFC are true in* $\mathrm{V}^{(\mathbb{B})}$*; in symbols,* $\mathrm{V}^{(\mathbb{B})} \models$ *"a theorem of ZFC" or* $\mathrm{V}^{(\mathbb{B})} \models$ *ZFC.*

This result is usually expressed by saying that $\mathrm{V}^{(\mathbb{B})}$ is a Boolean-valued model of ZFC.

Theorem 3.2 (*The Maximum Principle*). *For any formula $\varphi(u_0, u_1, \dots, u_n)$ of ZFC and any collection $x_1, \dots, x_n \in \mathrm{V}^{(\mathbb{B})}$ there exists $x_0 \in \mathrm{V}^{(\mathbb{B})}$ such that*

$$[\![(\exists u)\, \varphi(u, x_1, \dots, x_n)]\!] = [\![\varphi(x_0, x_1, \dots, x_n)]\!].$$

Theorem 3.3 (*The Mixing Principle*). *For every family $(x_\xi)_{\xi\in\Xi}$ in $\mathrm{V}^{(\mathbb{B})}$ and a partition of unity $(b_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ there exists a unique element x of $\mathrm{V}^{(\mathbb{B})}$ such that $b_\xi \leqslant [\![x = x_\xi]\!]$ for all $\xi \in \Xi$.* This element is called the *mixing* of a family (x_ξ) by (b_ξ) and is denoted as $x = \mathrm{mix}_{\xi\in\Xi}(b_\xi x_\xi) = \mathrm{mix}\{b_\xi x_\xi : \xi \in \Xi\}$.

There is a canonical embedding of the von Neumann universe V into the Boolean valued universe $\mathrm{V}^{(\mathbb{B})}$ which sends $x \in \mathrm{V}$ to its *standard name* $x^\wedge \in \mathrm{V}^{(\mathbb{B})}$. The standard name sends V onto $\mathrm{V}^{(2)}$, where $2 := \{\mathbb{O}, \mathbb{1}\}$. A formula is called *restricted* if bound variable x is restricted by a quantifier of the form $(\forall\, x \in y)$ or $(\exists\, x \in y)$.

Theorem 3.4 (*The Restricted Transfer Principle*). *For each restricted formula φ of ZFC and every collection $x_1, \dots, x_n \in \mathrm{V}$ the following equivalence holds:*

$$\varphi(x_1, \dots, x_n) \Longleftrightarrow \mathrm{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

Given an arbitrary element $X \in \mathrm{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}$ of X as

$$X{\downarrow} := \{x \in \mathrm{V}^{(\mathbb{B})} : [\![x \in X]\!] = \mathbb{1}\}.$$

Then $X{\downarrow}$ is a set. Assume that $X, Y, f, P \in \mathrm{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$ and $[\![P \subset X^2]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y and P is a binary relation on X inside $\mathrm{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ for which $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ $(x \in X{\downarrow})$ and $P{\downarrow}$ is a unique binary relation on $X{\downarrow}$ such that $(x_1, x_2) \in P{\downarrow} \Longleftrightarrow [\![(x_1, x_2)^{\mathbb{B}} \in P]\!] = \mathbb{1}$.

Boolean valued reals and Banach spaces. Transfer principle and Maximum principles guarantee the existence of various ‘Boolean-valued objects’: If the formula $(\exists u)\, \varphi(u, u_1, \dots, u_n)$ is a ZFC-theorem then by the Transfer Principle this theorem is also valid inside $\mathrm{V}^{(\mathbb{B})}$, i.e. $[\![(\exists u)\, \varphi(u, x_1, \dots, x_n)]\!] = \mathbb{1}$ and by Maximum Principle there exists $x_0 \in \mathrm{V}^{(\mathbb{B})}$ with $[\![\varphi(x_0, x_1 \dots, x_n)]\!] = \mathbb{1}$. Mixing Principle tells us how these ‘Boolean-valued objects’ may be constructed.

For example, applying the Transfer and Maximum Principles to the ZFC-theorem ‘There exists a field of reals’ we find an element $\mathscr{R} \in \mathrm{V}^{(\mathbb{B})}$ for which $[\![\,\mathscr{R}$ is a field of reals$\,]\!] = \mathbb{1}$. We call $\mathscr{R}$ the *reals* in $\mathrm{V}^{(\mathbb{B})}$. The following result due to Gordon [16] tells us that the interpretation of reals in $\mathrm{V}^{(\mathbb{B})}$ is a universally complete vector lattice with the Boolean algebra of band projections isomorphic to $\mathbb{B}$.

Theorem 3.5. *Let $\mathscr{R}$ be the field of reals inside $\mathrm{V}^{(\mathbb{B})}$. Then $\mathscr{R}{\downarrow}$ (with the descended operations and order) is a universally complete vector lattice with order unity $\mathbb{1}$. Moreover, there exists a Boolean isomorphism $\chi : \mathbb{B} \to \mathbb{P}(\mathscr{R}{\downarrow})$ such that*

$$\chi(b)x = \chi(b)y \iff b \leqslant [\![x = y]\!], \quad \chi(b)x \leqslant \chi(b)y \iff b \leqslant [\![x \leqslant y]\!].$$

Let Λ be the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$, i.e. Λ is the order-dense ideal in $\mathscr{R}{\downarrow}$ generated by the order-unit $\mathbb{1} := 1^{\wedge} \in \mathscr{R}{\downarrow}$. Take a Banach space $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ and denote

$$\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} \colon |x| \in \Lambda\}.$$

Then $\mathscr{X}{\Downarrow}$ is a Banach–Kantorovich space called the *bounded descent* of $\mathscr{X}$. Since Λ is a Dedekind complete AM-space with unit, $\mathscr{X}{\Downarrow}$ is a Banach space with mixed norm over Λ, and hence, $\mathbb{B}$-cyclic Banach space, see [22, 7.3.3]. If $\mathscr{Y}$ is another Banach space and $\mathscr{T} : \mathscr{X} \to \mathscr{Y}$ is a bounded linear operator inside $\mathrm{V}^{(\mathbb{B})}$ then the bounded descent of $\mathscr{T}$ is the restriction of $\mathscr{T}{\downarrow}$ to $\mathscr{X}{\Downarrow}$. Clearly, the bounded descent of $\mathscr{T}$ is a bounded linear operator from $\mathscr{X}{\Downarrow}$ to $\mathscr{Y}{\Downarrow}$.

The bounded part of the space $\mathscr{X}{\downarrow}$ is named the *bounded descent* of $\mathscr{X}$.

Theorem 3.6. *A bounded descent of a Banach space from the model* $\mathrm{V}^{(\mathbb{B})}$ *is a* $\mathbb{B}$*-cyclic Banach space. Conversely, if* X *is a* $\mathbb{B}$*-cyclic Banach space, then in the model* $\mathrm{V}^{(\mathbb{B})}$ *there exists up to the isometric isomorphism exactly one Banach space* $\mathscr{X}$ *whose bounded descent is isometrically* $\mathbb{B}$*-isomorphic to* X*.*

Let $\mathscr{X}$ and $\mathscr{Y}$ be the Boolean-valued representations of $\mathbb{B}$-cyclic Banach spaces X and Y, respectively. Let $\mathscr{L}(\mathscr{X},\mathscr{Y})$ denote an element in $\mathrm{V}^{(\mathbb{B})}$ which represents the space of all bounded linear operators from $\mathscr{X}$ into $\mathscr{Y}$.

Theorem 3.7. *A bounded descent of the Banach space* $\mathscr{L}(\mathscr{X},\mathscr{Y})$ *and a* $\mathbb{B}$*-cyclic Banach space* $\mathscr{L}_{\mathbb{B}}(X,Y)$ *are isometrically* $\mathbb{B}$*-isomorphic. The isomorphism is set up by assigning to any bounded* $\mathbb{B}$*-linear operator* $T : X \to Y$ *the element* $\tau := T{\uparrow}$*, defined by the relations* $[\![\tau \text{ is a linear operator from } \mathscr{X} \text{ to } \mathscr{Y}]\!] = \mathbb{1}$ *and* $[\![\tau(x) = T(x)]\!] = \mathbb{1}$ $(x \in X)$*.*

Boolean valued ordinals and cardinals. It is customary to refer to the standard names of ordinals and cardinals as *standard ordinals* and *standard cardinals* inside $\mathrm{V}^{(\mathbb{B})}$. Let $\mathrm{Ord}(\alpha)$ and $\mathrm{Card}(\alpha)$ denote the formulas that declare α an ordinal and a cardinal, respectively.

Clearly, $\mathrm{Ord}(\alpha)$ is a bounded formula, while $\mathrm{Card}(\alpha)$ is not. Since $\lim(\alpha) \leqslant \alpha$ for every ordinal α, the formula $\mathrm{Ord}(x) \wedge x = \lim(x)$ may be rewritten as $\mathrm{Ord}(x) \wedge (\forall\, t \in x)(\exists\, s \in x)(t \in s)$. Hence, $\mathrm{Ord}(x) \wedge x = \lim(x)$ is a bounded formula as well. Finally, the record

$$\mathrm{Ord}(x) \wedge x = \lim(x) \wedge (\forall\, t \in x)(t = \lim(t) \to t = 0)$$

convinces us that the "least limit ordinal" is a bounded formula too. Hence α is the least limit ordinal if and only if $\mathrm{V}^{(\mathbb{B})} \models$"$\alpha^{\wedge}$ is the least limit ordinal." Since ω is the least limit ordinal, $\mathrm{V}^{(\mathbb{B})} \models$ "$\omega^{\wedge}$ is the least limit ordinal." Using the similar argument we can ensure that $\mathrm{V}^{(\mathbb{B})} \models$ "$\mathbb{N}^{\wedge}$ is the set of naturals and $\mathbb{Q}^{\wedge}$ is the field of rationals."

Proposition 3.8. *Each ordinal inside* $\mathrm{V}^{(\mathbb{B})}$ *is a mixing of some set of standard ordinals. In other words, given* $x \in \mathrm{V}^{(\mathbb{B})}$*, we have* $\mathrm{V}^{(\mathbb{B})} \models \mathrm{Ord}(x)$ *if and only if there are an ordinal* β *and a partition of unity* $(b_\alpha)_{\alpha\in\beta}$ *in* $\mathbb{B}$ *such that* $x = \mathrm{mix}_{\alpha\in\beta}\, b_\alpha \alpha^{\wedge}$*.*

Proposition 3.9. *Given* $x \in \mathrm{V}^{(\mathbb{B})}$*, we have* $\mathrm{V}^{(\mathbb{B})} \models \mathrm{Card}(x)$ *if and only if there are nonempty set of cardinals* Γ *and a partition of unity* $(b_\gamma)_{\gamma\in\Gamma}$ *in* $\mathbb{B}$ *such that*

$V^{(\mathbb{B})} \models \mathrm{Card}(\gamma^\wedge)$ *for all* $\gamma \in \Gamma$ *and* $x = \mathrm{mix}_{\gamma\in\Gamma} b_\gamma \gamma^\wedge$. *In other words, each Boolean valued cardinal is a mixing of some set of standard cardinals.*

Proposition 3.10. *The standard name of a finite cardinal is a finite cardinal.*

4. Boolean Valued Banach Lattices

Theorem 4.1. *A bounded descent of a Banach lattice from the model* $V^{(\mathbb{B})}$ *is a* $\mathbb{B}$*-cyclic Banach lattice. Conversely, if* X *is a* $\mathbb{B}$*-cyclic Banach lattice, then in the model* $V^{(\mathbb{B})}$ *there exists up to the isometric isomorphism a unique Banach lattice* $\mathscr{X}$ *whose bounded descent is isometrically* $\mathbb{B}$*-isomorphic to* X*. Moreover,* $\mathbb{B} = \mathbb{M}(X)$ *if and only if* $[\![$*there is no* M*-projection in* $\mathscr{X}$ *other than* 0 *and* $I_{\mathscr{X}}]\!] = \mathbb{1}$.

$\lhd$ The Banach part of the claim follows from Theorem 3.6. Assume that X is a $\mathbb{B}$-cyclic Banach lattice and put $\mathscr{X}_+ := X{\uparrow}$. For any extensional mapping f we have $f(A){\uparrow} = f{\uparrow}(A{\uparrow})$ where $A \subset \mathrm{dom}(f)$, see [23, Theorem 3.3.11 (3)] and [22, A.10 (2,4)]. Applying this relation to addition $f : (x, y) \mapsto x + y$ $(x, y \in X)$ with $A := X_+ \times X_+$ and Λ-multiplication $f : (\lambda, x) \mapsto \lambda x$ $(\lambda \in \Lambda, x \in X)$ with $A := \Lambda_+ \times X_+$ we find $[\![\mathscr{X}_+ + \mathscr{X}_+ = \mathscr{X}]\!] = \mathbb{1}$ and $[\![\mathscr{R}_+ \cdot \mathscr{X}_+ = \mathscr{X}_+]\!] = \mathbb{1}$, i. e $[\![\mathscr{X}_+$ is a convex cone$]\!] = \mathbb{1}$. Moreover, $[\![\mathscr{X}_+$ is pointed$]\!] = \mathbb{1}$, since $[\![\pm x \in \mathscr{X}_+$ and $\|x\| \leqslant 1]\!] = \mathbb{1}$ imply $\pm x \in \mathscr{X}_+{\downarrow} \cap X \subset X_+$. Now, define an ordering in $\mathscr{X}$ by $[\![(\forall x, y \in \mathscr{X})(x \leqslant y \leftrightarrow y - x \in \mathscr{X}_+)]\!] = \mathbb{1}$. By the Transfer Principle $(\mathscr{X}, \mathscr{X}_+)$ is an ordered Banach space inside $V^{(\mathbb{B})}$. Moreover, for any $x, y \in X$ the relations $x \leqslant y$ and $[\![x \leqslant y]\!] = \mathbb{1}$ are equivalent.

