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Цель настоящей работы — получить новые сведения о строении ортосимметрических билинейных операторов в векторных решетках и, в частности, доказать теорему типа Радона — Никодима для этого класса операторов. Основные средства, используемые и развиваемые в работе, базируются на однородном функциональном исчислении в векторных решетках и понятии степени векторной решетки. В этой связи собраны также некоторые полезные факты о взаимодействии однородного функционального исчисления и квадратов векторных решеток.

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The purpose of this paper is to provide new information about the structure of orthosymmetric bilinear operators and, in particular, to prove a Radon–Nikodým type theorem for this class of operators. The main tools used and developed in the paper are based on homogeneous functional calculus on vector lattices and squares of vector lattices. Therefore, some useful facts concerning an interplay between squares of vector lattices and homogeneous functional calculus are collected.

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ORTHOSYMMETRIC BILINEAR OPERATORS[1]

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Introduction

The class of orthosymmetric bilinear operators on vector lattices was introduced by G. Buskes and A. van Rooij in [14] and received much attention in succeeding years, see [5, 8, 13, 16, 21, 28, 29]. An inseparable companion of the orthosymmetric bilinear operators turns out to be the concept of square of vector lattice, developed by the same authors in another paper [16]. Different approaches to the construction of the square of a vector lattice are presented in [5, 16].

As was observed in [13, 29], in the theory of positive orthosymmetric bilinear operators, the role played by square of Archimedean vector lattice is as important as that of Fremlin's tensor product of Archimedean vector lattices [19] in the theory of general positive bilinear operators. Both, the Fremlin tensor product and the square, possess the following universal property: the sets of positive bilinear operators on Cartesian product of two Archimedean vector lattices and positive orthosymmetric bilinear operators on Cartesian square of an Archimedean vector lattice with values in a relatively uniformly complete vector lattice are bijective with the sets of positive linear operators on the Fremlin tensor product and on the square of given vector lattices, respectively.

At the same time there is a significant difference: the mentioned correspondence between positive linear and bilinear operators do not preserve order continuity in the case of the Fremlin tensor product, while it does preserve in the case of the square. Thus, the square of a vector lattice lends itself to a transfer of known results on regular order continuous linear operators to regular order continuous orthosymmetric bilinear operators. This fact is crucial, in particular, for the validity of a Radon–Nikodým type theorem for orthosymmetric bilinear operators.

The aim of this paper is to provide new information about the structure of orthosymmetric bilinear operators and, in particular, to prove a Radon–Nikodým type theorem for this class of operators. The paper is organized as follows. The main purpose of the first two sections is to fix the notation and terminology and give a brief outline of some useful results which are of particular importance to this paper. Section 1 deals with general bilinear operators on products of vector lattices while Section 2 introduces the class of orthosymmetric bilinear operators, the main subject of the paper.

The purpose of the next three sections is to consider some interplay between squares of vector lattices and homogeneous functional calculus. We also collect some useful facts on homogeneous functional calculus which despite of their simplicity does not seem appeared in the literature. In Section 3 we introduce homogeneous functional calculus on relatively uniformly complete vector lattices. Section 4 deals

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with Hölder type inequalities for orthosymmetric bilinear operators. In Section 5 we prove further Hölder type inequalities containing homogeneous expressions of the form $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N}$ with $0 \le p_1, \ldots, p_N \in \mathbb{R}$, $p_1 + \cdots + p_N = 1$.

Section 6 provides some new information concerning the structure of the square of a vector lattice. In Section 7 we characterize orthoregular bilinear operators that may be presented as differences of symmetric lattice bimorphisms. In Section 8 we introduce order interval preserving order continuous bilinear operators and prove that an orthosymmetric positive bilinear operator is order interval preserving if and only if its linearization via square is also order interval preserving.

A Radon–Nikodým type theorem and Hahn Decomposition Theorem for order continuous orthoregular bilinear operators are discussed in Section 9. Finally, Section 10 contains concluding remarks outlining some further perspectives.

The main tools used and developed in this paper are based on two fundamental concepts, namely homogeneous functional calculus on vector lattices and powers of vector lattices. Both go back to G. Ya. Lozanovskiĭ, see [40, 41, 42, 43, 44, 45].

For the theory of vector lattices and positive operators we refer to the books [4] and [27]. Throughout the paper a vector lattice means an Archimedean vector lattice over the field of real numbers. We use the symbol := if an equality is taken as a definition; $\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural numbers and reals, respectively.

1. Bilinear operators on vector lattices

In this section we introduce the classes of bilinear operators on products of vector lattices. The main purpose is to fix the notation and terminology and give a brief outline of some useful facts. For an extended discussion of this subject see the forthcoming survey paper [11].

1.1. Let E, F, and G be vector lattices. A bilinear operator $b : E \times F \to G$ is called *positive* if $b(x, y) \ge 0$ for all $0 \le x \in E$ and $0 \le y \in F$, and *regular* if it can be represented as the difference of two positive bilinear operators. Denote by $BL_r(E, F; G)$ and $BL_+(E, F; G)$ respectively the sets of all regular and positive bilinear operators from $E \times F$ to G. For any positive bilinear operator b we have $|b(x, y)| \le b(|x|, |y|)$ $(x \in E,\ y \in F)$.

A bilinear operator $b : E \times F \to G$ is said to be *of order bounded variation* if for all $0 \le x \in E$ and $0 \le y \in F$ the set

$$\Sigma b[x; y] := \Bigg\{ \sum_{k=1}^{n} \sum_{l=1}^{m} b(x_k, y_l) :\ 0 \le x_k \in E\,(1 \le k \le n \in \mathbb{N}),$$
$$0 \le y_l \in E\,(1 \le l \le m \in \mathbb{N}),\ x = \sum_{k=1}^{n} x_k,\ y = \sum_{l=1}^{m} y_l \Bigg\}$$

is order bounded in G. The set of all bilinear operators $b : E \times F \to G$ that are of order bounded variations (order bounded) is denoted by $BL_{bv}(E, F; G)$ $\big(BL^{\sim}(E, F; G)\big)$ and forms an ordered vector space with the positive cone $BL_+(E, F; G)$. Obviously, $BL_r(E, F; G) \subset BL_{bv}(E, F; G) \subset BL^{\sim}(E, F; G)$ and $BL_r(E, F; G)$ is considered with the induced ordering. The converse inclusion may be false. Order bounded variation was first introduced in [51], see also [11, 17].

If G is Dedekind complete then $BL_r(E, F; G) = BL_{bv}(E, F; G)$ and this space is a Dedekind complete vector lattice, see [17]. In particular, every regular bilinear operator $b \in BL_r(E, F; G)$ has the modulus $|b|$ and

$$|b|(x, y) = \sup \Sigma b[x; y] \quad (0 \leq x \in E,\ 0 \leq y \in F),$$
$$|b(x, y)| \leq |b|(|x|, |y|) \quad (x \in E,\ y \in F).$$

1.2. A bilinear operator $b : E \times F \to G$ is said to be *lattice bimorphism* if the mappings $b_e : y \mapsto b(e, y)$ $(y \in F)$ and $b_f : x \mapsto b(x, f)$ $(x \in E)$ are lattice homomorphisms for all $0 \leq e \in E$ and $0 \leq f \in F$, see [19]. Evidently, every lattice bimorphism is positive. The following characterization of lattice bimorphism was given in [54].

For a positive bilinear operator b the following assertions are equivalent:

(1) *b is a lattice bimorphism;*

(2) *$|b(x, y)| = b(|x|, |y|)$ for all $x \in E$ and $y \in F$;*

(3) *if $0 \leq x, u \in E$ and $0 \leq y, v \in F$ satisfy $x \wedge u = 0$ and $y \wedge v = 0$, then $b(x, y) \wedge b(u, v) = 0$.*

The lattice bimorphisms are simple in structure modulo the lattice homomorphisms, as is shown in [35]: *Any lattice bimorphism $b : E \times F \to G$ admits the representation $b(x, y) = S(x)T(y)$ $(x \in E,\ y \in F)$, where $S : E \to G^u$ and $T : F \to G^u$ are lattice homomorphisms with values in the universal completion G^u of G and G^u is equipped with an f-algebra multiplication uniquely determined by a choice of an order unit in G^u.*

The following fundamental result was established by D. Fremlin in [19; Theorem 4.2]. (Different approaches to the Fremlin tensor product see [22, 54].)

1.3. Theorem. *Let E and F be vector lattices. Then there exist a unique up to isomorphism vector lattice $E \bar{\otimes} F$ and a bimorphism $\phi : E \times F \to E \bar{\otimes} F$ such that:*

(1) *whenever G is a vector lattice and $\psi : E \times F \to G$ is a lattice bimorphism, there is a unique lattice homomorphism $T : E \bar{\otimes} F \to G$ with $T \circ \phi = \psi$;*

(2) *ϕ induces an embedding of the algebraic tensor product $E \otimes F$ into $E \bar{\otimes} F$;*

(3) *$E \otimes F$ is dense in $E \bar{\otimes} F$ in the sense that for every $v \in E \bar{\otimes} F$ there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\varepsilon > 0$ there is an element $u \in E \otimes F$ with $|v - u| \leq \varepsilon x_0 \otimes y_0$;*

(4) *if $0 < v \in E \bar{\otimes} F$, then here exist $x \in E_+$ and $y \in F_+$ with $0 < x \otimes y \leq v$.*

The lattice bimorphism ϕ is conventionally denoted by $\otimes$ and the algebraic tensor product $E \otimes F$ is regarded as actually embedded into $E \bar{\otimes} F$.

1.4. Let ψ and T be the same as in Theorem 1.3 (1). Suppose that for any $x \in E_+$ and $y \in E_+$ the equality $\psi(x, y) = 0$ implies $x = 0$ or $y = 0$. In this case T is injective and thus maps $E \bar{\otimes} F$ onto a vector sublattice of G generated by $\operatorname{im} \psi := \psi(E \times F)$. In particular, if E_0 and F_0 are vector sublattices in E and F, respectively, then the tensor product $E_0 \bar{\otimes} F_0$ is isomorphic to the vector sublattice in $E \bar{\otimes} F$ generated by $E_0 \otimes F_0$. Therefore, $E_0 \bar{\otimes} F_0$ is regarded as a vector sublattice of $E \bar{\otimes} F$, see [19; Corollaries 4.4 and 4.5].

D. Fremlin [19; Theorem 5.3] proved also the following important universal property of $E \bar{\otimes} F$: *if G is a relatively uniformly complete vector lattice, then for every positive bilinear operator $b : E \times F \to G$ there exists a unique positive linear operator $T : E \bar{\otimes} F \to G$ such that $b = T\otimes$.*

Let $L_r(H, G)$ and $L^\sim(H, G)$ stand respectively for the spaces of all linear regular operators and linear order bounded operators from H to G.

1.5. Theorem. *Let E, F, and G be vector lattices with G relatively uniformly complete. Then the mapping $T \mapsto T\otimes$ constitutes an isomorphism of the following pairs of ordered vector spaces:*

(1) $L_r(E \mathbin{\bar{\otimes}} F, G)$ *and* $BL_r(E, F; G)$*;*

(2) $L^{\sim}(E \mathbin{\bar{\otimes}} F, G)$ *and* $BL_{bv}(E, F; G)$*.*

$\lhd$ The first assertion is immediate from the above mentioned universal property of the Fremlin tensor product and the second one was established in [17]. $\rhd$

Thus, the Fremlin tensor product lends itself to a transfer of known results on regular linear operators to regular bilinear operators as well as on order bounded linear operators to bilinear operators of order bounded variation. This and certain other aspects of bilinear operators on products of vector lattices are presented in [11]. Concerning dominated bilinear operators in lattice normed spaces see [36].

1.6. A bilinear operator b is called *separately order continuous* if b_e and b_f are order continuous for each $e \in E$ and each $f \in F$. *Order continuity* of b means that the net $\big(b(x_\alpha, y_\beta)\big)$ is order convergent to $b(x, y)$ whenever (x_α) is order convergent to x in E and $\big(y_\beta\big)$ is order convergent to y in F. As was observed in [57] a regular bilinear operator b is order continuous if and only if b is separately order continuous. The set of all order continuous regular bilinear operators with the linear operations and order relation induced from $BL_r(E, F; G)$ is denoted by $BL_n(E, F; G)$.

An operator $b \in BL_r(E, F; G)$ is called *singular* if it vanishes on some order dense ideal in $E \times F$. We say that b is *supersingular* if b vanishes on an order dense ideal of the form $E_0 \times F$ or $E \times F_0$ where E_0 and F_0 are order dense ideals in E and F, respectively. The sets of all singular and all order continuous bilinear operators comprise disjoint order ideals in $BL_r(E, F; G)$.