Consider a sentence $\sigma \equiv (\forall\, a \in \{0, 1\})(\forall x, y \in \mathscr{X})\,(ax \leqslant ay \leftrightarrow (a \neq 1 \vee x \leqslant y))$ which is a very simple ZF-theorem. By the Transfer Principle $[\![\sigma]\!] = \mathbb{1}$. Calculating the Boolean truth values for quantifiers we find the following equivalent form: For all $a \in \{0, 1\}{\downarrow}$ and $x, y \in \mathscr{X}{\downarrow}$ $[\![ax \leqslant ay]\!] = [\![a = 1]\!]^* \vee [\![x \leqslant y]\!]$. Making use of the Boolean isomorphism $\chi : \mathbb{B} \to \{0, 1\}{\downarrow}$ we may replace $a \in \{0, 1\}{\downarrow}$ by $\chi(b)$ for $b \in \mathbb{B}$: $b^* \vee [\![x \leqslant y]\!] = [\![\chi(b)x \leqslant \chi(b)y]\!]$. Now it is easy to see that

$$b \leqslant [\![x \leqslant y]\!] \iff \chi(b)x \leqslant \chi(b)y \quad (b \in \mathbb{B}; x, y \in \mathscr{X}{\downarrow}). \tag{1}$$

The last relation allows us to treat the interplay between X and $\mathscr{X}$. As an example we prove that $\mathscr{X}$ is a vector lattice, i. e. the sentence $(\forall x \in \mathscr{X})(\exists y \in \mathscr{X})y = \sup\{x, -x\}$ is true inside $V^{(\mathbb{B})}$. Using the the rules for calculating Boolean truth values (see [23, 3.3.2] and [22, A.10 (1)]) and the Maximum Principle we have to prove that for every $x \in X$ there exists $y \in X$ for which $[\![y = \sup\{x, -x\}]\!] = \mathbb{1}$. Put $y = |x|$ and note that $[\![\pm x \leqslant y]\!] = \mathbb{1}$. Thus, it remains to check that $[\![(\forall u \in \mathscr{X})(\pm x \leqslant u \to y \leqslant u)]\!] = \mathbb{1}$. Again by [22, A 4, A 10 (1)] it is equivalent to the relation $[\![\pm x \leqslant u]\!] \leqslant [\![y \leqslant u]\!]$ $(u \in X)$. Now, if $b = [\![\pm x \leqslant u]\!]$ then $\pm\chi(b)x \leqslant \chi(b)u$ and $\chi(b)y \leqslant \chi(b)u$. It follows that $b \leqslant [\![y \leqslant u]\!]$.

The Λ-valued norm $\lvert\!\cdot\!\rvert$ of X is the descent of the norm $\|\cdot\|_{\mathscr{X}}$ of $\mathscr{X}$. Therefore, $\|\cdot\|_{\mathscr{X}}$ is a lattice norm if and only if $|x| \leqslant |y|$ implies $\lvert x\rvert \leqslant \lvert y\rvert$ for all $x, y \in X$. If $\lvert x\rvert > \lvert y\rvert$ then $\pi\lvert x\rvert > \pi(\lvert y\rvert + \varepsilon\mathbb{1})$ for some $\pi \in$ and $0 < \varepsilon \in \mathbb{R}$. Therefore, $\|\pi x\| = \|\pi\lvert x\rvert\|_\infty \geqslant \|\pi\lvert y\rvert\|_\infty + \varepsilon > \|\pi y\|$ which contradicts to $|x| \leqslant |y|$. Thus, $(\mathscr{X}, \mathscr{X}_+)$ is a Banach lattice.

Assume that π is an M-projection in $\mathscr{X}$ and Π is the restriction of $\pi\!\downarrow$ to X. Then $[\![\pi\circ\pi=\pi]\!]=\mathbb{1}$, $[\![0\leqslant \pi x\leqslant x\,(x\in\mathscr{X})]\!]=\mathbb{1}$, and $\|x\|=\max\{\|\pi x\|,\|\pi^\perp x\|\}\,(x\in\mathscr{X})$. By [22, A.9 (4,7)] $\pi\!\downarrow=(\pi\circ\pi)\!\downarrow=\pi\!\downarrow\circ\,\pi\!\downarrow$ and thus $\Pi=\Pi\circ\Pi$. Since $[\![\pi x=\Pi x]\!]=\mathbb{1}$ $(x\in X)$, we have $0\leqslant \Pi x\leqslant x$ for all $x\in X$. Finally, the relations $[\![\|x\|=\max\{\|\pi x\|,\|\pi^\perp x\|\}\,(x\in\mathscr{X})]\!]=\mathbb{1}$ and $|x|=\max\{|\Pi x|,|\Pi^\perp x|\}\,(x\in\mathscr{X})$ are equivalent, whence we deduce $\|x\|=\||\Pi x|\vee|\Pi^\perp x|\|_\infty=\max\{\|\Pi x\|,\|\Pi^\perp\|\}$. Thus, Π is an M-projection in X, i. e. $\Pi\in\mathbb{B}=\mathbb{M}(X)$ and, since $\mathbb{B}$ is the descent of the two-element Boolean algebra $\{0,I_{\mathscr{X}}\}$, we have $[\![$either $\pi=0$ or else $\pi=I_{\mathscr{X}}]\!]=\mathbb{1}$. The remaining details are obvious. $\rhd$

Definition 4.2. The element $\mathscr{X}\in\mathrm{V}^{(\mathbb{B})}$ from Theorem 2.3 is said to be the *Boolean-valued representation of* X. Let $\mathscr{X}$ and $\mathscr{Y}$ be the Boolean-valued representations of $\mathbb{B}$-cyclic Banach lattices X and Y, respectively. Let $\mathscr{L}(\mathscr{X},\mathscr{Y})$ and $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ denote the elements in $\mathrm{V}^{(\mathbb{B})}$ which represent respectively the spaces of all bounded linear operators and regular operators from $\mathscr{X}$ into $\mathscr{Y}$.

Corollary 4.3. *A Banach lattice X is linearly isometric to a bounded descent of some Banach lattice $\mathscr{X}$ from the model $\mathrm{V}^{(\mathbb{B})}$ if and only if it is $\mathbb{B}$-cyclic relative to the complete Boolean algebra of M-projections $\mathbb{B}$.*

Definition 4.4. Say that a downward directed set $A\subset X$ is $\mathbb{B}$-*convergent to zero* if for every $0<\varepsilon\in\mathbb{R}$ there exists a partition of unity $(\pi_a)_{a\in A}$ in $\mathbb{B}$ such that $\|\pi_a a\|\leqslant\varepsilon$ for all $a\in A$. The norm in X is said to be order $\mathbb{B}$-*continuous* if every downward directed set $A\subset X$ with $\inf A=0$ is $\mathbb{B}$-convergent to zero. (This should be compared with the notion of m-convergence by Takeuti [35].)

Theorem 4.5. *Suppose that X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X}\in\mathrm{V}^{(\mathbb{B})}$ is its Boolean valued representation. The following assertions hold:*

(1) *X is a Dedekind complete $\Longleftrightarrow$ $[\![\,\mathscr{X}$ is Dedekind complete$\,]\!]=\mathbb{1}$.*

(2) *X has a Fatou (Levi) norm $\Longleftrightarrow$ $[\![\,\mathscr{X}$ has a Fatou (Levi) norm$\,]\!]=\mathbb{1}$.*

(3) *X has an order $\mathbb{B}$-continuous norm $\Longleftrightarrow$ $[\![\,\mathscr{X}$ has an order continuous norm$\,]\!]=\mathbb{1}$.*

(4) *X has an order $\mathbb{B}$-continuous Levi norm $\Longleftrightarrow$ $[\![\,\mathscr{X}$ is a KB-space$\,]\!]=\mathbb{1}$.*

$\lhd$ (1) Just as in [22, Theorem 8.1.2] (see also [17]) one can prove that for $A\subset X_+$ there exists $a=\sup(A)$ if and only if $[\![$exists $\sup(A\!\uparrow)]\!]=\mathbb{1}$ and in this case $[\![a=\sup(A\!\uparrow)]\!]=\mathbb{1}$. Thus, the Dedekind completeness of $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ implies that X is Dedekind complete. Conversely, suppose that X is Dedekind complete and take a set $\mathscr{A}\subset\mathscr{X}_+$ bounded above by $u\in\mathscr{X}$. There is no loss of generality in assuming that $[\![\|u\|\leqslant 1]\!]=\mathbb{1}$. Then $A:=\mathscr{A}\!\downarrow$ is contained in X and, taking the cancelation rule $\mathscr{A}\!\downarrow\uparrow=\mathscr{A}$ (see [23, 3.3.3 (2)]) into account, we get the following: there exists $a=\sup(\mathscr{A}\!\downarrow)$ if and only if $[\![$exists $\sup(\mathscr{A})]\!]=\mathbb{1}$ and in this case $[\![a=\sup(\mathscr{A})]\!]=\mathbb{1}$.

(2) We may assume without loss of generality that the upward directed sets in the definitions of Fatou norm and Levi norm are taken from the unit balls $B(X)$ and $B(\mathscr{X})$. Moreover, if $A\subset X$ is upward directed then $[\![A\!\uparrow$ is upward directed$]\!]=\mathbb{1}$ and $[\![\mathscr{A}\subset\mathscr{X}$ is upward directed$]\!]=\mathbb{1}$ implies that $\mathscr{A}\!\downarrow$ is upward directed. Finally, observe that $B(\mathscr{X})\!\downarrow=\{x\in\mathscr{X}\!\downarrow:\,|x|\leqslant\mathbb{1}\}=B(X)$. Now, let $\mathscr{X}$ has a Levi norm and take an upward directed set $A\subset B(X)$. It follows that $\{|a|:\,a\in A\}\subset$

$[-\mathbb{1},\mathbb{1}]$ and thus $[\![\{\|a\| : a \in A\!\uparrow\} \subset [-1,1]]\!]$, i. e. $A\!\uparrow \subset B(\mathscr{X})$. By hypothesis $a = \sup(A\!\uparrow)$ exists in $\mathscr{X}$, whence $a = \sup(A)$. The argument for the converse is similar. To ensure the claim concerning the Fatou norm it is sufficient to observe that $|b| = \sup\{|a| : a \in A\}$ in Λ if and only if $\|b\| = \||b|\|_\infty = \sup\{\||a|\|_\infty : a \in A\}$, since the AM-space Λ has a Levi norm.

(3) Making use of the above remarks in (2) it is easy to see that $[\![\, \mathscr{X}$ has an order continuous norm $]\!] = \mathbb{1}$ if and only if for every downward directed set $A \subset X_+$ with $\inf(A) = 0$ we have $\inf\{|a| : a \in A\} = 0$ in Λ. By [22, Theorem 8.1.8] the latter property amounts to the following: for every $\varepsilon > 0$ there exists a partition of unity $(\pi_a)_{a\in A}$ in $\mathbb{B}$ such that $|\pi_a a| = \pi_a|a| < \varepsilon\mathbb{1}$ for all $a \in A$. Thus, we arrive at the desired result, since the relations $|\pi_a a| < \varepsilon\mathbb{1}$ and $\|\pi_a a\| < \varepsilon$ are equivalent.

(4) It is immediate from (1) and (2). $\rhd$

Definition 4.6. A $\mathbb{B}$-cyclic Banach lattice X is called $\mathbb{B}$-*separable,* if there is a sequence $(x_n) \subset X$ such that the norm closed $\mathbb{B}$-cyclic subspace, generated by the set $\{bx_n : n \in \mathbb{N},\, b \in \mathbb{B}\}$, coincides with X.

Theorem 4.7. *Let X and $\mathscr{X}$ be a $\mathbb{B}$-cyclic Banach lattice and its Boolean valued representation in $\mathrm{V}^{(\mathbb{B})}$, respectively. Then the following assertions hold.*

(1) *$[\![\mathscr{X}$ is injective$]\!] = \mathbb{1}$ and only if X is injective.*

(2) *$[\![\mathscr{X}$ is an AM-space$]\!] = \mathbb{1}$ if and only if X is an AM-space.*

(3) *$[\![\mathscr{X}$ is an AL-space$]\!] = \mathbb{1}$ if and only if X is injective and $\mathbb{B} = \mathbb{M}(X)$.*

(4) *$[\![\mathscr{X}$ is separable$]\!] = \mathbb{1}$ if and only if X is $\mathbb{B}$-separable.*

$\lhd$ (1) Theorem 2.4 is valid inside $\mathrm{V}^{(\mathbb{B})}$ by the Transfer principle. In view of Theorem 4.5 (2) we only have to show that $[\![\mathscr{X}$ has the splitting property$]\!] = \mathbb{1}$ if and only if X has the splitting property. It is easy to see that $[\![\mathscr{X}$ has the splitting property$]\!] = \mathbb{1}$ is equivalent to the following property: for any $x, y, z \in X_+$ with $|x| \leqslant \mathbb{1}$, $|y| \leqslant \mathbb{1}$, and $|x + y + z| \leqslant 2\mathbb{1}$, there exist $u, v \in X_+$ such that $z = u + v$, $|x + u| \leqslant \mathbb{1}$ and $|y + v| \leqslant \mathbb{1}$. But the latter amounts to the splitting property in X, since the relations $|x| \leqslant C\mathbb{1}$ and $\|x\| = \||x|\|_\infty \leqslant C$ are equivalent.