1.7. *Let E, F and G be vector lattices with G Dedekind complete. For any regular bilinear operator $b : E \times F \to G$ the following are equivalent:*

(1) *b is order continuous;*

(2) *b is separately order continuous;*

(3) *b is disjoint from all singular $b' \in BL_r(E, F; G)$;*

(4) *b is disjoint from each supersingular $b' \in BL_r(E, F; G)$.*

In particular, $BL_n(E, F; G)$ is a band in $BL_r(E, F; G)$.

$\lhd$ This fact was proved in [25; Proposition 4] and [57; Theorem 1]. $\rhd$

1.8. In spite of the nice universal property, Fremlin's tensor product has an essential disadvantage: the isomorphism from 1.5 do not preserve order continuity. For an order continuous $T \in L_r(E \mathbin{\bar{\otimes}} F \to G)$ the bilinear operator $T\otimes \in BL_r(E, F; G)$ is also order continuous but the converse may be false. An example can be extracted from [20].

D. Fremlin introduced also a construction for the "projective" tensor product $E \mathbin{\overset{\vartriangle}{\otimes}} F$ of Banach lattices E and F as the completion of $E \mathbin{\bar{\otimes}} F$ under "positive-projective" norm $\|\cdot\|_{|\pi|}$ [20; Theorem 1 E]. If $E = L^2([0, 1])$, then $E \mathbin{\bar{\otimes}} E$ is order dense in $E \mathbin{\overset{\vartriangle}{\otimes}} E$ but the norm of $E \mathbin{\overset{\vartriangle}{\otimes}} E$ is not order continuous, see [20; 4 B and 4 C]. Thus, there exists a (norm continuous) positive linear functional $l \in (E \mathbin{\overset{\vartriangle}{\otimes}} E)'$ which is not order continuous. Clearly, the restriction l_0 of l to $E \mathbin{\bar{\otimes}} E$ is not order continuous, too. At the same time the positive bilinear functional $b = l_0\otimes$ is separately order continuous, since E has an order continuous norm.

2. Orthosymmetric bilinear operators

As the title indicates in this section we introduce the main subject of the paper and outline some results needed in the sequel. More details can be found in [11, 13].

2.1. A bilinear operator $b : E \times E \to G$ is called *orthosymmetric* if $|x| \wedge |y| = 0$ implies $b(x, y) = 0$ for arbitrary $x, y \in E$, see [14]. The difference of two positive orthosymmetric bilinear operators is called *orthoregular*, see [13, 29]. Denote by $BL_{or}(E; G)$ the space of all orthoregular bilinear operators from $E \times E$ to G ordered by the cone of positive orthosymmetric operators. Recall also that b is said to be *symmetric* if $b(x, y) = b(y, x)$ for all $x, y \in E$, *positively semidefinite* if $b(x, x) \geq 0$ for every $x \in E$, and *positively definite* if it is positively semidefinite and $b(x, x) = 0$ implies $x = 0$ for all $x \in E$.

As an example of orthosymmetric bilinear operator we refer to the multiplication $(x, y) \mapsto xy$ $(x, y \in A)$ of an almost f-algebra A, since an *almost f-algebra* is by definition a lattice ordered algebra whose multiplication is a positive orthosymmetric bilinear operator, see [6]. The survey on certain aspects of f-algebras see in [10].

2.2. Denoted by $BL_o^{\sim}(E; G)$ the space of orthosymmetric order bounded bilinear operators from $E \times E$ to F with ordering induced from $BL^{\sim}(E, E; F)$. Put $BL_{obv}(E; G) := BL_{bv}(E, E; G) \cap BL_o^{\sim}(E; G)$. Then $BL_{or}(E; G) \subset BL_{obv}(E; G) \subset BL_o^{\sim}(E; G)$ and the converse inclusions may be false. If G is Dedekind complete, then these three classes of operators coincide and form a band in $BL_r(E, E; G)$. Thus, any positive bilinear operator $b : E \times E \to G$ admits a unique decomposition $b = c + d$ where $c \perp d$, c is orthosymmetric, and no nonzero orthosymmetric positive bilinear operator is majorized by d. The orthosymmetric component c of b can be described as follows:

$$c(x, y) := \inf \left\{ \sum_{\{k,l:\ x_k \wedge y_l \neq 0\}} b(x_k, y_l) :\ 0 \leq x_k, y_l \in E\ (k = 1, \dots, m \in \mathbb{N};\right.$$
$$\left. l = 1, \dots, n \in \mathbb{N}),\ x = \sum_{k=1}^{m} x_k,\ y = \sum_{l=1}^{n} y_l \right\},$$

for $x, y \in E_+$. Clearly, c admits a unique bilinear extension from $E_+ \times E_+$ to $E \times E$ by differences. In the case of bilinear forms (i. e. $G = \mathbb{R}$) this formula was established by O. van Gaans [21; Theorem 3.2]. S. N. Tabuev observed that the formula remains valid in general case (private communication).

The following important property of orthosymmetric bilinear operators was established in [14; Corollary 2]:

2.3. Theorem. *If E and F be vector lattices, then any orthosymmetric positive bilinear operator from $E \times E$ to F is symmetric.*

In particular, any Archimedean almost f-algebra is commutative [14]. It is easily seen that an orthosymmetric positive bilinear operator is positively semidefinite [21]. The converse is also true for lattice bimorphisms, see [13].

2.4. *If E and F be vector lattices, then for any lattice bimorphism $b : E \times E \to F$ the following are equivalent:*

(1) *b is symmetric;*

(2) *b is orthosymmetric;*

(3) *b is positively semidefinite.*

2.5. Theorem. *For an arbitrary vector lattice E there exists (unique up to isomorphism) a vector lattice $E^{\odot}$ and a lattice bimorphism $\odot : (x, y) \mapsto x \odot y$ from $E \times E$ to $E^{\odot}$ such that the following assertions hold:*

(1) *if b is a symmetric lattice bimorphism from $E \times E$ to some vector lattice F then there is a unique lattice homomorphism $\Phi_b : E^{\odot} \to F$ with $b = \Phi_b \odot$;*

(2) *given an arbitrary $u \in E^{\odot}$, there is $e_0 \in E_+$ such that, for every $\varepsilon > 0$, one can choose $x_1, \dots, x_n, y_1, \dots, y_n \in E$ with*

$$\left| u - \sum_{i=1}^{n} x_i \odot y_i \right| \leq \varepsilon e_0 \odot e_0;$$

(3) *for any $x, y \in E$ we have $x \odot y = 0$ if and only if $x \perp y$;*

(4) *given an element $0 < u \in E^{\odot}$, there exists $e \in E_+$ with $0 < e \odot e \leq u$.*

$\lhd$ See [13; Theorem 2.1] and [15; Theorem 4]. $\rhd$

2.6. The vector lattice $E^{\odot}$ (or the pair $(E^{\odot}, \odot)$) uniquely (up to lattice isomorphism) determined by an arbitrary vector lattice E is called the *square* of E. The lattice bimorphism $\odot : E \times E \to E^{\odot}$ is called the *canonical bimorphism.* The construction of $E^{\odot}$ was first introduced in [16] as follows.

Denote by J the smallest relatively uniformly closed order ideal in Fremlin's tensor product $E \bar{\otimes} E$ containing the set $\{x \otimes y : x, y \in E, x \perp y\}$. Define $E^{\odot} := E \bar{\otimes} E / J$ and $\odot := \phi \otimes$ where $\phi : E \bar{\otimes} E \to E^{\odot}$ is the quotient homomorphism. Then $E^{\odot}$ and $\odot$ meet the requirements of Theorem 2.5.

The pair $(E^{\odot}, \odot)$ is essentially unique, i.e. if for some vector lattice $E^{\circledcirc}$ and symmetric lattice bimorphism $\circledcirc : E \times E \to E^{\circledcirc}$ the pair $(E^{\circledcirc}, \circledcirc)$ obeys the said universal property, then there exists a lattice isomorphism i from $E^{\odot}$ onto $E^{\circledcirc}$ such that $i \odot = \circledcirc$ (and, of course, $i^{-1} \circledcirc = \odot$). Now we state a structural property of orthosymmetric regular bilinear operators [31].

2.7. Theorem. *Let E, F, and G be vector lattices with G uniformly complete. Let $\langle \cdot\,, \cdot \rangle : E \times E \to F$ be a positively definite lattice bimorphism and F_0 be the smallest vector sublattice in F containing the set $\{\langle x, y \rangle : x, y \in E\}$. Then for every orthosymmetric regular bilinear operator $b : E \times E \to G$ there exists a unique regular linear operator $\Phi_b : F_0 \to G$ such that*

$$b(x, y) = \Phi_b(\langle x, y \rangle) \quad (x, y \in E).$$

The correspondence $b \mapsto \Phi_b$ constitutes an isomorphism of the ordered vector spaces $BL_{or}(E, G)$ and $L_r(F_0, G)$.

By 2.5 (1) there exists a lattice isomorphism $h : E^{\odot} \to F_0$ such that $\langle \cdot\,, \cdot \rangle = h \odot$. Thus, if G is relatively uniformly complete, then for every bilinear orthoregular operator $b : E \times E \to G$ there exists a unique linear regular operator $\Phi_b : E^{\odot} \to G$ such that

$$b(x, y) = \Phi_b(x \odot y) \quad (x, y \in E).$$

Moreover, the correspondence $b \mapsto \Phi_b$ (called also the *linearization via square*) constitutes an isomorphism of the ordered vector spaces $BL_{or}(E, G)$ and $L_r(E^{\odot}, G)$, see [15; Theorem 9] and [13; Theorem 3.1]. The operator Φ_b is also called the *linearization* of b *via square.*

Thus, an orthoregular bilinear operator defined on a vector lattice and with values in a uniformly complete vector lattice is representable as a composition of the canonical bimorphism and some regular linear operator uniquely defined on the square of the given vector lattice. This approach leads us to extension and analytical representation results for orthoregular bilinear operators, see [13].

2.8. The following result was obtained in [15]: *The Dedekind completion of an almost f-algebra A can be endowed by an almost f-algebra multiplication that extends the multiplication on A.* This raises the question of whether an almost f-algebra multiplication given on a majorizing vector sublattice A can be extended to an almost f-algebra multiplication on the ambient vector lattice E. The positive answer was announced in [29] but the available proof is not correct as G. Buskes indicated (private communication).

3. Homogeneous functions on vector lattices

In this section we introduce homogeneous functional calculus on relatively uniformly complete vector lattices and state some useful facts. There are different ways to introduce the homogeneous functional calculus on vector lattices, see [12, 24, 39, 45, 55, 58]. We follow the approach [12, 16] going back to G. Ya. Lozanovskiĭ [45].

3.1. Denote by $\mathscr{H}(\mathbb{R}^N)$ the vector lattice of all continuous positively homogeneous functions $f:\mathbb{R}^N\to\mathbb{R}$. In accordance with [12] we say that $f(x_1,\dots,x_N)$ exists in E and write $y=f(x_1,\dots,x_N)$ if there is an element $y\in E$ such that $\omega(y)=f(\omega(x_1),\dots,\omega(x_N))$ for every $\mathbb{R}$-valued lattice homomorphism ω on the sublattice of E generated by $\{x_1,\dots,x_N,y\}$. The definition is correct in the sense that if L is any vector sublattice of E containing $\{x_1,\dots,x_N,y\}$ and $\omega(y)=f(\omega(x_1),\dots,\omega(x_N))$ $(\omega\in\Omega)$ for some point separating set Ω of $\mathbb{R}$-valued lattice homomorphisms on L, then $y=f(x_1,\dots,x_N)$. It is immediate from the definition that $f(x,\dots,x)=xf(1,\dots,1)$ whenever $0\le x\in E$. Define $dt_j\in\mathscr{H}(\mathbb{R}^N)$ by $dt_j(t_1,\dots,t_N)=t_j$ $(j:=1,\dots,N)$. A homogeneous functional calculus can be extended in the spirit of monotonic analysis [53] to the class of increasing positively homogeneous functions f defined on conic subsets of $\mathbb{R}^N$.

3.2. Theorem. *Let E be a relatively uniformly complete vector lattice and $x_1,\dots,x_N\in E$. Then $f(x_1,\dots,x_N)$ exists for any $f\in\mathscr{H}(\mathbb{R}^N)$ and the mapping*

$$f\mapsto f(x_1,\dots,x_N)\quad\big(f\in\mathscr{H}(\mathbb{R}^N)\big)$$

is a unique lattice homomorphism from $\mathscr{H}(\mathbb{R}^N)$ into E with $dt_j(x_1,\dots,x_N)=x_j$ $(j:=1,\dots,N)$.

In particular, if $f,g\in\mathscr{H}(\mathbb{R}^N)$ and $f\le g$, then $f(x_1,\dots,x_N)\le g(x_1,\dots,x_N)$ for all $(x_1,\dots,x_N)\in E^N$. Moreover, the inequality holds:

$$|f(x_1,\dots,x_N)|\le\|f\|\bigvee_{j=1}^N|x_j|,$$

where $\|f\|:=\sup\{f(t_1,\dots,t_N):\ (t_1,\dots,t_N)\in\mathbb{R}^N,\ \max_j|t_j|=1\}$.

3.3. Let $K,M,N\in\mathbb{N}$ and consider finite collections of positively homogeneous functions $f_1,\dots,f_M\in\mathscr{H}(\mathbb{R}^N)$ and $g_1,\dots,g_K\in\mathscr{H}(\mathbb{R}^M)$. Denote $f:=(f_1,\dots,f_M)$ and $g:=(g_1,\dots,g_K)$. Then $g_1\circ f,\dots,g_K\circ f\in\mathscr{H}(\mathbb{R}^N)$ and, for any $x:=(x_1,\dots,x_N)$ in E^N and $y:=(y_1,\dots,y_M)$ in E^M, the elements $f(x):=(f_1(x),\dots,f_M(x))\in E^M$ and $g(y):=(g_1(y),\dots,g_K(y))\in E^K$ are well defined. Moreover,

$$(g\circ f)(x)=g(f(x))\quad(x:==(x_1,\dots,x_N)\in E^N).$$

In particular, if $N = M = K$ and $g = f^{-1}$, then

$$f^{-1}(f(x)) = x, \quad f(f^{-1}(y)) = y \quad (x, y \in E^N).$$

We define also $f_1 \times g_1 : E^{N+M} \to E^2$ by $(f_1 \times g_1)(x, y) := (f_1(x), g_1(y))$.

3.4. Let E and F be relatively uniformly complete vector lattices, $h : E \to F$ be a lattice homomorphism, $x_1, \dots, x_N \in E$, and $f \in \mathscr{H}(\mathbb{R}^N)$. Then

$$h(f(x_1, \dots, x_N)) = f(h(x_1), \dots, h(x_N))).$$

If E is a relatively uniformly complete vector sublattice of F containing $x_1, \dots, x_N \in F$ and h is the inclusion map $E \hookrightarrow F$, then $f(x_1, \dots, x_N)$ relative to F is contained in E and its meaning relative to E is the same.

3.5. Assume that $f \in \mathscr{H}(\mathbb{R}^N)$ possesses the following property:

$$(\forall\, t_1, \dots, t_N \in \mathbb{R})\, t_1 t_2 \cdot \ldots \cdot t_N = 0 \;\Rightarrow\; f(t_1, t_2, \dots, t_N) = 0.$$

Then for any $u, x_1, \dots, x_N \in E$ and fixed integer $1 \leq k \leq N$ we have

$$x_k \perp u \;\Rightarrow\; f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_N) \perp u.$$

Moreover, for any band $L \subset E$ there holds $f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_N) \in L$ whenever $x_k \in L$. If L admits a band projection π, then

$$\pi f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_N) = f(x_1, \dots, x_{k-1}, \pi x_k, x_{k+1} \dots, x_N).$$

Now, we consider concrete examples of homogeneous functions.

3.6. Homogeneous functional calculus is used to introduce the so called p-convexification and p-concavification procedures for a Banach lattice, see [39, 58]. Consider three functions $\sigma_{\alpha,N}, \sigma'_{\alpha,N} : \mathbb{R}^N \to \mathbb{R}$, and $J : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\sigma_{\alpha,N}(t_1, \dots, t_N) := \theta_\alpha^{-1}(\theta_\alpha(t_1) + \dots + \theta_\alpha(t_N)),$$
$$\sigma'_{\alpha,N}(t_1, \dots, t_N) := \theta_\alpha(\theta_\alpha^{-1}(t_1) + \dots + \theta_\alpha^{-1}(t_N)),$$
$$J(r, s) := \theta_2^{-1}(rs) \quad (r, s, t_1, \dots, t_N \in \mathbb{R}),$$

where $0 < \alpha \in \mathbb{R}$ and $\theta_\alpha : t \mapsto \operatorname{sgn}(t) t^\alpha$ is an order preserving bijection of $\mathbb{R}$. Obviously, $\sigma_{\alpha,N}, \sigma'_{\alpha,N}$ belong to $\mathscr{H}(\mathbb{R}^N)$ and J belongs to $\mathscr{H}(\mathbb{R}^2)$, so that $\sigma_{\alpha,N}(x_1, \dots, x_N), \sigma'_{\alpha,N}(x_1, \dots, x_N)$, and $J(x, y)$ are well defined for all $x, y, x_1, \dots, x_N$ in a relatively uniformly complete vector lattice E. From the above definitions the following implication is easily deduced

$$(\forall\, x, y \in E)\, |x| \wedge |y| = 0 \;\Rightarrow\; \sigma_{\alpha,2}(x, y) = \sigma'_{\alpha,2}(x, y) = x + y,$$

since it is true in the real context. Denote for brevity $\theta := \theta_2$, $\sigma = \sigma_{2,2}$, and $\sigma' = \sigma'_{2,2}$.

Given a relatively uniformly complete vector lattice E, the square $(E^\odot, \odot)$ can be defined as $E^\odot := (E, \tilde{+}, *, \leq)$ and $\odot := J$, where $x \tilde{+} y := \sigma(x, y)$, $\lambda * x := \theta^{-1}(\lambda) x$, and $\leq$ is the given ordering in E, see [16; Theorem 9] and 6.1 below.

3.7. We say that a function $f \in \mathscr{H}(\mathbb{R}^N)$ is *multiplicative* and *modulus preserving* if $f(s_1 t_1, \dots, s_N t_N) = f(s_1, \dots, s_N) f(t_1, \dots, t_N)$ and $f(|t_1|, \dots, |t_N|) = |f(t_1, \dots, t_N)|$ for all $s_1, t_1, \dots, s_N, t_N \in \mathbb{R}$. The general form of a positively homogeneous multiplicative and modulus preserving function is given by

$$t_1 t_2 \cdot \ldots \cdot t_N = 0 \;\Rightarrow\; f(t_1, t_2, \dots, t_N) = 0,$$
$$f(t_1, \dots, t_N) = f(|t_1|, \dots, |t_N|) \operatorname{sgn} f(t_1, \dots, t_N),$$
$$f(|t_1|, \dots, |t_N|) = \exp(g_1(\ln |t_1|)) \cdot \ldots \cdot \exp(g_N(\ln |t_N|)),$$

where $g_1, \dots, g_N$ are some additive functions in $\mathbb{R}$ (i. e. solutions to Cauchy functional equation, see [2]) with $\sum_{i=1}^N g_i = I_{\mathbb{R}}$. In the case of continuous $g_1, \dots, g_N$ we get a Kobb–Duglas type function f and if, in addition, f is nonnegative, then $f(t_1, \dots, t_N) = c|t_1^{p_1}| \cdot \ldots \cdot |t_N^{p_N}|$ with $0 \le c, p_1, \dots, p_N \in \mathbb{R}$ and $\sum_{i=1}^N p_i = 1$. Therefore, the expression $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N}$ is well defined in E. Moreover,

$$|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N} \le p_1|x_1| + \cdots + p_N|x_N|$$

by the inequality between the weighted arithmetic and geometric means.

4. Gauges and Hölder type inequalities

Now we consider some interplay between squares of vector lattices and homogeneous functional calculus and deduce some Hölder type inequalities. In the sequel E denotes a relatively uniformly complete vector lattice.

4.1. A *gauge* is a nonnegative sublinear function defined on a convex cone contained in $\mathbb{R}^N$. The *polar* k° of a gauge k defined by

$$k^\circ(t) := \inf\{\lambda > 0 : \ (\forall\, s \in \mathbb{R}^N)\, \langle s, t\rangle \le \lambda k(s)\} \quad (t \in \mathbb{R}^N)$$

is also a gauge. (Hereafter $\langle s, t\rangle := s_1 t_1 + \cdots + s_N t_N$). Moreover, $k^{\circ\circ} := (k^\circ)^\circ = k$ if and only if k is lower semicontinuous (for more details see [52]).

A gauge $k : \mathbb{R}^N \to \mathbb{R}$ is *strictly positive* provided that $k(s) > 0$ for every $s \ne 0$. Here we consider only strictly positive gauges defined everywhere on $\mathbb{R}^N$. The totality of such gauges on $\mathbb{R}^N$ will be denoted by $\mathscr{G}(\mathbb{R}^N)$. Every gauge from $\mathscr{G}(\mathbb{R}^N)$ is continuous. The polar of a gauge $k \in \mathscr{G}(\mathbb{R}^N)$ is also contained in $\mathscr{G}(\mathbb{R}^N)$ and can be calculate by

$$k^\circ(t) = \sup_{0 \ne s \in \mathbb{R}^N} \frac{\langle s, t\rangle}{k(s)} = \sup\{\langle s, t\rangle : \ s \in \mathbb{R}^N, \ k(s) \le 1\} \quad (t \in \mathbb{R}^N).$$

Since $\mathscr{G}(\mathbb{R}^N) \subset \mathscr{H}(\mathbb{R}^N)$, there exist $k(x_1, \dots, x_N) \in E$ and $k^\circ(x_1, \dots, x_N) \in E$ for any $x_1, \dots, x_N \in E$. Moreover, the mapping $(x_1, \dots, x_N) \mapsto k(x_1, \dots, x_N)$ is a sublinear operator from E^N to E and

$$|k(x_1, \dots, x_N) - k(y_1, \dots, y_N)| \le \|p\| \bigvee_{i=1}^N |x_i - y_i|.$$

4.2. *If $k \in \mathscr{G}(\mathbb{R}^N)$ and $x_1, \dots, x_N \in E$, then*

$$k^\circ(x_1, \dots, x_N) = \sup\left\{ \sum_{i=1}^N \lambda_i x_i : \ (\lambda_1, \dots \lambda_N) \in \mathbb{R}^N, \ k(\lambda_1, \dots \lambda_N) \le 1 \right\}.$$

Moreover, $k^\circ(x_1, \dots, x_N)$ is a relatively uniform limit of an increasing sequence which is comprised of the finite suprema of linear combinations of the form $\sum_{i=1}^N \lambda_i x_i$ with $k(\lambda_1, \dots \lambda_N) \le 1$.

$\lhd$ Observe that the set $U := \left\{ \sum_{i=1}^N \lambda_i x_i \ : \ k(\lambda_1, \dots \lambda_N) \le 1 \right\}$ is norm totally bounded in the AM-space E_u, $u := |x_1| \vee \cdots \vee |x_N|$, since it is the image of the compact set $\{\lambda \in \mathbb{R}^N : \ k(\lambda) \le 1\}$ under the map $\lambda = (\lambda_1, \dots, \lambda_N) \mapsto \sum_{k=1}^N \lambda_i x_i$.

Denote by $U^{\vee}$ the subset of E consisting of the suprema of the finite subsets of U. Then by Krengel's Lemma (see [4; Theorem 12.29] or [1; Lemma 3.13]) $y := \sup U$ exists in E_u and belongs to the norm closure $\overline{U^{\vee}}$ of $U^{\vee}$. Since $U^{\vee}$ is upward directed, $U^{\vee}$ is norm convergent to y. Therefore, for any $\mathbb{R}$-valued homomorphism ω on E_u we have

$$\omega(y) = \lim_{u \in U^{\vee}} \omega(u) = \sup\{\omega(u) : \ u \in U^{\vee}\}$$
$$= \sup\{\omega(u) : \ u \in U\} = k^{\circ}(\omega(x_1), \dots, \omega(x_N)).$$

Thus, $y = k^{\circ}(x_1, \dots, x_N)$ by [12; Corollary 3.4]. $\rhd$

4.3. Take a gauge $k_{p,N} : (t_1, \dots, t_N) \mapsto \left(\sum_{i=1}^{N} |t_i|^p\right)^{\frac{1}{p}}$ with $1 \le p \le \infty$. For the corresponding mapping from E^N into E an expressive notation is used, see [39, 55, 58]:

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} := k_{p,N}(x_1, \dots, x_N) \quad (x_1, \dots, x_N \in E).$$

For $p = \infty$, we define $k_{p,N}(t_1, \dots, t_N) = \max\big\{|t_i| : \ i := 1, \dots, N\big\}$ and, obviously, $k_{p,N}(x_1, \dots, x_N) = |x_1| \vee \ldots \vee |x_N|$. Of course, $k_{p,N} \in \mathscr{H}(\mathbb{R}^N)$ and the mapping $(x_1, \dots, x_N) \mapsto k_{p,N}(x_1, \dots, x_N) \in E$ is well defined even if $0 < p < 1$, but in this case $k_{p,N} \notin \mathscr{G}(\mathbb{R}^N)$ and the corresponding mapping is not sublinear.