(2) Since the Λ-valued norm $|\cdot|$ in X is the bounded descent of the norm $\|\cdot\| = \|\cdot\|_{\mathscr{X}}$ and the lattice operation $(x, y) \mapsto x \vee y$ in X is the descent of the similar operation in $\mathscr{X}$, it follows that $[\![\, \|\cdot\|_{\mathscr{X}}$ is an M-norm$]\!] = \mathbb{1}$ if and only if $|x \vee y| = |x| \vee |y|$ for all $x, y \in X_+$. Since $(\Lambda, \|\cdot\|_\infty)$ is an AM-space, we deduce $\|x \vee y\| = \||x \vee y|\|_\infty = \||x|\|_\infty \vee \||y|\|_\infty = \|x\| \vee \|y\|$.

(3) By the Transfer Principle and Theorem 2.4 we can claim that $[\![\mathscr{X}$ is an AL-space if and only if $\mathbb{M}(\mathscr{X}) = \{0, I_X\}]\!] = \mathbb{1}$. Therefore, the result follows immediately from Theorems 2.4, 4.1, and 4.5 (2). $\rhd$

5. Representation of Injective Banach Lattices

The results above allow us to state a representation result and to produce new examples of injective Banach lattices.

Theorem 5.1. *Let X be a Banach lattice with the complete Boolean algebra $\mathbb{B} = \mathbb{M}(X)$ of M-projections, Λ be a Dedekind complete AM-space with unit such that $\mathbb{P}(\Lambda)$ is isomorphic to $\mathbb{B}$. Then the following assertions are equivalent:*

(1) *X is injective.*

(2) *X is lattice $\mathbb{B}$-isometric to the bounded descent of some AL-space from* $\mathrm{V}^{(\mathbb{B})}$.

(3) *There exists a strictly positive Maharam operator $\Phi : X \to \Lambda$ with the Levi property such that $X = L^1(\Phi)$ and $\|x\| = \|\Phi(|x|)\|_\infty$ for all $x \in X$.*

(4) *There is a Λ-valued additive norm on X such that $(X, |\!\cdot\!|)$ is a Banach–Kantorovich lattice and $\|x\| = \||x|\|_\infty$ for all $x \in X$.*

$\lhd$ (1) $\Longleftrightarrow$ (2) follows from Corollary 4.2 and Theorem 4.7 (3).

(2) $\Longrightarrow$ (3) Assume that the Boolean valued representation $\mathscr{X}$ of X is an AL-space inside $\mathrm{V}^{(\mathbb{B})}$. Working inside $\mathrm{V}^{(\mathbb{B})}$ and using the Transfer Principle, we can find a strictly positive order continuous functional $\phi : \mathscr{X} \to \mathscr{R}$ with the Levi property such that $\|x\| = \phi(|x|)$ for all $x \in \mathscr{X}$. The descent $\Phi' := \phi{\downarrow}$ as well as its restriction $\Phi := \Phi'|_X : X \to \Lambda$ is a strictly positive Maharam operator with the Levi property, see [21] and [22, Theorem 3.6.4 (4)]. Since $|\!\cdot\!| = (\|\cdot\|_{\mathscr{X}}){\downarrow}$ we have $|x| = \Phi(|x|)$ for all $x \in X$. By definition of the resrticted descent $\|x\|_X = \||x|\|_\infty = \|\Phi(|x|)\|_\infty$.

(3) $\Longrightarrow$ (4) If (3) is true then a Λ-valued additive norm on X is defined by $|x| := \Phi(|x|)$ $(x \in X)$. The fact that $(X, |\!\cdot\!|)$ is a Banach–Kantorovich space follows from [22, 3.5.1–3.5.3].

(4) $\Longrightarrow$ (2) This is immediate from 3.6, 4.1, and 4.7 (3). $\rhd$

Corollary 5.2. *If Φ is a strictly positive Maharam operator with the Levi property taking values in a Dedekind complete AM-space Λ with unit and $|\!|\!|x|\!|\!| = \|\Phi(|x|)\|_\infty$ $(x \in L^1(\Phi))$, then $(L^1(\Phi), |\!|\!|\cdot|\!|\!|)$ is an injective Banach lattice with $\mathbb{M}(L^1(\Phi)) = \mathbb{P}(\Lambda)$.*

Conversely, any injective Banach lattice X is lattice $\mathbb{B}$-isometric to $(L^1(\Phi), |\!|\!|\cdot|\!|\!|)$ for some strictly positive Maharam operator Φ with the Levi property taking values in a Dedekind complete AM-space Λ with unit, where $\mathbb{B} = \mathbb{M}(L^1(\Phi)) = \mathbb{P}(\Lambda)$.

Corollary 5.3. *An injective Banach lattice has an order $\mathbb{B}$-continuous norm with $\mathbb{B}$ being the complete Boolean algebra of its M-projections.*

$\lhd$ It is immediate from 4.5 (3) and 5.1 (2). $\rhd$

Corollary 5.4. *In an injective Banach lattice X with $\mathbb{B} := \mathbb{M}(X)$ there is a $\mathbb{B}$-invariant closed sublattice which is lattice $\mathbb{B}$-isometric to $\Lambda = \Lambda(\mathbb{B})$.*

$\lhd$ There exists an order continuous lattice isomorphism $S : \Lambda \to X$ such that $\Phi \circ S = I_\Lambda$, see [22, Theorem 3.4.10]. Thus, $\|\lambda\|_\infty = \|\Phi(S(|\lambda|))\|_\infty = \|S(\lambda)\|$ for all $\lambda \in \Lambda$. Moreover, for $\pi \in \mathbb{B}$ and $\lambda \in \Lambda$ we have $S(\pi\lambda) = S(\pi\Phi(S\lambda)) = (S \circ \Phi)(\pi S\lambda)) = \pi S\lambda$, since $S \circ \Phi$ is a positive projections onto $S(\Lambda)$. It follows that $S(\Lambda)$ is $\mathbb{B}$-invariant. $\rhd$

Corollary 5.5. *An injective Banach lattice X has an order continuous norm if and only if X is a finite l_∞-product of AL-spaces.*

$\lhd$ It is clear from the representation $\|x\| = \|\Phi(|x|)\|_\infty$ $(x \in X)$ that X has an order continuous norm if and only if Λ has an order continuous norm. But the latter occurs only if Λ is finite dimensional. $\rhd$

The construction of the Maharam's extension of positive operators [22, §4.5] together with Corollary 5.2 supplies plenty of injective Banach lattices. Given a set M

in a vector lattice X, denote by $M^{\downarrow}$ the collection of all elements $x \in X$ that can be written as $x = \inf(A)$, where A is a downward directed subset of M. The set $M^{\uparrow}$ is defined similarly on using upward-directed sets. We also put $M^{\downarrow\uparrow} := (M^{\downarrow})^{\uparrow}$.

Theorem 5.6. *Let E be a vector lattice, Λ be a Dedekind complete AM-space with unit, and $\Phi : E \to \Lambda$ be a strictly positive operator. Then there exists a unique (up to lattice isometry) injective Banach lattice X such that the following hold:*

(1) $\mathbb{M}(X) = \mathbb{P}(\Lambda)$.

(2) *there are a lattice isomorphism $\iota : E \to X$ and an f-algebra isomorphism h from $\mathscr{Z}(\Lambda)$ into $\mathscr{Z}(X)$ such that $\|\sigma\Phi(x)\|_\infty = \|h(\sigma)\iota(x)\|$ $(x \in E_+,\ \sigma \in \mathscr{Z}(\Lambda)_+)$.*

(3) *for every $x \in X$ and $0 < \varepsilon \in \mathbb{R}$ there are $x_\varepsilon \in X$, a partition $(\pi_\xi)_{\xi\in\Xi}$ of unity in $\mathbb{M}(X)$, and a family $(e_\xi)_{\xi\in\Xi}$ in E such that $\|x - x_\varepsilon\| \leqslant \varepsilon$ and $\pi_\xi x = \pi_\xi \iota(e_\xi)$ for all $\xi \in \Xi$.*

(4) *$X = X_0^{\downarrow\uparrow}$, where X_0 comprises finite sums $\sum_{k=1}^n \pi_k \iota(x_k)$ with $\pi_k \in \mathbb{M}(X)$, $x_k \in E$ $(k = 1, \dots, n \in \mathbb{N})$.*

$\lhd$ The Maharam extension $\tilde{\Phi}$ of Φ is a strictly positive Maharam operator with Levi property, see [22, Propositions 3.5.1, 3.5.3 and Theorem 3.5.2]. If $X = L^1(\tilde{\Phi})$ is the domain of $\tilde{\Phi}$ and $\|x\| = \|\tilde{\Phi}(|x|)\|_\infty$ $(x \in X)$ then X is injective Banach lattice by Theorem 5.1. The properties (1–4) are immediate from [22, 3.5.1 and 3.5.8 (1)].

Another proof can be given by interpreting in Boolean valued universe $\mathrm{V}^{(\mathbb{B})}$ the well known construction permitting us to produce an AL-space $(E^\varphi, \|\cdot\|^\varphi)$ from a strictly positive linear functional $\varphi \in E^\sim$, see [29, Proposition 2.4.16]. To this end, we have to observe that: a) the standard name $E^\wedge$ of E is a vector lattice over the rationals $\mathbb{Q}^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$; b) the modified ascent $\varphi = \Phi\!\uparrow$ of Φ is a strictly positive $\mathbb{Q}^\wedge$-linear functional on $E^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$; c) Proposition 2.4.16 of [29] remains valid for vector lattices over $\mathbb{Q}$ and $\mathbb{Q}$-linear φ. $\rhd$

Theorem 5.7. *Let X be an injective Banach lattice and let X_0 be a vector sublattice of X. Denote by $\mathbb{B}_0$ the part of $\mathbb{M}(X)$ consisting of the projections leaving X_0 invariant. If $\mathbb{B}_0$ is order closed subalgebra of $\mathbb{M}(X)$ and X_0 is $\mathbb{B}_0$-cyclic Banach lattice then X_0 is an injective Banach lattice with $\mathbb{B}_0 = \mathbb{M}(X_0)$.*

$\lhd$ By Corollary 5.2 there exists a strictly positive Maharam operator $\Phi : X \to \Lambda$ with the Levi property such that $X = L^1(\Phi)$ and $\|x\| = \|\Phi(|x|)\|_\infty$ $(x \in X)$. Denote by Λ_0 the order closed sublattice of Λ with $\mathbb{P}(\Lambda) = \mathbb{B}_0$. Under the hypothesis the restriction Φ_0 of Φ onto X_0 is also a strictly positive Maharam operator from X_0 to Λ_0 with the Levi property. $\rhd$

Let E and F be vector lattices and G be a majorizing sublattice of E. Fix a positive operator $S : G \to F$ and denote by Λ_S the order ideal in $L^r(G, F)$ generated by S with the order unit norm $\|\cdot\|_\infty = \|\cdot\|_S$ with respect to S. Put

$$X_S := \{T \in L^r(E, F) :\ (\exists \lambda \in \mathbb{R}_+)\, |Tx| \leqslant \lambda S|g|\ (x \in E,\ g \in G,\ |x| \leqslant |g|)\},$$
$$\|T\|_S := \inf\{\lambda \in \mathbb{R}_+ : |Tx| \leqslant \lambda S|g|\ (x \in E,\ g \in G\, |x| \leqslant |g|)\}.$$

Theorem 5.8. *Let E and F be vector lattices with F Dedekind complete and G be a majorizing sublattice of E. Then $(X_S, \|\cdot\|_S)$ is an injective Banach lattice.*

$\lhd$ This is immediate from Theorem 5.1, since the restriction operator $\Phi : T \mapsto T|_G$ acting from X_S to Λ_S is a Maharam operator with the Levi property see [22, Example 3.4.2 (6) and Theorem 3.4.11]. $\rhd$

If $F = G$ and $S = I_G$ then operators from X_S are called *central with respect to* G and the notation $X_S = \mathscr{Z}(E|G)$ is used, see [37, Definitions 2.1 and 2.2]. For this particular case we have the following improved version of Wickstead's theorem [37, Theorem 5.2].

Corollary 5.9. *Let E be a vector lattice and G be a Dedekind complete majorizing sublattice of E. Then $\mathscr{Z}(E|G)$ with the natural norm, is an injective Banach lattice.*

Remak 5.10. (1) Theorem 5.8, Corollaries 5.2 and 5.9 suggest a slightly more general construction. Let X and Y be Dedekind complete vector lattices and $\Phi : X \to Y$ be a strictly positive Maharam operator with the Levi property. Denote by $Y(e)$ and $\|\cdot\|_e := \|\cdot\|_\infty$ the order ideal generated by $e \in Y_+$ and an order unit norm on $Y(e)$, respectively. Then the vector space $L_1(\Phi, e) := \Phi^{-1}(Y(e))$ with the mixed norm $\|x\| := \|\Phi(|x|)\|_e$ is an injective Banach lattice.