4.4. *For any $k \in \mathscr{G}(\mathbb{R}^N)$ and $x_1, \dots, x_N, y_1, \dots, y_N \in E$ the inequality holds*

$$\sum_{i=1}^{N} x_i \odot y_i \le k(x_1, \dots, x_N) \odot k^{\circ}(y_1, \dots, y_N).$$

$\lhd$ It is an easy exercise to check that the inequality (see 3.6)

$$\sigma_{2,N}\big(J(s_1, t_1), \dots, J(s_N, t_N)\big) \le J\big(k(s_1, \dots, s_N), k^{\circ}(t_1, \dots, t_N)\big).$$

is equivalent to the well known property of gauges (see, [52]):

$$\langle s, t\rangle \le k(s)k^{\circ}(t) \quad (s = (s_1, \dots, s_N), \ t = (t_1, \dots, t_N) \in \mathbb{R}^N).$$

Combining this with 3.3 and 3.6 we obtain the desired inequality. $\rhd$

In the special case of $k := k_{p,N}$, $k^{\circ} = k_{q,N}$, $1 \le p,\ q \le \infty$, $1/p + 1/q = 1$, we have

$$\sum_{i=1}^{N} |x_i \odot y_i| \le \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}} \odot \left(\sum_{i=1}^{N} |y_i|^q\right)^{\frac{1}{q}}.$$

4.5. *If $b : E \times E \to G$ is a positive orthosymmetric bilinear operator and $x_i, y_i \in E$, $i := 1, \dots, N$, then*

$$\sum_{k=1}^{N} |b(x_k, y_k)| \le b\big(k(x_1, \dots, x_N), k^{\circ}(y_1, \dots, y_N)\big).$$

$\lhd$ Apply Φ_b to 4.4 and use 2.7. $\rhd$

Again, if $1 \le p,\ q \le \infty$ and $1/p + 1/q = 1$, then

$$\sum_{k=1}^{N} |b(x_k, y_k)| \le b\left(\left(\sum_{k=1}^{N} |x_k|^p\right)^{\frac{1}{p}}, \left(\sum_{k=1}^{N} |y_k|^q\right)^{\frac{1}{q}}\right).$$

4.6. *Let $b : E \times E \to G$ be a positive orthosymmetric bilinear operator and $b = \Phi_b \odot$ for a positive linear operator Φ_b from $E^{\odot}$ to G. Then for $x_1, y_1, \dots, x_N, y_N \in E$ and $k \in \mathscr{G}(\mathbb{R}^N)$ we have*

$$k(b(x_1, y_1), \dots, b(x_N, y_N)) \le \Phi_b\big(k(x_1 \odot y_1, \dots, x_N \odot y_N)\big).$$

In particular, if $1 \le p,\ q \le \infty$ and $1/p + 1/q = 1$, then

$$\left(\sum_{i=1}^{N} |b(x_i, y_i|)^p\right)^{\frac{1}{p}} \le \Phi_b\left(\left(\sum_{i=1}^{N} |x_i \odot y_i|^p\right)^{\frac{1}{p}}\right).$$

$\lhd$ Taking into consideration 2.7, 4.2 and positivity of Φ_b we deduce

$$\sum_{i=1}^{N} \lambda_i b(x_i, y_i) = \Phi_b\left(\sum_{i=1}^{N} \lambda_i x_i \odot y_i\right) \le \Phi_b\big(k(x_1 \odot y_1, \dots, x_N \odot y_N)\big)$$

for any finite collection $\lambda_1, \dots \lambda_N \in \mathbb{R}$ with $k^{\circ}(\lambda_1, \dots \lambda_N) \le 1$. It remains to apply 4.2 again. $\rhd$

4.7. *If in 4.6 b is a lattice bimorphism, then*

$$k(b(x_1, y_1), \dots, b(x_N, y_N)) = \Phi_b\big(k(x_1 \odot y_1, \dots, x_N \odot y_N)\big).$$

In particular, if $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, then we have

$$\left(\sum_{i=1}^{N} |b(x_i, y_i|^p\right)^{\frac{1}{p}} = \Phi_b\left(\left(\sum_{i=1}^{N} |x_i \odot y_i|^p\right)^{\frac{1}{p}}\right).$$

$\lhd$ Since Φ_b is a lattice homomorphism by 2.5 (1), we only need to apply 3.4 and 2.7. $\rhd$

5. Inequalities with monomials

In this section we prove several inequalities containing homogeneous expressions of the form $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N}$ with $0 \le p_1, \dots, p_N \in \mathbb{R}$, $p_1 + \dots + p_N = 1$, see 3.7.

5.1. *Assume that a homogeneous function $f \in \mathscr{H}(\mathbb{R}^N)$ is multiplicative and modulus preserving. Then for all $x_1, y_1, \dots, x_N, y_N \in E$ we have*

$$f(x_1 \odot y_1, \dots, x_N \odot y_N) = f(x_1, \dots, x_N) \odot f(y_1, \dots, y_N).$$

In particular, if $0 \le p_1, \dots, p_N \in \mathbb{R}$, $p_1 + \dots + p_N = 1$, then

$$\prod_{i=1}^{N} |x_i \odot y_i|^{p_i} = \left(\prod_{i=1}^{N} |x_i|^{p_i}\right) \odot \left(\prod_{i=1}^{N} |y_i|^{p_i}\right).$$

$\lhd$ If f is multiplicative and modulus preserving, then $\theta(f(s_1, \dots, s_N)) = f(\theta(s_1), \dots, \theta(s_N))$ and the equality $f \circ (J \times \dots \times J) = J \circ (f \times f)$ holds, see 3.7. Applying 3.3 and 3.6 we come to the desired inequalities. $\rhd$

5.2. Теорема (The generalized Hölder inequality). *Assume that E and G be relatively uniformly complete vector lattices. If a mapping $f : E \to G$ is sublinear $\big(f(x+y) \le f(x) + f(y),\ f(\lambda x) = \lambda f(x);\ x, y \in E,\ 0 \le \lambda \in \mathbb{R}\big)$ and increasing on E_+, then*

$$f\left(\prod_{i=1}^{N} |x_i|^{p_i}\right) \le \prod_{i=1}^{N} f(|x_i|)^{p_i}$$

for $x_1, \dots, x_N \in E$ and $0 \le p_1, \dots, p_N \in \mathbb{R}$ with $p_1 + \dots + p_N = 1$. Equality holds if f is order continuous lattice homomorphism.

$\lhd$ Without loss of generality we may assume that $0 \le x_i$ and $0 < p_i < 1$ for all $i := 1, \dots, N$. Indeed, if $\{i_1, \dots, i_k\} = \{j \le N :\ p_j \ne 0\}$, then $|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N} = |x_{i_1}|^{p_{i_1}} \cdot \ldots \cdot |x_{i_k}|^{p_{i_k}}$. Now we observe that, for $0 \le x, y \in E$ and $0 < p < 1$, the representation holds

$$x^p y^{1-p} = \inf\{p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y :\ 0 < \lambda \in \mathbb{R}\}.$$

Indeed, by 3.7 for an arbitrary $0 < \lambda \in \mathbb{R}$ the inequality is valid:

$$x^p y^{1-p} = \big(\lambda^{1/p}x\big)^p \big(\lambda^{-1/(1-p)}y\big)^{1-p} \le p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y.$$

Assume that $v \le \varphi_\lambda := p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y$ for all $0 < \lambda \in \mathbb{R}$. By the Kreĭns–Kakutani Representation Theorem we can view the principal ideal E_u generated by $u = x+y+|v|$ as $C(S)$ for some compact space S. Then v, x, y, and $x^p y^{1-p}$ lie in $C(S)$ and for $0 < \lambda \in \mathbb{R}$ the pointwise inequality $v(s) \le \varphi_\lambda(s)$ $(s \in S)$ is true. If $x(s) = 0$, then trivially $v(s) \le \inf\{(1-p)\lambda^{-1/(1-p)}y(s) :\ 0 < \lambda \in \mathbb{R}\} = 0 = x(s)^p y(s)^{1-p}$. If $x(s) \ne 0$, then for $\lambda := (y(s)/x(s))^{p(1-p)}$ we have $\varphi_\lambda(s) = x(s)^p y(s)^{1-p} \ge v(s)$. Thus, $v \le x^p y^{1-p}$ and the desired representation for $x^p y^{1-p}$ follows.

Now, taking into consideration that f is sublinear and increasing, we deduce

$$f(x^p y^{1-p}) \le \inf\{f(p\lambda^{1/p}x + (1-p)\lambda^{-1/(1-p)}y) :\ 0 < \lambda \in \mathbb{R}\} \le$$
$$\le \inf\{p\lambda^{1/p}f(x) + (1-p)\lambda^{-1/(1-p)}f(y) :\ 0 < \lambda \in \mathbb{R}\} = f(x)^p f(y)^{1-p}.$$

The general case is handled by induction. Suppose

$$f(x_1^{q_1} \cdot \ldots \cdot x_{N-1}^{q_{N-1}}) \le f(x_1)^{q_1} \cdot \ldots \cdot f(x_{N-1})^{q_{N-1}},$$

whenever $q_1 + \dots + q_{N-1} = 1$. Put $p := p_1 + \dots + p_{N-1}$, $q_i := p_i/p$ $(i := 1, \dots, N-1)$, and $u := \big(x_1^{p_1} \cdot \ldots \cdot x_{N-1}^{p_{N-1}}\big)^{1/p} = x_1^{q_1} \cdot \ldots \cdot x_{N-1}^{q_{N-1}}$. Then

$$f\big(x_1^{p_1} \cdot \ldots \cdot x_N^{p_N}\big) = f(u^p x_N^{p_N}) \le f(u)^p f(x_N)^{p_N}$$
$$= f\big(x_1^{q_1} \cdot \ldots \cdot x_{N-1}^{q_{N-1}}\big)^p f(x_N)^{p_N} \le f(x_1)^{p_1} \cdot \ldots \cdot f(x_N)^{p_N},$$

and the required inequality follows. The remaining part is obvious. $\rhd$

5.3. We can take in 5.2 an arbitrary increasing gauge $k \in \mathscr{G}(\mathbb{R}^M)$ instead of f and consider the corresponding sublinear operator from E^M to E. Suppose that $M \in \mathbb{N}$ and for every $j := 1, \dots, M$ a finite collection of elements $(x_{1j}, \dots, x_{Nj}) \in E^M$ is given. Replacing f, for example, by $k_{p,M}$ $(1 \le p \le \infty)$ we arrive at the following version of Hölder inequality:

$$\left(\sum_{j=1}^{M} \Big(|x_{1j}|^{p_1} \cdot \ldots |x_{Nj}^{p_N}|\Big)^p\right)^{1/p} \le \left(\sum_{j=1}^{M} |x_{1j}|^p\right)^{p_1/p} \cdot \ldots \cdot \left(\sum_{j=1}^{M} |x_{Nj}|^p\right)^{p_N/p}.$$

5.4. Let (Ω, Σ, μ) be a measure space with a σ-finite positive measure μ and F be a Banach lattice. Let $\mathscr{L}^1(\Omega, \Sigma, \mu, F)$ be the space of all Bochner integrable functions on Ω with values in F and $E := L^1(\mu, F) := \mathscr{L}(\Omega, \Sigma, \mu, F)/\sim$ denotes the space of all equivalence classes (of almost everywhere equal) functions from $\mathscr{L}^1(\Omega, \Sigma, \mu, F)$. Then $E = L^1(\mu, F)$ is also a Banach lattice and hence $f(x_1, \dots, x_N)$ is well defined in E for $f \in \mathscr{H}(\mathbb{R}^M)$ and $x_1, \dots, x_N \in E$. Denote by $\tilde{x}$ the equivalence class of $x \in \mathscr{L}^1(\Omega, \Sigma, \mu, F)$. Making use of the continuity of functional calculus (see [16; Theorem 7]) one can deduce that for any finite collection $x_1, \dots, x_N \in \mathscr{L}^1(\Omega, \Sigma, \mu, F)$ the equality $f(\tilde{x}_1, \dots, \tilde{x}_N)(\omega) = f(x_1(\omega), \dots, x_N(\omega))$ is true for almost all $\omega \in \Omega$. Since the Bochner integral defines a linear and increasing operator from E to F, we can replace f in 5.2 by the Bochner integral. Thus, we get the following Hölder inequality:

$$\int_\Omega \left(\prod_{i=1}^N |x_i(\omega)|^{p_i} \right) d\mu \leq \prod_{i=1}^N \left(\int_\Omega |x_i(\omega)|\, d\mu \right)^{p_i}$$

for $x_1(\cdot), \dots, x_N(\cdot) \in \mathscr{L}^1(\Omega, \Sigma, \mu, F)$, $0 \leq p_1, \dots, p_N \in \mathbb{R}$, $p_1 + \dots + p_N = 1$.

5.5. *Let E and G be relatively uniformly complete vector lattices, $f, g : E \to G$ be ingreasing sublinear operators, and $b : E \times E \to G$ be a positive orthosymmetric bilinear operator. Then*

$$b\big(f(|x_1|^{p_1} \cdot \ldots \cdot |x_N|^{p_N}),\ g(|y_1|^{p_1} \cdot \ldots \cdot |y_N|^{p_N})\big) \leq \prod_{i=1}^N b\big(f(|x_i|), g(|y_i|)\big)^{p_i}.$$

for all $x_1, y_1, \dots, x_N, y_N \in E$ and $0 \leq p_1, \dots, p_N \in \mathbb{R}$ with $p_1 + \dots + p_N = 1$.

$\triangleleft$ By applying 5.2 to f and g and using 5.1 we obtain

$$f\left(\prod_{i=1}^N |x_i|^{p_i} \right) \odot g \left(\prod_{i=1}^N |y_i|^{p_i} \right) \leq \prod_{i=1}^N \big(f(|x_i|) \odot g(|y_i|) \big)^{p_i}.$$

It remains to apply Φ_b to the last inequality, use again 5.2 with $f := \Phi_b$, and take 2.7 into account. $\triangleright$

6. Squares of vector lattices

In this section we shall take a close look at the square of a vector lattice defined in 2.6. First, in 6.1 and 6.2 (1, 2) below we present briefly some results from [16]. The following theorem is essentially a paraphrase of [16; Theorem 9].

6.1. Theorem. *Let E be a relatively uniformly complete vector lattice. Then the map $\iota : x \mapsto x \odot |x|$ constitutes a modulus preserving orthogonally additive order isomorphism of E onto $E^\odot$. Moreover, for any order bounded orthosymmetric bilinear operator b from $E \times E$ to any vector lattice F the formula*

$$(\Phi_b \circ \iota)(x) := b(x, |x|) \quad (x \in E)$$

defines a unique order bounded linear operator Φ_b from $E^\odot$ to F with $b = \Phi_b \circ \odot$.

$\triangleleft$ Use the p-convexification procedure with $p = 1/2$, see [16] ($p = 2$ in notation of [39, 58]). Define an addition $\tilde{+}$ and a scalar multiplication $*$ on E by

$$x \tilde{+} y := \sigma(x, y), \quad \lambda * x := \theta^{-1}(\lambda) x \quad (\lambda \in \mathbb{R};\ x, y \in E)$$

and put $E^\bullet := (E, \tilde{+}, *, \leq)$, where $\leq$ is the given ordering in E. It was established in [16; Theorem 9 (iii)] that the homogeneous function J from 3.6, considered as an operator from $E \times E$ to $E^\bullet$, is a symmetric bimorphism and $(E^\bullet, J)$ is a square of E. By 2.5 the square $(E^\odot, \odot)$ is essentially unique and thus $j \circ J = \odot$ for some lattice isomorphism $j : E^\bullet \to E^\odot$. Denote by ι the lattice homomorphism j considered as a bijection of E onto $E^\odot$. According to 3.6 $x \perp y$ implies $x + y = x \tilde{+} y$ and hence $\iota(x+y) = j(x \tilde{+} y) = j(x) + j(y) = \iota(x) + \iota(y)$; therefore, we conclude that ι is orthogonally additive. Since J is an orthosymmetric bimorphism, we have $J(x, |x|) = J(x^+, x^+) - J(x^-, x^-) = (x^+ - x^-)J(1, 1) = x$ and thus, $\iota(x) = x \odot |x|$. Now, if a bilinear operator $b : E \times E \to F$ is order bounded and orthosymmetric, then, according to [16; Theorem 9], $x \mapsto \Phi_b^\bullet(x) := b(x, |x|)$ $(x \in E^\bullet)$ is an order bounded linear operator and $b = \Phi_b^\bullet \circ J$. Clearly, $\Phi_b := \Phi_b^\bullet \circ \iota^{-1}$ is the required linear order bounded operator. $\rhd$

6.2. Thus, $E^\bullet$ is the square of E with the canonical bimorphism J. Moreover E and $E^\bullet$ coincide as ordered sets and the vector lattice structure on $E^\bullet$ is transplanted from $E^\odot$ by means of ι. The inverse of the addition, the modulus of an element, and the disjointness relation have the same meaning in E and $E^\bullet$, since $(-1) * x = (-1)x$ for all $x \in E$. The sum of two disjoint elements is the same in E and $E^\bullet$ because of orthogonal additivity of ι. Continuity of the addition $\tilde{+}$ relative to relatively uniform convergence implies that relatively uniformly convergent nets are the same in E and $E^\bullet$. Thus, we arrive at the following corollary, see [16; Corollaries 10 and 11].

(1) *If a vector lattice is relatively uniformly complete (laterally complete, Dedekind σ-complete, Dedekind complete) then so is its square.*

Denote by $BL_o^\sim(E; F)$ the space of all order bounded orthosymmetric bilinear operators from $E \times E$ to F ordered by the cone of positive operators. (Note that if F is Dedekind complete then $BL_o^\sim(E; F)$ and $BL_{or}(E; F)$ coincide.) The following proposition is an immediate consequence of 6.1, cf. 2.7.

(2) *Let E be a relatively uniformly complete vector lattice and F be an arbitrary vector lattice. The correspondence $b \mapsto \Phi_b$ from 6.1 is an isomorphism of ordered vector spaces $BL_o^\sim(E; F)$ and $L^\sim(E^\odot, F)$. Moreover, b is a lattice bimorphism if and only if Φ_b is a lattice homomorphism.*

Note that $|J(x, y)| \leq |x| \vee |y|$ $(x, y \in E)$, as $|J(s, t)| \leq \max\{|s|, |t|\}$ $(s, t \in \mathbb{R})$. From this we deduce $|x \odot y| = |j \circ J(x, y)| \leq \iota(|x|) \vee \iota(|y|)$ and thus, the following useful fact is valid.

(3) *The canonical bimorphism $(x, y) \mapsto x \odot y$ is order continuous.*

6.3. Theorem. *Let F be a relatively uniformly complete vector lattice. Then there exists a relatively uniformly complete vector lattice F° such that F is the square of F°; in symbols, $(F^\circ)^\odot = F$. The vector lattice F° with this property is unique up to lattice isomorphism.*

$\lhd$ The first part follows readily from 6.1 and [58; Proposition 4.8 (ii)]. Define an addition $\dot{+}$ and a scalar multiplication $\star$ on F by (see 3.6)

$$x \dot{+} y := \sigma'(x, y), \quad \lambda \star x := \theta(\lambda)x \quad (\lambda \in \mathbb{R};\ x, y \in F)$$

and put $F^\circ := (F, \dot{+}, \star, \leq)$ where $\leq$ is the given ordering in F. Then F° is a relatively uniformly complete vector lattice and $(F^\circ)^\bullet = F$.

Now, suppose that for some vector lattice E there is a lattice isomorphism h of $E^\bullet$ onto F. Consider the homogeneous function σ' on $E^\bullet$ (with respect to $\tilde{+}$) and on

F (with respect to $+$). By 3.4 $h(\sigma'(x,y)) = \sigma'(h(x),h(y))$ and taking into account the definitions of $\tilde{+}$ and $\dot{+}$ we get $h(x+y) = h(x) \dot{+} h(y)$. Moreover, the relations $h(\lambda * x) = \lambda h(x)$ and $h(\lambda x) = \lambda \star h(x)$ are identical. Since E and $E^\bullet$ as well as F and F° have the same ordering, h is a lattice isomorphism between E and F°. Thus, the uniqueness of F° follows and the proof is complete. $\triangleright$

Now, we present some further properties of the square of a vector lattice. Denote by $\mathfrak{B}(E)$ and $\mathfrak{P}(E)$ the Boolean algebras of all bands and all band projections in E, respectively. Let $\mathscr{S}_{\mathrm{uc}}(E)$ denotes the inclusion-ordered set of relatively uniformly complete sublattices of E.

6.4. Theorem. *Let E be a relatively uniformly complete vector lattice. For $L \in \mathscr{S}_{\mathrm{uc}}(E)$ denote by $\hat{\iota}(L)$ the set $\iota(L) := \{x \odot |x| : x \in L\}$ with the ordering and vector operations induced from $E^\odot$. Then the following statements hold:*

(1) *the mapping $L \mapsto \hat{\iota}(L)$ constitutes an order isomorphism of the inclusion-ordered sets $\mathscr{S}_{\mathrm{uc}}(E)$ and $\mathscr{S}_{\mathrm{uc}}(E^\odot)$;*

(2) *$L \in \mathscr{S}_{\mathrm{uc}}(E)$ is an order (dense) ideal in E if and only if $\hat{\iota}(L)$ is an order (dense) ideal in $E^\odot$;*

(3) *$\hat{\iota}$ defines a Boolean isomorphism of $\mathfrak{B}(E)$ onto $\mathfrak{B}(E^\odot)$;*

(4) *$\hat{\iota}(L)$ admits a band projection π if and only if L admits a band projection π' and in this case $\pi(x \odot y) = (\pi'x) \odot y = x \odot (\pi'y) = (\pi'x) \odot (\pi'y)$;*

(5) *E has the projection property simultaneously with $E^\odot$ and in this case there exists a Boolean isomorphism $\pi \mapsto \pi'$ of $\mathfrak{P}(E^\odot)$ onto $\mathfrak{P}(E)$ such that $\pi(x \odot y) = (\pi'x) \odot y = x \odot (\pi'y) = (\pi'x) \odot (\pi'y)$ for all $x, y \in E$ and $\pi \in \mathfrak{P}(E)$;*

(6) *if E is Dedekind complete then there exists an f-algebra isomorphism $\alpha \mapsto \alpha'$ of $\mathrm{Orth}^\infty(E^\odot)$ onto $\mathrm{Orth}^\infty(E)$ such that $\alpha(x \odot y) = (\alpha'x) \odot y = x \odot (\alpha'y)$ for all $\alpha \in \mathrm{Orth}^\infty(E^\odot)$ and $x, y \in \mathscr{D}(\alpha)$, where $\mathscr{D}(\alpha)$ denotes the domain of α.*

$\triangleleft$ According to 6.1 and 6.2 we may assume that $E^\odot = E^\bullet$ and $\odot = J$. Then ι is the identity map in E and $\hat{\iota}(L) = L^\bullet$ for every $L \in \mathscr{S}_{\mathrm{uc}}(E)$.

(1): By 6.1 the correspondence $L \mapsto L^\bullet$ is an inclusion preserving injection from $\mathscr{S}_{\mathrm{uc}}(E)$ into $\mathscr{S}_{\mathrm{uc}}(E^\bullet)$. To prove that it is bijection take a relatively uniformly complete sublattice K in $E^\bullet$. Then by 6.3 $L := K^\circ$ is a relatively uniformly complete sublattice in E and $L^\bullet = K$.

(2): In addition to (1) it should be noted that, by virtue of 6.2, the sublattices $L \subset E$ and $L^\bullet \subset E^\bullet$ are order (dense) ideals or not simultaneously, since the modulus and the disjointness relation have the same meaning in E and $E^\bullet$.

(3): In accordance with the remarks at the beginning of 6.2 the Boolean algebras $\mathfrak{B}(E)$ and $\mathfrak{B}(E^\bullet)$ are identical as inclusion-ordered sets.

(4): Denote by $\oplus$ and $\tilde{\oplus}$ the direct sums relative to $+$ and $\tilde{+}$, respectively. It can be easily deduced from (2) and 6.1 that the equalities $L^\bullet \tilde{\oplus} (L^\bullet)^\perp = E^\bullet$ and $L \oplus L^\perp = E$ are identical and the band projection in $E^\bullet$ onto $L^\bullet$ coincides with the band projection in E onto L. This proves the first part of the statement. Now, if π is the band projection in $E^\bullet$ onto $L^\bullet$ and π' is the same map considered as the band projection in E onto L, then according to 3.4 and 3.6 we have $\pi J(x,y) = J(\pi'x, \pi'y)$ for all $x, y \in E$. Since J is orthosymmetric (see 6.1), $J(\pi'x, \pi'y) = J(\pi'x, y)$.