(2) As an illustration consider the space of Bochner summable functions $X := L_1(\mu, Y) := L_1(\Omega, \Sigma, \mu; Y)$, where (Ω, Σ, μ) is a finite measure space and Y is a Banach lattice with an order continuous norm. The latter hypothesis implies that $L_1(\mu, Y)$ has an order continuous norm [12], whence the Bochner integral $I_\mu : x \mapsto \int_\Omega x(\omega)\, d\mu(\omega)$ $(x \in L_1(\mu, Y))$ is order continuous. If, in addition, Y has a Levi norm, then $L_1(\mu, Y)$ also has a Levi norm [10] and, as a consequence, Bochner integral has the Levi property. Now we can state the following result which is a direct consequence of (1):

Let (Ω, Σ, μ) be a finite measure space, Y be a KB-space and $e \in Y_+$. Then $L_1(\mu, Y, e) := \{x \in X : I_\mu(|x|) \in Y(e)\}$ with the norm $\|x\| := \|I_\mu(|x|)\|_e$ is an injective Banach lattice.

Definition 5.11. The norm on a Banach lattice X is said to have the *Nakano property* if, for every upward directed order bounded subset $A \subset X_+$ the identity $\inf\{\|b\| : b \in B\} = \sup\{\|a\| : a \in A\}$ holds, where B is the set of all upper bounds for A.

Theorem 5.12. *Let X and Y be $\mathbb{B}$-cyclic Banach spaces and, in addition, Y be an AM-space in which the norm has the Nakano property. Suppose that there is a linear $\mathbb{B}$-isometry Φ from Y onto $X^\#$ and denote $X_+ := \{x \in X : (\forall y \in Y_+)\, \langle x, \Phi(y)\rangle \geqslant 0\}$. Then X_+ is a convex cone and, being equipped with the induced ordering, X is an injective Banach lattice with $\mathbb{B} = \mathbb{M}(X)$.*

$\lhd$ Let $\mathscr{X}$ and $\mathscr{Y}$ be Boolean valued representations of X and Y, respectively. Put $\varphi := \Phi{\uparrow}$ and observe that $[\![\varphi$ is a linear isometry from $\mathscr{Y}$ onto $\mathscr{X}^*]\!] = \mathbb{1}$. Moreover, $[\![\mathscr{Y}$ is an AM-space with the Nakano property$]\!] = \mathbb{1}$ and $[\![\mathscr{Y}$ is lattice isometric to $\mathscr{C}_0(K)$ for some locally compact Hausdorff space $K]\!] = \mathbb{1}$ by virtue of the Nakano's isometric characterization of the space $\mathscr{C}_0(K)$ of all continuous real-valued functions on K vanishing at infinity under the supremum norm [30]. Now the Grothendieck theorem [34, Theorem 27.4.1] tells us that $\mathscr{X}$ is an AL-space and it remains to apply Theorem 5.1. $\rhd$

Theorem 5.13. *Let Λ be a Dedekind complete AM-space with unit and let $\mathscr{A}$ be a Boolean algebra with unit $\mathbb{1}$. Then the vector lattice* $\mathrm{ba}(\mathscr{A},\Lambda)$ *of Λ-valued order bounded finitely additive measures on $\mathscr{A}$ equipped with the norm $\|\nu\|:=\||\nu|(\mathbb{1})\|_\infty$ ($\nu\in\mathrm{ba}(\mathscr{A}\Lambda)$) is an injective Banach lattice.*

$\lhd$ Clearly, $\mathrm{ba}(\mathscr{A},\Lambda)$ is a Dedekind complete vector lattice, since it is order isomorphic to a space of regular operators. By 5.1 (4) we have to show that $\mathrm{ba}(\mathscr{A},\Lambda)$ is a Banach–Kantorovich space with additive Λ-valued norm. The Λ-valued norm defined as $\{\nu\}:=|\nu|(\mathbb{1})$ is evidently additive on the positive cone. It is also decomposable. Indeed, if $|\nu|=\lambda_1+\lambda_2$ then $\lambda_k=\alpha_k|\nu|$ for some orthomorphism $0\leqslant\alpha_k\leqslant I_\Lambda$ and, putting $\nu_k:=\alpha_k\nu$ we get $\nu=\nu_1+\nu_2$ and $|\nu_k|=|\alpha_k\nu|(\mathbb{1})=\alpha_k|\nu|(\mathbb{1})=\lambda_k$ ($k:=1,2$). For every order bounded family $(\nu_\xi)_{\xi\in\Xi}$ in $\mathrm{ba}(\mathscr{A},\Lambda)$ and every partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{P}(\Lambda)$, the sum $\nu(A):=o\text{-}\sum_{\xi\in\Xi}\pi_\xi\nu_\xi(A)$ exists. Moreover, ν is a unique element in $\mathrm{ba}(\mathscr{A},\Lambda)$ satisfying the relations $\pi_\xi\nu=\pi_\xi\nu_\xi$ ($\xi\in\Xi$). In view of [22, Theorem 2.2.3] it remains to ensure the uniform completeness of $\mathrm{ba}(\mathscr{A},\Lambda)$ which can be proved by standard arguments. $\rhd$

6. Operators on Injective Banach Lattices

Denote by $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ the space of all regular $\mathbb{B}$-linear operators from X to Y equipped with the *regular norm* $\|T\|_r:=\inf\{\|S\|:\,S\in\mathscr{L}_{\mathbb{B}}(X,Y\},\,\pm T\leqslant S\}$. Let $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ stands for the space of all regular operators from $\mathscr{X}$ to $\mathscr{Y}$ with the regular norm inside $\mathrm{V}^{(\mathbb{B})}$.

Theorem 6.1. *Assume that X and Y are $\mathbb{B}$-cyclic Banach lattices, while $\mathscr{X}$ and $\mathscr{Y}$ are their respective Boolean-valued representation. The space $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is lattice $\mathbb{B}$-isometric to the bounded descent of the space $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$.*

$\lhd$ According to Theorem 4.1 we may assume without loss of generality that X and Y are the bounded descents of some internal Banach lattices $\mathscr{X}$ and $\mathscr{Y}$. Moreover, $\mathscr{L}_{\mathbb{B}}(X,Y)$ and $\mathscr{L}(\mathscr{X},\mathscr{Y}){\Downarrow}$ are $\mathbb{B}$-isometric by Theorem 3.7. Since $T(X_+){\uparrow}=T{\uparrow}(X_+{\uparrow})=\tau(\mathscr{X}_+)$, it follows that $T(X_+)\subset Y_+$ if and only if $[\![\tau(\mathscr{X}_+)\subset\mathscr{Y}_+]\!]=\mathbb{1}$. This means that the bijection $T\leftrightarrow\tau=T{\uparrow}$ preserves positivity and hence is an order $\mathbb{B}$-isomorphism between $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ and $\mathscr{L}^r(\mathscr{X},\mathscr{Y}){\Downarrow}$. Since for $S\in\mathscr{L}^r_{\mathbb{B}}(X,Y)$ and $\sigma:=S{\uparrow}$ the relations $\pm T\leqslant S$ and $[\![\pm\tau\leqslant\sigma]\!]=\mathbb{1}$ are equivalent, we have $[\![\|\tau\|_r=\lvert T\rvert_r]\!]=\mathbb{1}$, where $\lvert T\rvert_r=\inf\{\lvert S\rvert:\,S\in\mathscr{L}^r_{\mathbb{B}}(X,Y),\,\pm T\leqslant S\}$ and $\lvert S\rvert:=\sup\{\lvert Sx\rvert:\,\lvert x\rvert\leqslant\mathbb{1}\}$. Thus, we have to show that the regular norm in $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is the mixed norm, i.e. $\|T\|_r=\|\lvert T\rvert_r\|_\infty$ ($T\in\mathscr{L}^r_{\mathbb{B}}(X,Y)$).

If $\pm T\leqslant S$ then $\|\lvert T\rvert\|_\infty\leqslant\|\lvert S\rvert\|_\infty=\|S\|$ and hence $\|T\|_r\geqslant\|\lvert T\rvert_r\|_\infty$. To prove the reverse inequality take an arbitrary $0<\varepsilon\in\mathbb{R}$ and choose a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(S_\xi)_{\xi\in\Xi}$ in $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ such that $S_\xi\geqslant\pm T$ and $\pi_\xi\lvert S_\xi\rvert\leqslant(1+\varepsilon)\lvert T\rvert_r$ for all $\xi\in\Xi$. Define an operator $S\in\mathscr{L}^r_{\mathbb{B}}(X,Y)$ by $Sx:=\mathrm{mix}_{\xi\in\Xi}\,Sx$ ($x\in X$), where the mixing is taken in Y and is equal to the sum of the family $(S_\xi x)$ in Λ-valued norm on Y. Then $S\geqslant\pm T$ and $\lvert S\rvert\leqslant(1+\varepsilon)\lvert T\rvert_r$, whence $\|T\|_r\leqslant\|S\|=\|\lvert S\rvert\|_\infty\leqslant(1+\varepsilon)\|\lvert T\rvert_r\|_\infty$.$\rhd$

Theorem 6.2. *Let X be an injective Banach lattice and an operator $T\in\mathscr{L}(X)$ commutes with all M-projections. Then there exist pair-wise disjoint M-projections*

π_0, π_1, and π_2 in X such that $\pi_0+\pi_1+\pi_2 = I_X$ and the operator $\pi_1\circ T \pm \pi_0\circ T - \pi_2\circ T$ satisfies the Daugavet equation. Moreover, for any nonzero M-projections $\rho_k \leqslant \pi_k$ $(k = 1, 2)$ the operators $-\rho_2 \circ T$ and $\rho_1 \circ T$ fail to satisfy the Daugavet equation.

$\lhd$ Let $\mathscr{X}, \mathscr{T} \in \mathrm{V}^{(\mathbb{B})}$ be the Boolean-valued representations of X and T, respectively. By Theorem 4.7 $[\![\mathscr{X}$ is an AL-space and $\mathscr{T} \in \mathscr{L}(\mathscr{X})]\!] = \mathbb{1}$. Let the formula $\psi(T)$ formalize the sentence 'T satisfies the Daugavet equation' and put $\tilde{\pi}_1 = [\![\psi(\mathscr{T})]\!]$, $\tilde{\pi}_2 = [\![\psi(-\mathscr{T})]\!]$, $\pi_0 = \tilde{\pi}_1 \circ \tilde{\pi}_2$, and $\pi_i = \tilde{\pi}_i - \pi_0$. It was shown in Abramovich [2] and Schmidt [33] that if $\mathscr{T}$ is a continuous operator on an AL-space, then either $\mathscr{T}$ or $-\mathscr{T}$ satisfies the Daugavet equation, see [4, Theorem 11.23]. It follows from the Transfer Principle that $\tilde{\pi}_1 \vee \tilde{\pi}_2 = [\![\psi(\mathscr{T}) \vee \psi(-\mathscr{T})]\!] = \mathbb{1}$, whence $\pi_1 + \pi_0 + \pi_2 = \mathbb{1}$. Denote by $\mathscr{S}$ the mixing of $(\mathscr{T}, \mathscr{T}, -\mathscr{T})$ by (π_1, π_0, π_2), i. e. $\pi_0+\pi_1 \leqslant [\![\mathscr{S} = \mathscr{T}]\!]$ and $\pi_2 \leqslant [\![\mathscr{S} = -\mathscr{T}]\!]$. If $S := \mathscr{S}{\Downarrow}$ then $S := \pi_1\circ T+\pi_0\circ T-\pi_2\circ T$. By applying [22, A.5 (6)] we have $\pi_0 + \pi_1 \leqslant [\![\psi(\mathscr{T})]\!] \wedge [\![\mathscr{S} = \mathscr{T}]\!] \leqslant [\![\psi(\mathscr{S})]\!]$ and $\pi_2 \leqslant [\![\psi(-\mathscr{T})]\!] \wedge [\![\mathscr{S} = -\mathscr{T}]\!] \leqslant [\![\psi(S)]\!]$ which imply $[\![\psi(\mathscr{S})]\!] = \mathbb{1}$. Since $[\![\|\mathscr{S}\| = |S|]\!] = \mathbb{1}$, we have $|I + S| = \mathbb{1} + |S|$ and taking into account the easy relation $\|\mathbb{1} + \lambda\|_\infty = 1 + \|\lambda\|_\infty$ for all $\lambda \in \Lambda$ we arrive at the required equality $\|I + S\| = 1 + \|S\|$. The operator $\pi_1 \circ T - \pi_0 \circ T - \pi_2 \circ T$ is handled similarly. $\rhd$

Theorem 6.3. *Let T be a positive operator on an injective Banach lattice X which commutes with all projections in $\mathbb{M}(X)$. Assume that norms $\|T^n\|$ are bounded and the arithmetic means $(x+Tx+\ldots+T^n x)/n$ are order bounded above for every $x \in X_+$. Then the sequence of these means $\mathbb{B}$-converges in norm for all $x \in X$.*