(5): Follows immediately from (4).

(6): The Boolean isomorphism $\pi \mapsto \pi'$ from (5) is uniquely extended to an f-algebra isomorphism of $\mathrm{Orth}^\infty(E)$ onto $\mathrm{Orth}^\infty(E^\odot)$. Denote this isomorphism by $\alpha \mapsto \alpha'$. If $\alpha := \sum_{l=1}^{n} \lambda_l \pi_l$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ and $\{\pi_1, \dots, \pi_n\}$ is a partition of unity in $\mathfrak{P}(G)$, then, obviously, $(\pi'_l \circ \alpha')(x \odot y) = \pi'_l((\lambda_l \pi_l x) \odot y) = \pi_l((\alpha x) \odot y))$ for all l and $x, y \in E$. Summing over l yields $\alpha'(x \odot y) = (\alpha x) \odot y = x \odot (\alpha y)$. Finally, if $\alpha \in \mathrm{Orth}^\infty(G)_+$ then $\alpha = \sup(\alpha_\xi)$ for some upward-directed family (α_ξ) in $\mathscr{Z}(G)$. Whereas the elements of $\mathscr{Z}(E)$ are the relatively uniform limits of orthomorphisms of the form $\sum_{l=1}^n \lambda_l \pi_l$. Thus, to complete the proof, it remains to appeal to o-continuity of the operator $\odot$, see 6.2 (3). $\rhd$

7. Differences of symmetric bimorphisms

By way of illustration consider an application of Theorem 6.4 to the description of differences of symmetric lattice bimorphisms.

7.1. Let E and G be vector lattices with G Dedekind complete. As was observed in [37] a linear operator $T : E \to G$ is, in a sense, determined up to an orthomorphism from the family of the kernels of the *strata* πT $(\pi \in \mathfrak{P}(G))$ of T.

Theorem. *Let S and T be linear operators from E to G. Then* $\ker(\pi S) \supset \ker(\pi T)$ *for all $\pi \in \mathfrak{P}(G)$ if and only if there is an orthomorphism α of G such that $S = \alpha T$.*

Making use of this observation S. S. Kutateladze [37] proved the following characterization of order bounded linear operator T representable as the difference of two lattice homomorphisms.

7.2. Theorem. *An order bounded operator T from E to G may be presented as the difference of some lattice homomorphisms if and only if the kernel of each stratum πT of T is a vector sublattice of E for all $\pi \in \mathfrak{P}(G)$.*

In view of 2.7 a similar approach should be successful for orthoregular bilinear operators and we gain a possibility of studying some properties of b in terms of the kernels of the strata. For an orthoregular bilinear operator b the set

$$\mathrm{sker}(b) := \{x \in E : \ b(x, |x|) = 0\}$$

will be called the *symmetric kernel* of b.

7.3. Theorem. *Let b and ϕ be orthoregular bilinear operators from $E \times E$ to G. Then* $\mathrm{sker}(\pi b) \supset \mathrm{sker}(\pi\phi)$ *for all $\pi \in \mathfrak{P}(G)$ if and only if there is an orthomorphism α of F such that $b = \alpha\phi$.*

$\lhd$ Apply 7.1 to Φ_b and Φ_ϕ. In view of 6.1 we have $\ker(\pi(\Phi_b \circ \iota)) = \iota^{-1}(\ker(\pi\Phi_b))$ and the inclusion $\mathrm{sker}(\pi b) \supset \mathrm{sker}(\pi\phi)$ is equivalent to $\ker(\pi\Phi_b) \supset \ker(\pi\Phi_\phi)$. Therefore, by 7.1 there exists an orthomorphism α of G such that $b = \Phi_b \odot = \alpha\Phi_\phi \odot = \alpha\phi$. $\rhd$

7.4. Theorem. *Let E and G be vector lattices with E relatively uniform complete and G Dedekind complete. An orthoregular bilinear operator $b : E \times E \to G$ may be presented as the difference of two symmetric lattice bimorphisms if and only if the symmetric kernel of each stratum πb of b is a vector sublattice of E for all $\pi \in \mathfrak{P}(G)$.*

$\lhd$ According to 6.2 (2) $b = \Phi_b \odot$ for some order bounded linear operator $\Phi_b : E^\odot \to G$. Obviously, b is representable as the difference of two symmetric lattice bimorphisms if and only if Φ_b is representable as the difference of two lattice homomorphisms. By the above mentioned Kutateladze theorem the latter is equivalent to the following: the kernel of each stratum $\pi\Phi_b$ of Φ_b is a vector sublattice of $E^\odot$ for all $\pi \in \mathfrak{P}(G)$. Now, it remains to observe that by 6.2 (2) the kernel $\ker(\pi(\Phi_b \circ \iota)) = \iota^{-1}(\ker(\pi\Phi_b))$ coincides with the symmetric kernel of πb and by 6.4 (1) it is a vector sublattice in E if and only if $\ker(\pi\Phi_b)$ is a vector sublattice in $E^{\odot}$. $\rhd$

7.5. S. S. Kutateladze [37] proved also that the modulus of an order bounded operator $T : E \to G$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $\pi \in \mathfrak{P}(G)$ is a Grothendieck subspace of the ambient vector lattice E. A similar result for order bounded orthosymmetric bilinear operators is of interest. In [34] bilinear operators representable as finite sums of disjointness preserving bilinear operators are also characterized.

8. Bilinear Maharam operators

In this section we prove that a positive orthosymmetric bilinear operator is order interval preserving (order continuous) if and only if its linearization via square is also order interval preserving (order continuous).

8.1. Let E and G be vector lattices and let b be a positive bilinear operator from $E \times E$ into G. Say that b is *order interval preserving* or possesses the *Maharam property* if, for every $x, y \in E_+$ and $0 \le g \le b(x, y) \in G_+$, there exist $0 \le u \le x$ and $0 \le v \le y$ such that $g = b(u, v)$ or, in short, $b\,(\,[0, x] \times [0, y]\,) = [0, b(x, y)]$ for all $x, y \in E_+$. A positive order continuous bilinear operator with the Maharam property is called a *bilinear Maharam operator*.

The order interval preserving phenomena is not sufficiently understood in the bilinear context. In [9] the notion of almost right (or left) interval preserving bilinear operator was considered and a bilinear version of Arendt's Theorem on duality between lattice homomorphisms and interval preserving operators was proved [9; Theorem 14], cf. [4; Theorem 7.4].

Let ϕ be another positive bilinear operator from $E \times E$ into G. Then b is said to be *absolutely continuous* with respect to ϕ whenever $b(x, y) \in \phi(x, y)^{\perp\perp}$ for all $0 \le x, y \in E$. Evidently, any $b \in \phi^{\perp\perp}$ is absolutely continuous with respect to ϕ.

8.2. Theorem. *Let E and G be Dedekind complete vector lattices, $b : E \times E \to G$ be a positive orthosymmetric bilinear operator and $b = \Phi_b \odot$ for a uniquely defined positive linear operator $\Phi_b : E^{\odot} \to G$. The following conditions are equivalent:*

(1) *b is order interval preserving;*

(2) *$b(x, \cdot)$ is order interval preserving for any $0 \le x \in E$;*

(3) *$b(\cdot, y)$ is order interval preserving for any $0 \le y \in E$;*

(4) *for any $0 \le x \in E$ and $0 \le u \le b(x, x)$ there exists $y \in E$, $0 \le y \le x$ such that $u = b(y, y)$;*

(5) *Φ_b is order interval preserving.*

$\lhd$ The equivalence (2) $\Leftrightarrow$ (3) is trivial, since b is symmetric, see 2.3. The implication (2) $\Rightarrow$ (1) is also obvious. Take $0 \le x \in E$ and $0 \le v \in G$ with $v \le b(x, x)$. By (1) we can choose $x_1, x_2 \in [0, x]$ such that $v = b(x_1, x_2)$. Put $x_0 := x_1^{1/2} x_2^{1/2}$ and observe that $x_0 \odot x_0 = x_1 \odot x_2$. Indeed, it suffices to apply the equality $(x \odot y)^p (u \odot v)^{1-p} = (x^p u^{1-p}) \odot (y^p v^{1-p})$ (see 5.1) with $p = 1/2$. Thus, $b(x_0, x_0) = \Phi_b(x_0 \odot x_0) = \Phi(x_1 \odot x_2) = b(x_1, x_2) = v$ and (1) $\Rightarrow$ (4) follows. The implication (4) $\Rightarrow$ (5) is immediate from the representation $(\Phi_b \circ \iota)(x) = b(x, |x|)$ $(x \in E)$, see 6.1.

Finally, assuming Φ_b to be order interval preserving, it suffices to prove that so is $b(\cdot, y)$ for every $0 \le y \in E$. Let $x, y \in E_+$ and $v \le b(x, y) = \Phi_b(x \odot y)$. By (5) there exists $0 \le z \le x \odot y$ such that $v = \Phi_b(z)$. By the Kreĭns–Kakutani Representation Theorem there is an isomorphism h from the principal ideal E_u with $u := x + y + z$ onto $C(S)$ for some extremally disconnected compact Hausdorff space S. Consider two mappings $P, Q : C(S) \to C(S)$ given by

$$P(f) := \theta \circ f = \operatorname{sgn}(f) f^2 = f|f|,$$
$$Q(f) := \theta^{-1} \circ f = \operatorname{sgn}(f)\sqrt{|f|}.$$

It is easily seen that P and Q are multiplicative and order preserving bijections with $P = Q^{-1}$. Denote $f = h(x)$, $g = h(y)$, and $k = h(\iota^{-1}(z))$. Then by 3.4 $h(J(h^{-1}(f), h^{-1}(g))) = J \circ (f, g) = Q(fg)$. Since $k \le Q(fg)$, we have $P(k) \le fg$. Choose a function $f_0 \in C(S)$ with $0 \le f_0 \le 1$ and $P(k) = f_0 fg$ and denote $f_1 := f_0 f$. Then $h(\iota^{-1}(z)) = k = Q(f_1 g) = h(J(x_1, y)$ where $x_1 := h^{-1}(f_1)$ and, taking into consideration 6.1, we get $z = \iota \circ J(x_1, y) = x_1 \odot y$. It remains to apply Φ_b to the last equality and use again 6.1: $b(x_1, y) = \Phi_b(x_1 \odot y) = \Phi_b(z) = v$. Thus, (5) $\Rightarrow$ (2) and the proof is complete. $\rhd$

8.3. Consider vector lattices E and G with E relatively uniformly complete and G Dedekind complete and an orthoregular operator $b : E \times E \to G$. The *symmetric null ideal* $\mathscr{N}_b$ and the *symmetric carrier* $\mathscr{C}_b$ of b are defined by

$$\mathscr{N}_b := \{x \in E : \ |b|(|x|, |x|) = 0\},$$
$$\mathscr{C}_b := \mathscr{N}_b^{\perp} = \{x \in E : x \perp \mathscr{N}_b\}.$$

Since $b(x, x) = b(|x|, |x|)$ $(x \in E)$ for any orthosymmetric b, the symmetric null ideal $\mathscr{N}_b$ can be defined also by $\mathscr{N}_b := \{x \in E : \ |b|(x, x) = 0\}$. The symmetric null ideal is indeed an order ideal by virtue of 6.1 and 6.4 (2), since $\mathscr{N}_{\Phi_b} = \iota(\mathscr{N}_b)$. If b is order continuous then $\mathscr{N}_b$ is a band in E. If $b \ge 0$ and $\mathscr{N}_b = \{0\}$ then b is called *strictly positive*. An orthosymmetric positive bilinear operator b is strictly positive if and only if it is positively definite, see 2.1.

An orthoregular bilinear operator b is singular if $\mathscr{N}_b$ contains some order dense ideal or, equivalently, $\mathscr{N}_b$ is an order dense ideal in E. Indeed, if b vanishes on $E_1 \times E_2$ for some order dense ideals E_1 and E_2 in E, then $E_0 := E_1 \cap E_2$ is an order dense ideal which is contained in $\mathscr{N}_b$. Conversely, if an order dense ideal $E_0 \subset E$ is contained in $\mathscr{N}_b$, then b vanishes on $E_0 \times E_0$.

8.4. *Let E, G, b, and Φ_b be as in 8.2. The following assertions hold:*

(1) *b is order continuous if and only if Φ_b is order continuous;*

(2) *b is singular if and only if Φ_b is singular;*

(3) *b is strictly positive if and only if Φ_b is strictly positive;*

(4) *b is a Maharam operator if and only if Φ_b is a Maharam operator.*

$\lhd$ (1): If Φ_b is order continuous then b is also order continuous as the composite of two order continuous maps, see 6.1 and 6.2 (3). Conversely, assume that b is order continuous and a net (u_α) in $E^{\odot}$ decreases to zero, i.e. $u_\alpha \downarrow 0$ in $E^{\odot}$. Put $x_\alpha := \iota^{-1}(u_\alpha)$ and note that $x_\alpha \downarrow 0$ in E. Thus, by 6.1 we have

$$\Phi_b(u_\alpha) = (\Phi_b \circ \iota)(x_\alpha) = b(x_\alpha, x_\alpha) \to 0.$$

(2), (3): The equality $\mathscr{N}_{\Phi_b} = \iota(\mathscr{N}_b)$ and 6.4 (2) imply that $\mathscr{N}_b$ and $\mathscr{N}_{\Phi_b}$ are order dense ideals in E and $E^{\odot}$ simultaneously. Moreover, $\mathscr{N}_b = \{0\}$ if and only if $\mathscr{N}_{\Phi_b} = \{0\}$. Note, that order completeness of E is superfluous in (1)–(3).

(4): Follows from (1) and 7.2. $\rhd$

8.5. The class of linear Maharam operators was first studied by D. Maharam in [48] (see also the survey paper [50]). W. A. J. Luxemburg and A. R. Schep [47] extended a portion of Maharam's theory to the case of positive operators in Dedekind complete vector lattices. The terms "Maharam property" and "Maharam operator" were introduced in [47] and [26], respectively (more details see in [27]). The Maharam property transplanted to the entourage of convex operators is presented in [32]. Every linear Maharam operator is an interpretation of some order continuous linear functional in an appropriate Boolean-valued model, see [27]. This Boolean-valued status of the concept of Maharam operator was established in [26].

9. A Radon–Nikodým type theorem

Making use of Theorem 8.4 we are able to transfer some results from linear Maharam operators to orthosymmetric bilinear Maharam operators. In this section we show that the versions of the Radon–Nikodým Theorem and the Hahn Decomposition Theorem are valid for orthosymmetric Maharam operators. For a bilinear operator $\phi \in BL_o^\sim(E;G)$ denote $E_\phi := \mathscr{C}_\phi$ and $G_\phi := \phi(E \times E)^{\perp\perp}$.

9.1. Theorem. *Let E and G be Dedekind complete vector lattices and let ϕ be a positive orthosymmetric bilinear Maharam operator from $E \times E$ into G. Then there exists an f-algebra isomorphism h of the universally complete f-algebra* $\mathrm{Orth}^\infty(G_\phi)$ *onto an order closed f-subalgebra in* $\mathrm{Orth}^\infty(E_\phi)$ *such that the following holds:*

(1) *h induces a Boolean isomorphism of $\mathfrak{P}(G_\phi)$ onto an order complete subalgebra of the Boolean algebra $\mathfrak{P}(E_\phi)$;*

(2) *h induces an f-algebra isomorphism of $\mathscr{Z}(G_\phi)$ onto an order closed f-subalgebra in $\mathscr{Z}(E_\phi)$;*

(3) *for every positive orthosymmetric order continuous bilinear operator b from $E \times E$ to G absolutely continuous with respect to ϕ we have*

$$\pi \circ b(x,y) = b(h(\pi)x,y) = b(x,h(\pi)y) \quad (\pi \in \mathrm{Orth}^\infty(G_\phi)_+;\ x,y \in \mathscr{D}(\pi))$$

and in this case b is a Maharam operator.

$\lhd$ Without loss of generality, we may assume that $G = G_\phi$ and $E = E_\phi$. Put $\Phi := \Phi_\phi$. By [47; Theorem 2.2] (see also [27; Theorem 3.4.3]) there exists a Boolean isomorphism h' of $\mathfrak{P}(G)$ onto some order complete subalgebra in $\mathfrak{P}(E^\odot)$ such that $\pi \circ \Phi_b = \Phi_b \circ h'(\pi)$ for all $\pi \in \mathfrak{P}(G)$. Take a Boolean isomorphism $h'' : \pi \mapsto \pi'$ between $\mathfrak{P}(E^\odot)$ and $\mathfrak{P}(E)$ as in 6.4 (5) and put $h = h'' \circ h'$. Then h is a Boolean isomorphism of $\mathfrak{P}(G)$ onto an order complete subalgebra of $\mathfrak{P}(E)$ and

$$\begin{aligned}\pi \circ b(x,y) = \pi \circ \Phi_b(x \odot y) = \Phi_b \circ h'(\pi)(x \odot y)\\ = \Phi_b((h'(\pi)'x \odot y) = b(h(\pi)x,y) \quad (x,y \in E;\ \pi \in \mathfrak{P}(G)).\end{aligned}$$

The Boolean isomorphism h is uniquely extended to an order continuous f-algebra isomorphism from $\mathrm{Orth}^\infty(G)$ onto the order complete f-subalgebra in $\mathrm{Orth}^\infty(E)$ constituted by those elements in $\mathrm{Orth}^\infty(E)$ whose spectral functions take their values in the Boolean algebra $h(\mathfrak{P}(G))$. Observe that b is absolutely continuous with respect to ϕ if and only if Φ_b is absolutely continuous with respect to Φ. Finally, we repeat the arguments from 6.4 (6) replacing $\odot$ by b and appealing to order continuity of b. $\rhd$

9.2. Theorem (**Radon–Nikodým**). *Let E and G be Dedekind complete vector lattices and let b and ϕ be orthosymmetric order continuous positive bilinear operators from $E \times E$ to G with ϕ possessing the Maharam property. Then the following assertions are equivalent:*

(1) $b \in \{\phi\}^{\perp\perp}$;

(2) *b is absolutely continuous with respect to ϕ;*

(3) *there exists an orthomorphism $0 \leq \rho \in \mathrm{Orth}^{\infty}(E)$ such that*

$$b(x,y) = \phi(\rho x, y) = \phi(x, \rho y) \quad (x, y \in \mathscr{D}(\rho));$$

(4) *there exists an increasing sequence of positive orthomorphisms (ρ_n), $\rho_n \in \mathrm{Orth}(E)$, such that*

$$b(x,y) = \sup_n \phi(\rho_n x, y) = \sup_n \phi(x, \rho_n y) \quad (x, y \in E_+).$$

Moreover, if $0 \leq b \leq \phi$, then $b = \phi \circ (\rho \times I_E) = \phi \circ (I_E \times \rho)$ for some orthomorphism $\rho \in \mathrm{Orth}(E)$ with $0 \leq \rho \leq I_E$.

$\lhd$ By virtue of 6.1, 6.2 (2), 6.4 (6), 8.5, and 9.1 each of the conditions (1)–(4) is equivalent to the corresponding condition for the linear operators Φ_b and Φ_ϕ. Thus, it remains to apply the Radon–Nikodým type theorem for linear Maharam operators established by W. A. J. Luxemburg and A. Schep in [47; Theorem 3.3], see also [27; Theorem 3.4.9]. $\rhd$

9.3. Theorem (**Nakano**). *Let E and G be Dedekind complete vector lattices and let ϕ be an orthosymmetric bilinear Maharam operator from $E \times E$ to G. Orthoregular bilinear operators $b_1, b_2 \in \{\phi\}^{\perp\perp}$ are disjoint if and only if their symmetric carriers $\mathscr{C}_{b_1}$ and $\mathscr{C}_{b_2}$ are disjoint.*

$\lhd$ The proof goes along the same lines taking into consideration the linear version of the required statement. Put $\Phi_i := \Phi_{b_i}$ and $\Phi := \Phi_\phi$. By 6.2 (2) b_1 and b_2 are disjoint if and only if Φ_1 and Φ_2 are disjoint and, moreover, Φ_1 and Φ_2 belong to $\Phi^{\perp\perp}$. According to 8.4 (3) Φ is a linear Maharam operator. By [27; Theorem 3.4.6 (1)] the disjointness of Φ_1 and Φ_2 is equivalent to the disjointness of their carriers $\mathscr{C}_{\Phi_1}$ and $\mathscr{C}_{\Phi_1}$. Now it suffices to observe that, by 6.1 and 6.4 (3), the carrier $\mathscr{C}_{\Phi_i}$ coincides with the symmetric carrier $\mathscr{C}_{b_i}$ $(i = 1, 2)$. $\rhd$

Worthy of mention is the following corollary to Theorem 9.3: *The correspondence $\pi \mapsto \phi \circ (\pi \times \pi)$ is an isomorphism of $\mathfrak{P}(\mathscr{C}_\phi)$ onto the Boolean algebra $\mathfrak{E}(\phi)$ of all components of ϕ*. In particular, by applying this result to $\phi := |b|$, the following variant of Hahn Decomposition Theorem is easily deduced.

9.4. Theorem (**Hahn Decomposition Theorem**). *Let E and G be Dedekind complete vector lattices and let $b : E \times E \to G$ be an order bounded orthosymmetric bilinear operator with $|b|$ a Maharam operator. Then there exists a band projection $\pi \in \mathfrak{P}(E)$ such that $b^+ = b \circ (\pi \times \pi)$ and $b^- = b \circ (\pi^\perp \times \pi^\perp)$.*

10. Concluding remarks

Denote by $BL_{on}(E;G)$ the set of orthoregular order continuous bilinear operator from $E \times E$ to G. According to 6.2 (2) and 8.4 (1) the bijection $b \mapsto \Phi_b$ constitutes an isomorphism of vector lattices $BL_{on}(E;G)$ and $L_n(E^{\circledcirc}, G)$. This fact enables us to transfer known results on regular order continuous linear operators to orthoregular order continuous bilinear operators. Consider some further possibilities assuming that E is relatively uniformly complete and G is Dedekind complete.

10.1. From 6.2 (2), 8.4 (1), and [4; Theorem 4.6] one can obtain the following description of the order continuous component b_n of $b \in BL_{or}(E;G)$:

$$b_n(x,x) = \inf\{\sup_\alpha b(x_\alpha, x_\alpha) :\ 0 \le x_\alpha \uparrow x\} \quad (0 \le x \in E);$$
$$b_n(x,y) = \frac{1}{2}\big(b_n(x+y,x+y) - b_n(x,x) - b_n(y,y)\big) \quad (0 \le x, y \in E);$$
$$b_n(x,y) = b_n(x^+,y^+) - b_n(x^+,y^-) - b_n(x^-,y^+) + b_n(x^-,y^-) \quad (x,y \in E).$$

Along similar lines, some other formulas for calculating specific components of an orthosymmetric positive bilinear operator can be derived, see [27; Section 3.2].

10.2. The orthoregular order continuous bilinear operators can be characterized in terms of symmetric null ideals similar to that of the linear operators, see [4; Theorem 4.8]: *a bilinear operator $b \in BL_{or}(E;G)$ is order continuous if and only if the symmetric null ideal $\mathscr{N}_d$ is a band for every bilinear operator d in the principal ideal generated by b in $BL_{or}(E;G)$.*

10.3. By the same arguments as in 6.2 (1) one can observe that a vector lattice E has the Egorov property if and only if $E^\odot$ has the Egorov property. Denote by $BL_{os}(E;G)$ the subset of $BL_{or}(E;G)$ consisting of all singular operators, see 1.6. Now, making use of 8.4 and [27; Theorem 4.4.10], we arrive at the following strong form of the Yosida–Hewitt decomposition: *If E is a vector lattice with the Egorov property and G is a Dedekind complete vector lattice with the countable sup property* (see [4; p. 52]), *then $BL_{os}(E;F) = BL_{on}(E;F)^\perp \cap BL_{or}(E;G)$ or, which is the same,*

$$BL_{or}(E;G) = BL_{on}(E;F) \oplus BL_{os}(E;F).$$

10.4. Let G be a Dedekind σ-complete weakly σ-distributive vector lattice, F be a majorizing sublattice of E, and b be an orthosymmetric positive bilinear operator from $F \times F$ to G. We say that b is *sequentially E-continuous* if $\inf_n b(x_n,x_n) = 0$ for any decreasing sequence (x_n) in F such that $\inf_n x_n = 0$ in E.

The isomorphism $\hat{\imath}$ between the inclusion-ordered sets $\mathscr{S}_{\mathrm{uc}}(E)$ and $\mathscr{S}_{\mathrm{uc}}(E^\odot)$ (see 6.4 (1)) sends σ-closed sublattices to σ-closed sublattices and hence for any $F \in \mathscr{S}_{\mathrm{uc}}(E)$ we have $(F^\sigma)^\odot = (F^\odot)^\sigma$, where F^σ denotes the σ-closed sublattice of E generated by F. By applying the Kantorovich–Mattes–Wright Extension Theorem [27; Theorem 4.5.3] to Φ_b, we get the following conclusion: *every sequentially E-continuous $b \in BL_{or}(F;G)$ has an extension to an orthosymmetric sequentially E-continuous positive bilinear operator from $F^\sigma \times F^\sigma$ to G.*

10.5. With the use of similar arguments, the Maharam extension and its functional representation (well known in the linear case, see [27; Sections 4.5 and 6.3] and [46]) can be also developed for an orthosymmetric positive bilinear operator.