$\lhd$ The proof is similar to that of the previous theorem. The ergodic theorem by G. Birkhoff and Kakutani (see [20]) tells us that the claim is true in the case of AL-space X, i.e. whenever $\mathbb{M}(X) = \{0, I_X\}$. By the Transfer principle it is also true in $\mathrm{V}^{(\mathbb{B})}$ for $\mathscr{X}$ and $\mathscr{T}$. If $C = \sup_n \|T^n\|$ then $[\![\sup_n \|\mathscr{T}\| \leqslant C^\wedge]\!] = \mathbb{1}$ and $v_n := (x+Tx+\ldots+T^n x)/n \leqslant a$ $(n \in \mathbb{N})$ implies $[\![u_n := (x+\mathscr{T}x+\ldots+\mathscr{T}^n x)/n \leqslant a$ $(n \in \mathbb{N}^\wedge)]\!] = \mathbb{1}$. Thus, the sequence of means u_n is convergent to some $u \in \mathscr{X}$, i.e $[\![\lim_n \|u_n - u\| = 0]\!] = \mathbb{1}$. This amounts to saying that $|v_n - u|$ is order convergent to zero. The latter implies that v_n is norm $\mathbb{B}$-convergent to u. $\rhd$

Theorem 6.4. *A Dedekind complete $\mathbb{B}$-cyclic Banach lattice Y has a Levi norm if and only if for every injective Banach lattice X with $\mathbb{B} = \mathbb{M}(X)$ we have $\mathscr{L}_{\mathbb{B}}(X, Y) = \mathscr{L}^r_{\mathbb{B}}(X, Y)$.*

$\lhd$ In fact, a Dedekind complete Banach lattice F is monotonically complete if and only if $\mathscr{L}(E, F) = \mathscr{L}^r(E, F)$ for every AL-space E, see [5]. $\rhd$

Theorem 6.5. *Suppose that the $\mathbb{B}$-cyclic Banach lattices X and Y satisfy $\mathscr{L}(X, Y) = \mathscr{L}^r(X, Y)$ and $\|T\| = \|T\|_r$ for all $T \in \mathscr{L}(X, Y)$. Then there exists an M-projection π_1 and π_2 such that $\pi_1 X$ is an injective Banach lattice, $\pi_2 Y$ is an AM-space, and $\pi_1 \vee \pi_2 = \mathbb{1}$.*

$\lhd$ By Transfer Principle we have that if $\mathscr{L}(\mathscr{X}, \mathscr{Y}) = \mathscr{L}^r(\mathscr{X}, \mathscr{Y})$ and $\|\mathscr{T}\| = \|\mathscr{T}\|_r$ for all $\mathscr{T} \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ then either $\mathscr{X}$ is AL-space or $\mathscr{Y}$ is an AM-space, see [4, Theorem 11]. Consequently, if $\pi_1 := [\![\mathscr{X}$ is an AL-space$]\!]$ and $\pi_2 := [\![\mathscr{Y}$ is an AM-space$]\!]$, then $\pi_1 \vee \pi_2 = \mathbb{1}$. Now, the claim follows from Theorem 4.7. $\rhd$

Theorem 6.6. *Let X and Y be $\mathbb{B}$-cyclic Banach lattices neither of which is the zero space with Y Dedekind complete. Then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an injective Banach lattice under the regular norm with $\mathbb{B} = \mathbb{M}(\mathscr{L}^r_{\mathbb{B}}(X,Y))$ if and only if X is an AM-space and Y is an injective Banach lattice with $\mathbb{B} = \mathbb{M}(Y)$.*

$\lhd$ This a Boolean-valued interpretation of the following result by Wickstead [38, Theorem 2.1]: If $\mathscr{X}$ and $\mathscr{Y}$ are Banach lattices, neither of which is the zero space, with $\mathscr{Y}$ Dedekind complete then $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AL-space under the regular norm if and only if $\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space. $\rhd$

Theorem 6.7. *Let Y be a nonzero $\mathbb{B}$-cyclic Dedekind complete Banach lattices. Then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an AM-space under the regular norm for every injective Banach lattice X with $\mathbb{B} = \mathbb{M}(X)$ if and only if Y is an AM-space with a Fatou norm.*

$\lhd$ This a Boolean-valued interpretation of the following result by Wickstead [38, Theorem 2.3]: If $\mathscr{Y}$ be a nonzero Dedekind complete Banach lattices then $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AM-space under the regular norm for every AL-space $\mathscr{X}$ if and only if $\mathscr{Y}$ is an AM-space with a Fatou norm. $\rhd$

Theorem 6.8. *If a Banach lattice X satisfies $\mathscr{L}_{\mathbb{B}}(X) = \mathscr{L}^r_{\mathbb{B}}(X)$, then there exists an M-projection $\pi \in \mathbb{B}$ such that πX is order $\pi\mathbb{B}$-isomorphic to an injective Banach lattice and $\pi^\perp X$ is order $\pi^\perp\mathbb{B}$-isomorphic to an AM-space.*

$\lhd$ This is a Boolean-valued interpretation of the following result: If a Banach lattice X satisfies $\mathscr{L}(X) = \mathscr{L}^r(X)$, then X is order isomorphic either to an AL- or AM-space, see Abramovich [3]. $\rhd$

Consider an injective Banach lattice represented as $L^1(\Phi)$. Denote by $L^0(\Phi)$ the universal completion of $L^1(\Phi)$ with a fixed order unit $\mathbb{1}$. Then there is a unique structure of a semiprime f-algebra on $L^0(\Phi)$ with ring unit $\mathbb{1}$. Consider also the set $X' \subset L^0(\Phi)$ defined as

$$X' := \{x' \in L^0(\Phi) : \ x' \cdot X \subset L^1(\Phi)\}.$$

Theorem 6.9. *Let X be an order $\mathbb{B}$-continuous $\mathbb{B}$-cyclic Banach lattice. In the universal completion X^u of X there is an order dense ideal which is an injective Banach lattice represented as $L^1(\Phi)$ for some order continuous Maharam operator with the Levi property. Moreover, the map assigning the operator $S_{x'} \in \mathscr{L}_{\Phi}(X,\Lambda)$ to an element $x' \in X'$ by*

$$S_{x'}(x) = \Phi(x \cdot x') \quad (x \in X).$$

is an lattice $\mathbb{B}$-isometry from $X^{\#}$ onto X'.

$\lhd$ The Boolean valued representation $\mathscr{X}$ of X is an order continuous Banach lattice inside $\mathrm{V}^{(\mathbb{B})}$, see Theorem 4.6. Working inside $\mathrm{V}^{(\mathbb{B})}$ we can find a strictly positive order continuous functional $\phi : L_1(\phi) \to \mathscr{R}$ with the Levi property such that $L^1(\phi)$ being an order dense ideal in the universal completion $\mathscr{X}^u$ of $\mathscr{X}$. Put $\mathscr{X}' := \{x \in \mathscr{X}^u : x\mathscr{X} \subset L^1(\phi)$. Then assigning to every element $x' \in \mathscr{X}'$ the functional $\sigma_{x'} : x \mapsto \phi(xx')$ $(x \in \mathscr{X})$ yields a lattice isometry from $\mathscr{X}'$ onto the dual $\mathscr{X}^*$. It is easy to see that $\mathscr{X}^u{\downarrow} = X^u$. Define Φ as the restriction of $\phi{\downarrow}$ to $L^1(\Phi) := \{x \in X^u : \phi{\downarrow}(x) \in \Lambda\}$. Clearly, Φ is a strictly positive Maharam operator with the Levi property and thus $L^1(\phi){\Downarrow} = L^1(\Phi)$. Now, it remains to observe that identifying X with a order dense ideal in $\mathscr{X}{\downarrow}$ we have $\mathscr{X}^*{\Downarrow} = X^\#$, $\mathscr{X}'{\Downarrow} = X'$, and $S_{x'} = \sigma_x{\downarrow}$. ▷

Corollary 6.10. *Let X be an injective Banach lattice. Let E and Λ be Dedekind complete AM-spaces with unit such that $\mathbb{P}(E) \simeq \mathbb{P}(X)$ and $\mathbb{P}(\Lambda) \simeq \mathbb{M}(X)$. Then X is lattice isometric to $\mathscr{L}^\times_{\mathbb{B}}(E, \Lambda)$.*

Remak 6.11. Corollary 6.10 is essentially the Haydon's main representation theorem in [19, Theorem 5C]. Haydon's another representation result [19, Theorem 6H] tells us that an injective Banach lattice can be represented as $L^1(\mathbf{m})$ for some Λ-valued modular Maharam measure $\mathbf{m}$. This result is immediate from our representation result 5.1, since the mapping $\Phi : L^1(\mathbf{m}) \to \Lambda$ defined by $\Phi : f \mapsto \int f\,d\mathbf{m}$ is a Maharam operator with the Levi property and $L^1(\mathbf{m}) = L^1(\Phi)$, see [22, Theorem 6.1.10].

7. Operators Factorable Through Injective Banach Lattices

Let X be a Banach lattice and Y be a $\mathbb{B}$-cyclic Banach space. Denote by $\mathrm{Prt}_\sigma := \mathrm{Prt}_\sigma(\mathbb{B})$ and $\mathscr{P}_{\mathrm{fin}}(X)$ the set of all countable partitions of unity in $\mathbb{B}$ and the collection of all finite subsets of X, respectively. For every $T \in \mathscr{L}(X, Y)$ put

$$\sigma(T) = \sup\left\{ \inf_{(\pi_k)\in\mathrm{Prt}_\sigma} \sup_{k\in\mathbb{N}} \sum_{i=1}^n \|\pi_k T x_i\| : \ (x_i) \in \mathscr{P}_{\mathrm{fin}}(X), \ \left\| \sum_{i=1}^n |x_i| \right\| \leqslant 1 \right\}.$$

Definition 7.1. An operator $T \in \mathscr{L}(X, Y)$ is said to be cone $\mathbb{B}$-*summing* if $\sigma(T) < \infty$. Thus, T is cone $\mathbb{B}$-*summing* if and only if there exists a positive constant C such that for any finite collection $x_1, \dots x_n \in X$ there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ with

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^n \|\pi_k T x_i\| \leqslant C \left\| \sum_{i=1}^n |x_i| \right\|.$$

Denote by $\mathscr{S}_{\mathbb{B}}(X, Y)$ the set of all cone $\mathbb{B}$-summing operators. Observe that if $\mathbb{B} = \{0, I_Y\}$ then $\mathscr{S}_{\mathbb{B}}(X, Y)$ is the space of cone absolutely summing operators, see [32, Ch. 4, §3, Proposition 3.3 (d)] or (which is the same) 1-concave operators, see [14, p. 330]. Cone absolutely summing operators were introduced by Levin [24], [25] and later independently by Schlotterbeck, see [32, Ch. 4]

Suppose that Q is a Stonean space and X is a Banach space. Let $C_\infty(Q, X)$ be the set of cosets of continuous vector-functions u that act from comeager subsets $\mathrm{dom}(u) \subset Q$ into X. (Recall that a set is called comeager if its complement is of first category.) Vector-functions u and v are equivalent if $u(q) = v(q)$ whenever $q \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$. The set $C_\infty(Q, X)$ is endowed, in a natural way, with the structure of a module over $C_\infty(Q)$. Moreover, the continuous extension of the pointwise norm $q \mapsto \|u(q)\|$ defines a decomposable norm $u \mapsto |u| \in C_\infty(Q)$ on $C_\infty(Q, X)$.

Denote by $C_\#(Q, X)$ the part of $C_\infty(Q, X)$ that consists of vector-functions u satisfying $|u| \in C(Q)$ with the norm given by $\|u\| := \||u|\|_\infty$ (cf. [22, 2.3.3]). Suppose that Q a Stonean space and X is a Banach lattice. Then the space $C_{\#}(Q, X)$ is a $\mathbb{B}$-cyclic Banach lattice with $\mathbb{B}$ being the Boolean algebra of clopen subsets of Q.

Proposition 7.2. *Suppose that X is a Banach lattice and $\mathscr{X}$ is the completion of the metric space $X^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$. Then $[\![\, \mathscr{X}$ is a Banach lattice$\,]\!] = \mathbb{1}$ and $\mathscr{X}{\Downarrow}$ is $\mathbb{B}$-isomorphic to $C_{\#}(Q, X)$. Moreover, if $\mathscr{Y}$ is the Boolean-valued representation of a $\mathbb{B}$-cyclic Banach space Y, then the map $T \mapsto T{\uparrow}$ is a linear isometry from $\mathscr{L}_{\mathbb{B}}(C_{\#}(Q, X), Y)$ onto $\mathscr{L}(\mathscr{X}, \mathscr{Y}){\Downarrow}$.*

$\lhd$ The proof is a duly modification of [22, 8.3.4], see also [17]. $\rhd$

Observe also that the map $T \mapsto T \circ h$ is a linear isometry from $\mathscr{L}(X, Y)$ onto $\mathscr{L}_{\mathbb{B}}(C_{\#}(Q, X), Y)$, where $h : X \to C_{\#}(Q, X)$ is defined as $h(x) := 1_Q x$.

Corollary 7.3. *X is an AL-space if and only if $C_{\#}(Q, X)$ is an injective Banach lattice.*

Let $\mathscr{S}(\mathscr{X}, \mathscr{Y})$ denotes the space of all cone absolutely summing operators from $\mathscr{X}$ to $\mathscr{Y}$ inside $\mathrm{V}^{(\mathbb{B})}$.

Theorem 7.4. *Let X, $\mathscr{X}$, Y, and $\mathscr{Y}$ be the same as in Proposition 7.3. For every $T \in \mathscr{S}_{\mathbb{B}}(X, Y)$ there exists a unique $\mathscr{T} := T{\uparrow} \in \mathrm{V}^{(\mathbb{B})}$ determined from the formulas*

$$[\![\, \mathscr{T} \in \mathscr{S}(\mathscr{X}, \mathscr{Y}) \,]\!] = \mathbb{1}, \quad [\![\, \mathscr{T} x^\wedge = Tx \,]\!] = \mathbb{1} \quad (x \in X).$$

The mapping $T \mapsto \mathscr{T}$ is a $\mathbb{B}$-isometry from $\mathscr{S}_{\mathbb{B}}(X, Y)$ onto the bounded descent $\mathscr{S}(\mathscr{X}, \mathscr{Y}){\Downarrow}$.

$\lhd$ Suppose that $T \in \mathscr{L}_{\mathbb{B}}(X, Y)$ and $\mathscr{T} \in \mathscr{S}(\mathscr{X}, \mathscr{Y}){\Downarrow}$ is the Boolean valued representation of T, see Theorem 3.7. Then $\sigma(\mathscr{T}) \in \Lambda$ and we can assume $\sigma(\mathscr{T}) \leqslant C\mathbb{1}$ for some $0 < C \in \mathbb{R}$. Moreover, the relation

$$\big(\forall \theta \in \mathscr{P}_{\mathrm{fin}}(\mathscr{X})\big) \sum\nolimits_{x \in \theta} \| \mathscr{T} x \| \leqslant C^\wedge \Big\| \sum\nolimits_{x \in \theta} |x| \Big\| \tag{2}$$

holds in $\mathrm{V}^{(\mathbb{B})}$ and hence, its Boolean truth value is $\mathbb{1}$. Replacing $\mathscr{X}$ by its dense sublattice $X^\wedge$ and making use of the formula $\mathscr{P}_{\mathrm{fin}}(X^\wedge) = \mathscr{P}_{\mathrm{fin}}(X)^\wedge$ we can replace the universal quantifier in (2) over finite subsets of $X^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$ by external infimum over $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$ and get

$$\mathrm{V}^{(\mathbb{B})} \models \sum\nolimits_{x \in \theta} \| \mathscr{T} x^\wedge \| \leqslant C^\wedge \Big\| \sum\nolimits_{x \in \theta} |x^\wedge| \Big\| \quad \big(\theta \in \mathscr{P}_{\mathrm{fin}}(\mathscr{X})\big) \tag{3}$$

for every $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$. Recall that $\mathbb{Q}^\wedge$ may be considered as the internal field of rationals. Denote by $U(\mathscr{Y})$ the unit ball of $\mathscr{Y}$. For arbitrary $0 < \varepsilon \in \mathbb{R}$ and $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$ the sentence

$$(\forall x \in \theta^\wedge)(\exists r_x \in \mathbb{Q}^\wedge)(r_x \leqslant (1 + \varepsilon^\wedge) \| \mathscr{T} x \|) \wedge (\mathscr{T} x \in r_x U(\mathscr{Y})).$$

Replacing quantifiers by infimum over θ and supremum over $\mathbb{Q}$ we deduce that for every $x \in \theta$ there exists a countable partition of unity $(\pi_{x,k})$ and a sequence $(r_{x,k})$ of rationals such that

$$\pi_{x,k} \leqslant [\![(r_{x,k}^\wedge \leqslant (1 + \varepsilon^\wedge) \| \mathscr{T} x^\wedge \|) \wedge (\mathscr{T} x^\wedge \in r_{x,k}^\wedge U(\mathscr{Y}))]\!] \quad (k \in \mathbb{N}).$$

Let (π_k) be a common refinement of the finite collection of partitions of unity $(r_{x,k})$ $(x \in \theta)$. Then for every $x \in \theta$ there is $k(x) \in \mathbb{N}$ such that

$$\pi_k \leqslant [\![r^{\wedge}_{x,k(x)} \leqslant (1+\varepsilon^{\wedge})\|\mathscr{T}x^{\wedge}\|]\!], \quad \pi_k \leqslant [\![\mathscr{T}x^{\wedge} \in r^{\wedge}_{x,k(x)} U(\mathscr{Y})]\!] \quad (k \in \mathbb{N}). \qquad (4)$$

Since $[\![\mathscr{T}x^{\wedge} = Tx]\!] = \mathbb{1}$ and $r_{x,k(x)}U(Y) = \big(r^{\wedge}_{x,k(x)}U(\mathscr{Y})\big){\Downarrow}$, the second relation in (4) implies $\pi_k Tx \in r_{x,k(x)}U(Y)$ or $\|\pi_k Tx\| \leqslant r_{x,k(x)}$. The last inequality together with (3) and the first relation in (4) yields

$$\Big(\sum_{x\in\theta} \|\pi_k Tx\|\Big)^{\wedge} \leqslant \Big(\sum_{x\in\theta} r_{x,k(x)}\Big)^{\wedge} = \sum_{x\in\theta} r^{\wedge}_{x,k(x)}$$

$$\leqslant (1+\varepsilon)^{\wedge} \sum_{x\in\theta} \|\mathscr{T}x^{\wedge}\| \leqslant ((1+\varepsilon)C)^{\wedge} \Big\| \sum\nolimits_{x\in\theta} |x^{\wedge}| \Big\| = \Big((1+\varepsilon)C \Big\| \sum\nolimits_{x\in\theta} |x| \Big\|\Big)^{\wedge}.$$

It follows that for any finite subset $\theta \subset X$ we have

$$\inf_{(\pi_k)\in \mathrm{Prt}_\sigma} \ \sup_{k\in\mathbb{N}} \sum_{x\in\theta} \|\pi_k Tx\| \leqslant (1+\varepsilon)C \Big\| \sum\nolimits_{x\in\theta} |x| \Big\|.$$

Thus, T is $\mathbb{B}$-summing and $\sigma(T) \leqslant C$, since $\varepsilon > 0$ is arbitrary.

Conversely, assume that $T \in \mathscr{S}_{\mathbb{B}}(X,Y)$ and C is a positive constant in Definition 7.1. Then for a finite subset $\theta \subset X$ there is a countable partition of unity (π_k) in $\mathbb{B}$ such that

$$\bigvee_{k\in\mathbb{N}} \sum_{x\in\theta} |\pi_k Tx| \leqslant \bigvee_{k\in\mathbb{N}} \sum_{x\in\theta} \|\pi_k Tx\| \pi_k \mathbb{1} \leqslant C \Big\| \sum_{x\in\theta} |x| \Big\| \mathbb{1}.$$

Taking into account the definition of $\mathscr{T}$ we deduce from the last inequality

$$\pi_k \leqslant \Big[\!\!\Big[\sum\nolimits_{x\in\theta} \|\mathscr{T}x^{\wedge}\| = \sum\nolimits_{x\in\theta} |Tx| \Big]\!\!\Big] \wedge [\![(\forall x \in \theta) Tx = \pi_k Tx]\!]$$

$$\leqslant \Big[\!\!\Big[\sum\nolimits_{x\in\theta} \|\mathscr{T}x^{\wedge}\| = \sum\nolimits_{x\in\theta} |\pi_k Tx| \leqslant C^{\wedge} \Big\| \sum\nolimits_{x\in\theta} |x^{\wedge}| \Big\| \Big]\!\!\Big].$$

Finally, for every $\theta \in \mathscr{P}_{\mathrm{fin}}(X)$ we have

$$\mathbb{1} = \bigvee_{k\in\mathbb{N}} \pi_k \leqslant \Big[\!\!\Big[\sum\nolimits_{x\in\theta} \|\mathscr{T}x^{\wedge}\| \leqslant C^{\wedge} \Big\| \sum\nolimits_{x\in\theta} |x^{\wedge}| \Big\| \Big]\!\!\Big]$$

and thus we arrive at (2) which implies that $\mathscr{T} \in \mathscr{S}(\mathscr{X},\mathscr{Y})$ and $[\![\sigma(\mathscr{T}) \leqslant C^{\wedge}]\!] = \mathbb{1}$. ▷

Corollary 7.5. *Let X be a Banach lattice and Y be a $\mathbb{B}$-cyclic Banach lattice. An operator $T \in \mathscr{L}(X,Y)$ is cone $\mathbb{B}$-summing with $\sigma(T) \leqslant C$ if and only if there exists $\lambda \in \Lambda$ such that $\|\lambda\|_\infty \leqslant C$ and for any finite collection $x_1, \dots x_n \in X$ we have*

$$\sum_{i=1}^{n} |Tx_i| \leqslant \lambda \Big\| \sum_{i-1}^{n} |x_i| \Big\|.$$

Theorem 7.6. *Let X be a Banach lattice and Y be a $\mathbb{B}$-cyclic Banach lattice. For an operator $T \in \mathscr{L}(X,Y)$ the following are equivalent:*

(1) *T is cone $\mathbb{B}$-summing and $\sigma(T) \leqslant C$.*

(2) *There exists a linear operator $S \in \mathscr{L}(X, \Lambda)$ such that $\|S\| \leqslant C$ and $\|\pi T x\| \leqslant \|\pi S(|x|)\|_\infty$ for all $x \in X$ and $\pi \in \mathbb{P}(\Lambda)$.*

(3) *There exist an injective Banach lattice L, a lattice homomorphism $T_1 \in \mathscr{L}(X, L)$ with $\mathbb{B}$-dense range, and $T_2 \in \mathscr{L}_{\mathbb{B}}(L, Y)$ such that $\|T_1\| \leqslant C$, $\|T_2\| \leqslant 1$, and $T = T_2 \circ T_1$.*

$\lhd$ (1) $\Longrightarrow$ (2) Let $T \in \mathscr{L}_{\mathbb{B}}(X, Y)$ with $\sigma(T) \leqslant C$ and $\mathscr{T}$ is defined as in Theorem 7.4. Then $\mathscr{T} \in \mathscr{S}_{\mathbb{B}}(\mathscr{X}, \mathscr{Y})$ and by [32, Ch. 4, §3, Proposition 3.3 (b)] there is $\sigma \in \mathrm{V}^{(\mathbb{B})}$ such that $[\![\, \sigma \in \mathscr{X}'$ and $\|\sigma\| \leqslant C^\wedge$ and $\|\mathscr{T} x\| \leqslant \langle |x|, \sigma\rangle$ for all $x \in \mathscr{X} \,]\!] = \mathbb{1}$. If S is the bounded descent of σ then $\|S\| \leqslant C$ and $|Tx| \leqslant S(|x|)$ for all $x \in X$. The last inequality is equivalent to $(\forall \pi \in \mathbb{P}(\Lambda))\, \|\pi T x\| \leqslant \|\pi S(|x|)\|_\infty$.

(2) $\Longrightarrow$ (3) Making use of Theorem 5.6 with $\Phi := S$ we only have to put $L := X$, $T_1 := \iota$ and define $T_2 : L \to Y$ by $T_2 x := \lim_{\varepsilon \to 0} T_2 x_\varepsilon$ and $\pi_\xi T_2 x_\varepsilon = \pi_\xi T x_\xi$ $(\xi \in \Xi)$. Evidently, by 5.6 (2,3) we have $\|T_1\| \leqslant C$, $T_2 \in \mathscr{L}_{\mathbb{B}}(L, Y)$ and $\|T_2\| \leqslant 1$. Moreover, $T = T_1 \circ \iota = T_2 \circ T_1$ by definition and $T_1(X)$ is $\mathbb{B}$-dense in L by 4.3 (3).

(3) $\Longrightarrow$ (1) Let $T = T_2 \circ T_1$ be a factorization claimed in (3). Observe that the relation $|Tx| \leqslant S(|x|)$ $(x \in X)$ implies $|T_2 x| \leqslant |x|$ $(x \in L)$. For any finite collection $x_1, \dots, x_n \in X_+$ we have

$$\sum_{i=1}^{n} |T_2 \circ T_1 x_i| \leqslant \sum_{i=1}^{n} |T_1 x_i| = \sum_{i=1}^{n} \Phi \circ T_1 x_i = \Phi \circ T_1\Big(\sum_{i=1}^{n} x_i\Big) \leqslant C \Big\| \sum_{i=1}^{n} x_i \Big\| \mathbb{1}$$

and the assertion (1) follows from Corollary 7.5. $\rhd$

Corollary 7.7. *Let X_0 be a Banach sublattice of a Banach lattice X and Y be a $\mathbb{B}$-cyclic Banach space. If $T_0 \in \mathscr{S}_{\mathbb{B}}(X_0, Y)$ then T_0 admits an extension $T \in \mathscr{S}_{\mathbb{B}}(X, Y)$ with $\sigma(T_0) = \sigma(T)$.*

Theorem 7.8. *Let X be a Banach lattice and Y be a $\mathbb{B}$-cyclic Banach lattice. The following are equivalent:*

(1) *$\mathscr{S}_{\mathbb{B}}(X, Y)$ is an injective Banach lattice with a Boolean algebra of M-projections isomorphic to $\mathbb{B}$.*

(2) *X is an AM-space and Y is an injective Banach lattice with $\mathbb{B} = \mathbb{M}(Y)$.*

$\lhd$ The proof can be carried out in similar lines by Boolean valued interpretation of the corresponding result for cone absolutely summing operators due to Schlotterbeck, see [32, Ch. 4, Proposition 4.5]. $\rhd$

8. Atomic Injective Banach Lattices

Definition 8.1. A positive element x of a $\mathbb{B}$-cyclic Banach lattice X is said to be $\mathbb{B}$-*indecomposable* or a $\mathbb{B}$-*atom* if for any pair of disjoint elements $x, y \in X_+$ with $y + z \leqslant x$ there exists a projection $\pi \in \mathbb{B}$ such that $\pi y = 0$ and $\pi^\perp z = 0$, while X is called $\mathbb{B}$-*atomic* if the only element of X disjoint from every $\mathbb{B}$-atom is the zero element. Denote by $\mathrm{at}(\mathscr{X})$ and $\mathbb{B}\text{-at}(X)$ the sets of atoms in $\mathscr{X}$ and $\mathbb{B}$-atoms in X.

Proposition 8.2. *Let X be a $\mathbb{B}$-cyclic Banach lattice identified with the bounded descent $\mathscr{X}{\Downarrow}$ of a Banach lattice, its Boolean valued representation $\mathscr{X} \in \mathrm{V}^{(\mathbb{B})}$. Then the following assertions hold:*

(1) $\mathbb{B}\text{-at}(X) = \text{at}(\mathscr{X})\Downarrow$.

(2) *X is $\mathbb{B}$-atomic if and only if* $[\![\mathscr{X}$ *is atomic*$]\!] = \mathbb{1}$.

$\lhd$ Observe that $x \in \text{at}(\mathscr{X})$ if and only if $x \in \mathscr{X}_+$ and for any two positive disjoint elements $x_1, x_2 \in \mathscr{X}$ with $x_1 + x_2 \leqslant x$ we have $x_1 = 0$ or $x_2 = 0$. Now, given $x \in \text{at}(\mathscr{X})\Downarrow$ with $y + z \leqslant x$ for some disjoint $y, z \in X_+$, we put $b := [\![y = 0]\!]$ and $\pi := \chi(b)$. Since $[\![y \neq 0 \to z = 0]\!] = \mathbb{1}$, we have $[\![y \neq 0]\!] \leqslant [\![z = 0]\!]$ and thus $b^* = [\![y \neq 0]\!] \leqslant [\![z = 0]\!]$. By Gordon Theorem $\pi y = 0$ and $\pi^\perp z = \chi(b^*)z = 0$. Thus, $\text{at}(\mathscr{X})\Downarrow \subset \mathbb{B}\text{-at}(X)$ and for the converse inclusion the argument is similar. The second claim is immediate from the first one, since disjoint complement and bounded desent commute: $(A^\perp)\Downarrow = (A\Downarrow)^\perp$. $\rhd$

Theorem 8.3. *A $\mathbb{B}$-cyclic Banach lattice X is $\mathbb{B}$-atomic and order $\mathbb{B}$-continuous if and only if $\mathscr{L}^r_{\mathbb{B}}(X, Y)$ is a vector lattice for every $\mathbb{B}$-cyclic Banach lattice Y.*

$\lhd$ For the two-element Boolean algebra $\mathbb{B}$ the result was established by van Rooij [31]. The general case is an easy Boolean valued interpretation of the latter. $\rhd$

Theorem 8.4. *If X is a $\mathbb{B}$-cyclic Banach lattice then the following are equivalent:*
(1) *X is $\mathbb{B}$-isomorphic to a $\mathbb{B}$-atomic injective Banach lattice.*
(2) *$\mathscr{L}_{\mathbb{B}}(X, Y) = \mathscr{L}^r_{\mathbb{B}}(X, Y)$ for all $\mathbb{B}$-cyclic Banach lattices Y.*
(3) *$\mathscr{L}_{\mathbb{B}}(X, Y)$ is a vector lattice for all $\mathbb{B}$-cyclic Banach lattices Y.*

$\lhd$ This result for two-element Boolean algebra see in Wickstead [38, Theorem 2.4]. The general case is again an easy Boolean valued interpretation. $\rhd$

Given a non-empty set Γ, denote by $l^1(\Gamma^\wedge)$ the internal Banach lattice of all summable families $x := (x_\gamma)_{\gamma \in \Gamma^\wedge} \subset \mathscr{R}$ with the norm $\|x\| := \sum_{\gamma \in \Gamma^\wedge} |x_\gamma|$. Let $l_1(\Gamma, \Lambda)$ stands for the vector lattice of all order summable families in Λ, i.e.

$$l_1(\Gamma, \Lambda) := \Big\{\mathbf{x} : \Gamma \to \Lambda : \ |\mathbf{x}| := \sum\nolimits_{\gamma \in \Gamma} |\mathbf{x}(\gamma)| \in \Lambda\Big\}$$

equipped with the norm $\|\mathbf{x}\| := \||\mathbf{x}|\|_\infty$, $\mathbf{x} = (x_\gamma)_{\gamma \in \Gamma}$. The map $\Phi : l_1(\Gamma, \Lambda) \to \Lambda$ defined by

$$\Phi(\mathbf{x}) = \sum_{\gamma \in \Gamma} \mathbf{x}(\gamma)$$

is a strictly positive Maharam operator with the Levi property, thus $l_1(\Gamma, \Lambda)$ is an injective Banach lattice by Corollary 5.2.

Proposition 8.5. *The Banach lattice $l^1(\Gamma^\wedge)$ is lattice isometric to the completion of $\mathbb{R}^\wedge$-normed space $l^1(\Gamma)^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$.*

$\lhd$ Denote by $\mathscr{L}_1$ the completion of $l^1(\Gamma)^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$. Let A be the set of all norm-one atoms in $l_1(\Gamma)$ which is of course bijective with Γ. Then $A^\wedge$ and $\Gamma^\wedge$ are also bijective and $A^\wedge$ can be considered as the set of all norm-one atoms in $l_1(\Gamma^\wedge)$. Denote by $\mathbb{Q}\text{-lin}(A)$ the set of all linear combinations of the members of A with rational coefficients. Then by [22, 8.4.10] we have $(\mathbb{Q}\text{-lin}(A))^\wedge = \mathbb{Q}^\wedge\text{-lin}(A^\wedge)$. Clearly, $\mathbb{Q}^\wedge$-lin$(A^\wedge)$ is dense sublattice in $l_1(\Gamma^\wedge)$, while $(\mathbb{Q}\text{-lin}(A))^\wedge$ is dense sublattice in $l^1(\Gamma)^\wedge$ and thus in $\mathscr{L}_1$, since $\mathbb{Q}\text{-lin}(A)$ is dense in $l^1(\Gamma)$. Moreover, the norms induced in $(\mathbb{Q}\text{-lin}(A))^\wedge$ by $l_1(\Gamma^\wedge)$ and $l^1(\Gamma)^\wedge$ coincide. Indeed, if $x \in (\mathbb{Q}\text{-lin}(A))^\wedge$ is of the form

$\sum_{k\in n} r(k)\,u(k)$ whith $n \in \mathbb{N}$, $r : n \to \mathbb{Q}$, and $u : n \to A$, then $r^\wedge : n^\wedge \to \mathbb{Q}^\wedge$, $u^\wedge : n^\wedge \to A^\wedge$ and $x^\wedge = \sum_{k\in n^\wedge} r^\wedge(k)u^\wedge(k)$; therefore,

$$\|x\|_{l_1(\Gamma)^\wedge} = \|x\|^\wedge = \Big(\sum\nolimits_{k\in n} |r(k)|\Big)^\wedge = \sum\nolimits_{k\in n^\wedge} |r^\wedge(k)| = \|x\|_{l_1(\Gamma^\wedge)}.$$

It follows that $\mathscr{L}_1$ and $l_1(\Gamma^\wedge)$ are lattice isometric. $\rhd$

Corollary 8.6. *Let Q be the Stone space of $\mathbb{B} = \mathbb{P}(\Lambda)$. Then the injective Banach lattices $l_1(\Gamma, \Lambda)$ and $C_\#(Q, l_1(\Gamma))$ are lattice $\mathbb{B}$-isometric.*

Corollary 8.7. *Given an arbitrary infinite cardinals γ_1 and γ_2, we may find a Stonean space Q so that the injective Banach lattices $C_\#(Q,\ l_1(\gamma_1)$ and $C_\#(Q, l_1(\gamma_2))$ are lattice $\mathbb{B}$-isometric.*

$\lhd$ There exists a complete Boolean algebra $\mathbb{B}$ such that the ordinals $\gamma_1^\wedge$ and $\gamma_2^\wedge$ have the same cardinality inside $\mathrm{V}^{(\mathbb{B})}$, see [22, A.19]). The claim follows from 7.2 and 8.5. $\rhd$

Corollary 8.8. *For every infinite cardinal γ, there exists a Stonean space Q such that the injective Banach lattice $C_\#(Q, l_1(\gamma))$ is $\mathbb{B}$-separable, with $\mathbb{B}$ standing for the Boolean algebra of the characteristic functions of clopen subsets of Q.*

Take some cardinal λ. Given $b \in \mathbb{B}$ and an ordinal β, denote by $b(\beta)$ the set of all partitions of b of the form $(b_\alpha)_{\alpha\in\beta}$. Define the $[\mathbb{O}, b]$-valued function d on $b(\beta)$ by

$$d(u, v) := \Big(\bigvee_{\alpha\in\beta} u_\alpha \wedge v_\alpha\Big)^*,\ \ \big(u = (u_\alpha),\ \ v = (v_\alpha) \in b(\beta)\big).$$

It is easy to observe that the function d is a Boolean valued metric, see [22, 7.4.11].

Definition 8.9. Given an ordinal γ, write $b(\beta) \simeq b(\gamma)$ if there is a bijection $\imath$ between $b(\beta)$ and $b(\gamma)$ which preserves the Boolean metric; i.e., $d(\imath(u), \imath(v)) = d(u, v)$. We call the Boolean algebra $\mathbb{B}$ as well as its Stone space λ-*stable* provided that $\lambda \leqslant \alpha$ for all nonzero $b \in B$ and each ordinal α with $b(\lambda) \simeq b(\alpha)$. A nonzero element $b \in \mathbb{B}$ is λ-stable by definition whenever $[\mathbb{O}, b]$ is a λ-stable Boolean algebra.

Proposition 8.10. *Suppose that the injective Banach lattices $C_\#(Q, l_1(\gamma))$ and $C_\#(Q, l_1(\delta))$ are lattice $\mathbb{B}$-isometric, where Q is the Stone space of $\mathbb{B}$, while γ and δ are infinite cardinals. If $\mathbb{B}$ is γ-stable and δ-stable then $\gamma = \delta$.*

$\lhd$ If $C_\#(Q, l_1(\Gamma))$ and $C_\#(Q, l_1(\Delta))$ are lattice $\mathbb{B}$-isometric then $\mathrm{V}^{(\mathbb{B})} \models$“ $l_1(\gamma^\wedge)$ and $l_1(\delta^\wedge)$ are lattice isometric” and thus $\mathrm{V}^{(\mathbb{B})} \models |\gamma^\wedge| = |\delta^\wedge|$. It remains to observe that $\mathbb{B}$ is γ-stable (δ-stable) if and only if $\mathrm{V}^{(\mathbb{B})} \models |\gamma^\wedge| = \gamma^\wedge$ ($|\delta^\wedge| = \delta^\wedge$). $\rhd$

Theorem 8.11. *Let X be a $\mathbb{B}$-atomic injective Banach lattice with $\mathbb{B} = \mathbb{M}(X)$. Then there is a family of Stonean spaces $(Q_\gamma)_{\gamma\in\Gamma}$, with Γ a set of cardinals, such that Q_γ is γ-stable for all $\gamma \in \Gamma$ and the following lattice $\mathbb{B}$-isometry holds:*

$$X \simeq_{\mathbb{B}} \Big(\sum\nolimits^{\oplus}_{\gamma\in\Gamma} C_\#\big(Q_\gamma, l_1(\gamma)\big)\Big)_{l_\infty}.$$

If some family $(P_\delta)_{\delta\in\Delta}$ of Stonean spaces satisfies the above conditions, then $\Gamma = \Delta$, and P_γ is homeomorphic with Q_γ for all $\gamma \in \Gamma$.

$\lhd$ By Theorem 4.7 (3) we may assume that X is the bounded descent of an AL-apace $\mathscr{X} \in \mathrm{V}^{(\mathbb{B})}$. Working inside $\mathrm{V}^{(\mathbb{B})}$ we observe that $\mathscr{X}$ is atomic by Proposition

8.2 and thus is lattice isometric to $l_1(\Delta)$ for some set Δ. Now, according to Proposition 3.9, the Boolean valued cardinal $|\Delta|$ is a mixing of some set of standard cardinals Γ, i.e. $|\Delta| = \operatorname{mix}_{\gamma\in\Gamma} \pi_\gamma\gamma^\wedge$. Denote by Q_γ the clopen subset of the Stone space of $\mathbb{B}$ which corresponds to $b_\gamma \in \mathbb{B}$ under the Stone representation. We next make use of the fact that X is the l_∞ direct sum of the spaces $b_\gamma X$, with $b_\gamma X$ lattice $\mathbb{B}_\gamma$-isometric to the restricted descent of the space $b_\gamma\mathscr{X}$ from $\mathrm{V}^{(\mathbb{B}_\gamma)}$, where $\mathbb{B}_\gamma = [\mathbb{O}, b_\gamma]$. By the argument similar to [22, Theorem 8.4.9], we can show that $b_\gamma \leqslant [\![\, b_\gamma\mathscr{X}$ is an atomic AL-space and the cardinality of the set of norm-one atoms is equal to $\gamma^\wedge\,]\!]$. Appealing to the Transfer Principle, we infer that $\mathrm{V}^{(\mathbb{B}_\gamma)} \models$ "$b_\gamma\mathscr{X}$ is lattice isometric to $l_1(\gamma^\wedge)$." By virtue of Propositions 7.2 and 8.5, the bounded descent of $l_1(\gamma^\wedge)$ from the model $\mathrm{V}^{(\mathbb{B}_\gamma)}$ is lattice $\mathbb{B}$-isometric to the injective Banach lattice $C_\#(Q_\gamma, l_1(\gamma))$. Suppose that $u_\gamma \in \mathrm{V}^{(\mathbb{B}_\gamma)}$ is a lattice isometry from $b_\gamma\mathscr{X}$ onto $l_1(\gamma^\wedge)$ inside $\mathrm{V}^{(\mathbb{B}_\gamma)}$, and U_γ is the bounded descent of u_γ. Then U_γ is a lattice $\mathbb{B}$-isometry between the injective Banach lattices $b_\gamma X$ and $C_\#(Q_\gamma, l_1(\gamma))$. By definition, the element $b_\gamma \in \mathbb{B}$, together with the compact space Q_γ, is γ-stable.

Assume now that some family Stonean spaces $(P_\delta)_{\delta\in\Delta}$ obeys the same conditions as $(Q_\gamma)_{\gamma\in\Gamma}$. Then P_δ is homeomorphic with some clopen subset P'_δ of the Stonean space of $\mathbb{B}$. Moreover, P'_δ is δ-stable. If $P_{\delta\gamma} := P'_\delta \cap Q_\gamma$ and $b_{\gamma\delta}$ is the corresponding element of $\mathbb{B}$ then the injective Banach lattices $C_\#(P_{\delta\gamma}, l_1(\delta))$ and $C_\#(P_{\delta\gamma}, l_1(\gamma))$ are lattice $[\mathbb{O}, b_{\delta\gamma}]$-isometric to the same band $b_{\delta\gamma}X$. Furthermore, the compact space $P_{\delta\gamma}$ must be δ- and γ-stable simultaneously.

According to 8.10 either $P_{\delta\gamma} = \varnothing$ or $l_2(\delta) \sim l_2(\gamma)$, implying $\delta = \gamma$. Therefore, $P'_\gamma = Q_\gamma$ $(\gamma \in \Gamma)$. $\rhd$

Corollary 8.12. *Let X is a injective Banach lattice and Q be the Stone space of $\mathbb{B} = \mathbb{M}(X)$. If X is $\mathbb{B}$-separable, then X is lattice $\mathbb{B}$-isometric to $C_\#(Q, l_1)$.*

Proposition 8.13. *A $\mathbb{B}$-cyclic Banach lattice is atomic if and only if it is $\mathbb{B}$-atomic and the Boolean algebra $\mathbb{B}$ is atomic.*

$\lhd$ The complete Boolean algebra $\mathbb{B}$ is atomic if and only if $\mathbb{B} = \mathscr{P}(A)$ for some set A and then X is the l_∞-sum of a family of Banach lattices $(X_a)_{a\in A}$. This l_∞-sum is evidently atomic if and only if X_a is atomic for all $a \in A$. $\rhd$

The following corollary should be compared with [13, Theorem 5.6].

Corollary 8.14. *An injective Banach lattice X is atomic if and only if there is a set of cardinals Γ such that the following lattice isometry holds:*

$$X \simeq \Big(\sum\nolimits_{\gamma\in\Gamma}^{\oplus} l_1(\gamma)\Big)_{l_\infty}.$$

$\lhd$ By Proposition 8.13 Q_γ is a one-point space and $C_\#\big(Q_\gamma, l_1(\gamma)\big) \simeq l_1(\gamma)$. $\rhd$

Definition 8.15. The *second* $\mathbb{B}$-*dual* of a $\mathbb{B}$-cyclic Banach space is defined by $X^{\#\#} := (X^\#)^\# := \mathscr{L}_{\mathbb{B}}(X^\#, \Lambda)$. A $\mathbb{B}$-cyclic Banach space is said to be $\mathbb{B}$-*reflexive* if X and $X^{\#\#}$ coincide under the *canonical embedding* $X \to X^{\#\#}$, see [22, 8.5.4].

Theorem 8.16. *Let X be a $\mathbb{B}$-reflexive injective Banach lattice with $\mathbb{B} = \mathbb{M}(X)$. Then there are a sequence of Stonean spaces $(Q_k)_{k\in\mathbb{N}}$, and an increasing sequence* *of naturals* (n_k) *such that the following lattice* $\mathbb{B}$*-isometry holds:*

$$X \simeq \Big(\sum\nolimits_{k\in\mathbb{N}}^{\oplus} C_{\#}\big(Q_k, l_1(n_k)\big)\Big)_{l_\infty}.$$

If some family $(P_k)_{k\in\mathbb{N}}$ *of Stonean spaces satisfies the above conditions, then* Q_k *and* P_k *are homeomorphic for all* $k \in \mathbb{N}$.

$\lhd$ Again identify X with $\mathscr{X}{\Downarrow}$, where $\mathscr{X}$ is an AL-space in $\mathrm{V}^{(\mathbb{B})}$. It follows from Theorem 3.7 that $\mathscr{X}^*{\Downarrow} = \mathscr{X}{\Downarrow}^{\#}$ and $\mathscr{X}^{**}{\Downarrow} = \mathscr{X}{\Downarrow}^{\#\#}$. Therefore, X is $\mathbb{B}$-reflexive if and only if $[\![\mathscr{X}$ is reflexive $]\!] = \mathbb{1}$. Since a reflexive AL-space is finite-dimensional, we have

$$\mathbb{1} = [\![(\exists n \in \mathbb{N}^{\wedge})\dim(\mathscr{X}) = n]\!] = \bigvee\nolimits_{n\in\mathbb{N}}[\![\dim(\mathscr{X}) = n^{\wedge}]\!].$$

This relation enables us to choose a countable partition of unity (b_n) in $\mathbb{B}$ such that $b_n \leqslant [\![\mathscr{X}$ is a $n^{\wedge}$-dimensional AL-space$]\!]$. Pick the sequence (n_k) of indices of nonzero projections in (b_n) and denote by Q_k the Stonean space of a Boolean algebra $\mathbb{B}_k := [\mathbb{O}, b_{n_k}]$. Now, by the Transfer Principle we conclude that $\mathrm{V}^{(\mathbb{B}_k)} \models$ "$b_{n_k}\mathscr{X}$ is lattice isometric to $l_1(n_k^{\wedge})$". The proof is concluded as in Theorem 8.11 taking into consideration Proposition 3.10 and the fact that a complete Boolean algebra is λ-*stable* for any finite cardinal λ. $\rhd$

References

1. Abramovich Y. A., Weakly compact sets in topological Dedekind complete vector lattices, *Teor. Funkciĭ, Funkcional. Anal. i Priložen.*, **15** (1972), 27–35.
2. Abramovich Y. A., A generalization of a theorem of J. Holub, *Proc. Amer. Math. Soc*, **108** (1990), 937–939.
3. Abramovich Y. A., On spaces of operators acting between Banach lattices, In: *Investigations on Linear Operators and Theory of Functions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **73**(8) (1977), 182–193.
4. Abramovich Y. A. and Aliprantis C. D., Positive operators, In: *Handbook of the Geometry of Banach Spaces.*, **1** (Eds. W. B. Johnson and J. Lindenstrauss). Elsevier, Amsterdam a.o. (2001), 85–122.
5. Abramovich Y. A. and Wickstead, A. W. When each continuous operator is regular, II, *Indag. Math.*, **8** (1997), 281–294.
6. Aliprantis C. D. and Burkinshaw O., *Positive Operators*, Acad. Press Inc., London (1985).
7. Arendt W., Factorization by lattice homomorphisms, *Math. Z.*, **185**(4) (1984), 567–571.
8. Bell J. L., *Boolean-Valued Models and Independence Proofs in Set Theory*, Clarendon Press, New York etc. (1985).
9. Behrends E., *M-Structure and Banach–Stone Theorem*, Springer, Berlin etc. (1979) (Lecture Notes in Math., **736**.)
10. Bukhvalov A. V. Geometric properties of Banach spaces of measurable vector-functions, *Dokl. Akad. Nauk SSSR*, **239**(6) (1978), 1279–1282.
11. Buskes G., Separably-injective Banach lattices are injective, *Proc. Roy. Irish Acad. Sect*, **A85**(2) (1985), 185–186.
12. Cartwright D. I., The order completeness of some spaces of vector-valued functions, *Bull. Austral. Math. Soc.*, **11** (1974), 57–61.
13. Cartwright D. I., Extension of positive operators between Banach lattices, *Memoirs Amer. Math. Soc.*, **164** (1975), 1–48.
14. Diestel J., Jarchow H., Tonge A., *Absolutely Summing Operators*, Cambridge Univ. Press, Cambridge etc. (1995).
15. *Fremlin D. H. Topological Riesz Spaces and Measure Theory*, Cambridge Univ. Press, London (1974).
16. Gordon E. I., Real numbers in Boolean-valued models of set theory and K-spaces, *Dokl. Akad. Nauk SSSR*, **237**(4) (1977), 773–775.
17. Gordon E. I., K-spaces in Boolean-valued models of set theory, *Dokl. Akad. Nauk SSSR*, **258**(4) (1981), 777–780.
18. Harmand P., Werner D. and Wener W. *M-Ideals in Banach Spaces and Banach Algebras*, Springer, Berlin etc. (1993) (Lecture Notes in Math, **1547**).
19. Haydon R. Injective Banach lattices, *Math. Z.*, **156** (1977), 19–47.
20. Kakutani S., Concrete representation of abstract (L)-spaces and the mean ergodic theorem, *Ann. of Math.*, **42** (1941), 523–537.
21. Kusraev A. G., Order continuous functionals in Boolean valued models of set theory, *Siberian Math. J.*, **25**(1) (1984), 57–65.
22. Kusraev A. G., *Dominated Operators*, Kluwer, Dordrecht (2000).
23. Kusraev A. G. and Kutateladze S. S., *Boolean Valued Analysis*, Nauka, Novosibirsk (1999); Kluwer, Dordrecht (1999).
24. Levin V. L., Tensor products and functors in categories of Banach spaces determined by *KB*-lineals, *Dokl. Acad. Nauk SSSR*, **163**(5) (1965), 1058–1060.
25. Levin V. L. Convex Analysis in Spaces of Measurable Functions and Its Application in Mathematics and Economics [in Russian], Nauka, Moscow (1985).
26. Lindenstrauss J. and Tzafriri L., On the isomorphic classification of injective Banach lattices, *Advances Math.*, **7B** (1981), 489–498.
27. Lotz H. P., Extensions and liftings of positive linear mappings on Banach lattices, *Trans. Amer. Math. Soc.*, **211** (1975), 85–100.
28. Mangheni P. J., The classification of injective Banach lattices, *Israel J. Math*, **48** (1984), 341–347.
29. Meyer-Nieberg P., *Banach Lattices*, Springer-Verlag, Berlin etc. (1991).

30. Nakano H., Über die Charakterizierung des allgemainen C-Raumen, *Proc. Imp. Acac. Tokyo*, **17** (1941), 301–307.
31. van Rooij A C. M. When do the regular operators between two Riesz spaces form a Riesz space, *Dokl. Technical Report 8410*, Katholike Universiteit, Nijmegen (1984).
32. Schaefer H. H., *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin etc. (1974).
33. Schmidt K. D. Daugavet's equation and orthomorphisms. *Proc. Amer. Math. Soc.*, **108** (1990), 905–911.
34. Semadeni Z., *Banach Spaces of Continuous Functions*, **1**, Polish Scientific Publishers, Warszawa (1971).
35. Takeuti G., Boolean completion and m-convergence, In: *Categorical Aspects of Topology and Analysis* (Ottawa, Ont. (1980)), Springer-Verlag, Berlin etc. (1982), 333–350. (Lecture Notes in Math., **915**.)
36. Takeuti G. and Zaring W. M. *Axiomatic set Theory*, Springer-Verlag, New York, 1973.
37. Wickstead A. W., Relatively central operators on Archimedean vector lattices, *Proc. Roy. Irish Acad. Sect.*, **A80**(2) (1980), 191–208.
38. Wickstead A. W. AL-spaces and AM-spaces of operators, *Positivity*, **4**(3) (2000), 303–311.
39. Wickstead A. W. Regular operators between Banach lattices, *Positivity* (Eds. K. Boulabiar, G. Buskes and A. Tiki), Birkhäuser, Basel a. o. (2007), 255–279.

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