10.6. Let $J(E,G)$ be the band of $L_r(E,G)$ generated by the set of finite-rank order continuous operators: $J(E,G) := (L_n(E,\mathbb{R}) \otimes G)^{\perp\perp}$. Similarly, denote by $BL_{ori}(E;G)$ the band in $BL_{or}(E;G)$ generated by $BL_{on}(E;\mathbb{R}) \otimes G$. The elements of $J(E,F)$ and $BL_{ori}(E;G)$ are called *almost integral operators*. By virtue of 6.2 (2) and 8.4 (1) the bijection $b \mapsto \Phi_b$ constitutes an isomorphism of vector lattices $BL_{ori}(E;G)$ and $J(E^\odot,G)$. In combination with Strizhevskiĭ's result [56] (see also [27; Theorem 2.5.3 (2)]) this fact leads to the following: *Let E and G*

be order complete vector lattices and let $BL_{on}(E;\mathbb{R})$ *separates the points of* E $\big(= (\forall\, 0 < x \in E)\,(\exists\, b \in BL_{on}(E;\mathbb{R}))\, b(x,x) > 0)\big)$. *If* E *is diffuse then every symmetric lattice bimorphism is disjoint from the band* $BL_{ori}(E;G)$. *If* F *is diffuse then every orthosymmetric bilinear Maharam operator is disjoint from* $BL_{ori}(E;G)$.

10.7. Every orthosymmetric bilinear Maharam operator is a Boolean-valued interpretation of an orthosymmetric order continuous positive bilinear functional, cf. [27; Theorem 4.6.10 (6)]. This approach provides us with the useful transfer principle from orthosymmetric order continuous positive bilinear functionals in Boolean-valued set theory to orthosymmetric bilinear Maharam operators.

10.8. A set of band projections $\mathscr{P}$ in $BL_{or}(E;G)$ is said to be *generating* if for all positive $b \in BL_{or}(E;G)$ we have

$$b(x^+, |x|) = \sup\{(pb)(x,|x|) :\ p \in \mathscr{P}\} \quad (x \in E).$$

Given a positive operator $b \in BL_{or}(E;G)$, denote by $\mathfrak{E}(b)$ the Boolean algebra of all components of b and define the set of *elementary components* $\mathscr{P}^{\vee}(b)$ of b by

$$\mathscr{P}^{\vee}(b) := \left\{ \sum_{k=1}^{n} \pi_k p_k b :\ p_1,\dots,p_n \in \mathscr{P};\ \pi_1,\dots,\pi_n \in \mathfrak{P}(G); \pi_k \circ \pi_l = 0\,(k \neq l) \right\}.$$

The following *up-down theorem* can be deduced from Theorem 6.1 and [33; Theorem 3.3]: *A set* $\mathscr{P}$ *of band projections in* $BL_{or}(E;G)$ *is generating if and only if for every orthosymmetric positive bilinear operator* $b : E \times E \to G$ *we have* $\mathfrak{E}(b) = \mathscr{P}^{\vee}(b)^{\uparrow\downarrow\uparrow}$. One can obtain different up-down formulas by specifying generating sets [33, 57]; further comments see in [4, 27].

10.9. Hölder type inequalities from Sections 5 can be applied to deduce some estimates for the Hadamard weighted geometric means of positive kernel operators in Banach function spaces. For example, the inequalities (1) of [18; Theorem 2.1] and (4) of [18; Theorem 2.2] are the easy consequences of 5.2 ($f(k) := \|K\|$) and 5.5 ($b(h,k) := h \cdot k$, $f(h) := H$, $g(k) := K$), respectively.

10.10. A bilinear operator $b : E \times E \to E$ is said to be *band preserving* if for all $x, y \in E$ we have $b(x,y) \in \{x\}^{\perp\perp} \cap \{y\}^{\perp\perp}$ or, equivalently, $b(x,y) \perp z$ for any $z \in E$ provided that $x \perp z$ or $y \perp z$. Evidently, a band preserving bilinear operator is orthosymmetric. Every order bounded band preserving bilinear operator $b : E \times E \to E$ admits a unique representation $b(x,y) = \beta(x \odot y)$ $(x, y \in E)$ with $\beta \in \mathrm{Orth}(E^{\odot})$, see 2.7 and 6.4 (3). The following question can be considered as a version of *Wickstead's problem* (see [27, 30]): *Under what conditions all band preserving bilinear operators in a vector lattice are order bounded?* A nice characterization of universally complete vector lattice with this property can be easily deduced from the corresponding result for linear operators due to A. E. Gutman [23]): *For an universally complete vector lattice* G *the following are equivalent:* (1) *every band preserving bilinear operator in* G *is order bounded;* (2) *the base* $\mathfrak{B}(G)$ *is a* σ*-distributive Boolean algebra.*

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APPENDIX[1]

On a Property of the Base of K-space of Regular Operators and Some of Its Applications

A. G. KUSRAEV

Abstract. An interplay between bases of Dedekind complete vector lattices of regular operators acting respectively from a vector lattice and from its majorizing sublattice into the same Dedekind complete vector lattice is studied. The existence of an order continuous simultaneous extension operator from a majorizing sublattice to the ambient vector lattice is established. Some applications to extension of positive bilinear operators and representation of regular functionals are also given.

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Key words: Boolean algebra, vector lattice, regular operator, lattice isomorphism, positive bilinear operator, simultaneous extension.

Introduction

Let $\hat{E}$ be an Archimedean vector lattice[2] and let E be its majorizing (= for every $x \in \hat{E}$ there is $x' \in E$ such that $|x| \leq x'$) sublattice[3]. In Section 1 we study the relationship between bases (= Boolean algebras of bands[4]) of Dedekind complete vector lattices[5] of regular operators acting respectively from E and $\hat{E}$ into the same Dedekind complete vector lattice. In Section 2 we construct an extension operator from $\mathscr{L}(E,F)$ to $\mathscr{L}(\hat{E},F)$ preserving not only positivity, that is possible by virtue of the Kantorovich Extension Theorem ([1; Theorem X.3.1]), but also the lattice operations in vector lattice of regular operators. In Section 3 we consider some aspects of the extension problem for bilinear operators. Finally, in Section 4 a representation result for the space of regular functionals on an arbitrary vector lattice is given.

1. Saturated bands

In the sequel we fix the following notation, unless otherwise stated: Let $\hat{E}$ be an Archimedean vector lattice, E its majorizing sublattice, and F an arbitrary Dedekind complete vector lattice. Denote by $\mathscr{L}(E,F)$ the space of all regular operators from E into F and let $\mathscr{L}^+(E,F)$ stands for the cone of positive operators. If K is a band in a Dedekind complete vector lattice, then Pr_K denotes the band projection onto K. The identity mapping on a set A will be denoted by id_A. Let $\mathfrak{E}$ and $\hat{\mathfrak{E}}$ stand for the bases of Dedekind complete vector lattices $\mathscr{L}(E,F)$ and $\mathscr{L}(\hat{E},F)$, respectively. The restriction of a regular operator $\hat{U} : \hat{E} \to F$ to E is denoted by $r\hat{U}$, i.e. $r\hat{U} := r(\hat{U}) := \hat{U}|_E$.

The following assertion is almost evident.

Lemma 1. *The restriction operator $\hat{r} : \mathscr{L}(\hat{E},F) \to \mathscr{L}(E,F)$ is linear, order continuous, and strictly isotonic.*

The latter means that $\hat{U} > 0$ implies $r\hat{U} > 0$.

Lemma 2. *If $(\hat{U}_\xi)_{\xi\in\Xi}$ is an increasing net of positive operators in $\mathscr{L}(\hat{E},F)$ and the net $(r\hat{U}_\xi)_{\xi\in\Xi}$ is order bounded in $\mathscr{L}(E,F)$, then $(\hat{U}_\xi)_{\xi\in\Xi}$ is likewise order bounded and*

$$r\left(\sup_{\xi\in\Xi} \hat{U}_\xi\right) = \sup_{\xi\in\Xi} r\big(\hat{U}_\xi\big).$$

[1]This is a translation of the original Russian work of the same title, Sobolev Institute Press ©1977, Novosibirsk. Translated by the author with a slight modification of notation and terminology. The vector lattices under study are denoted by E, F, and G instead of X, Y, and Z. The terms used in the original text are indicated in the footnotes.

[2]Vector lattice = K-lineal.

[3]Sublattice = Sublineal.

[4]Band = Component.

[5]Dedekind complete vector lattice = K-space = Kantorovich space.

Proof. Denote $U = \sup_\xi r(\hat{U}_\xi)$ and take $x \in \hat{E}^+$ and $x' \in E$ with $x \le x'$. Then $\hat{U}_\xi x \le \hat{U}_\xi x' = (r\hat{U}_\xi)x' \le Ux'$ for every $\xi \in \Xi$; therefore, the equality $\hat{U}x = \sup \hat{U}_\xi x$ $(x \in \hat{E})$ defines correctly a regular operator $\hat{U}$ which is the least upper bound of the net $(\hat{U}_\xi)_{\xi\in\Xi}$. The second assertion follows from order continuity of the restriction operator r and from the equalities $U = o\text{-}\lim_{\xi\in\Xi} r\hat{U}_\xi$ and $\hat{U} = o\text{-}\lim_{\xi\in\Xi} \hat{U}_\xi$. □

We also need the following easy corollary to the Hahn–Banach–Kantorovich Theorem (see [4]).

Lemma 3. *Let U be a positive operator from E to F and let $\hat{V}$ be a positive operator from $\hat{E}$ to F with $U \le r\hat{V}$. Then there exists a linear extension $\hat{U}$ of U to the whole of $\hat{E}$ such that $0 \le \hat{U} \le \hat{V}$.*

Also observe that if $U, V \in \mathscr{L}^+(E,F)$ and $\hat{U}$ and $\hat{V}$ are arbitrary positive extensions respectively of U and V to the whole of $\hat{E}$, then the disjointness of U and V implies the disjointness of $\hat{U}$ and $\hat{V}$ (see [1; Lemma X.4.1])

Proposition 1. *If K is a band in $\mathscr{L}(\hat{E},F)$ and K_0 is an order dense ideal*[6] *in K, then $r(K)$ is a band in $\mathscr{L}(E,F)$ and $r(K_0)$ is an order dense ideal in $r(K)$.*

Proof. First observe that if an operator $U = r\hat{U} \in r(K_0)$ is positive, then $U \le r|\hat{U}|$ and, by Lemma 3, it has a positive extension lying in K_0. Let $0 \le V \le \hat{U}$, where $V \in \mathscr{L}(E,F)$ and $0 \le \hat{U} \in K_0$, and let $\hat{V}$ be a positive extension of V with $\hat{V} \le \hat{U}$. Then $V \in r(K_0)$, since $\hat{V} \in K_0$. If an arbitrary regular operator $V \in r(K_0)$ has an extension $\hat{V}$ lying in K_0, then by Lemma 3 and above observation we have $V \in r(K_0)$, since $|V| \le r|\hat{V}|$ and $|\hat{V}| \in K_0$. Thus, $r(K)$ and $r(K_0)$ are order ideals[7] in $\mathscr{L}(E,F)$ and $r(K_0) \subset r(K)$.

Assume that $0 \le U \in r(K)^{dd}$ and denote $\mathfrak{U} = \{U' \in r(K_0) : 0 \le U' \le U\}$, $\hat{\mathfrak{U}} = \{\hat{U}' \in K_0 : r(\hat{U}') \in \mathfrak{U}\}$. The inclusion ordered set of upwards directed subsets of $\hat{\mathfrak{U}}$ satisfy the hypotheses of the Zorn Lemma and thus there exists a maximal upward directed subset $\hat{\mathfrak{U}}_0 \subset \hat{\mathfrak{U}}$. The set $\mathfrak{U}_0 = \{r\hat{U} : \hat{U} \in \hat{\mathfrak{U}}_0\} \subset \mathfrak{U}$ is bounded above and by Lemma 2 there exists $\sup \hat{\mathfrak{U}}_0 = \hat{U}_0 \in K$; moreover, $r(\hat{U}_0) = \sup \mathfrak{U}_0$. Suppose $U_0 = \sup \mathfrak{U}_0 < U$. Choose $V \in r(K_0)$ such that $0 < V \le U - U_0$ and a positive extension $\hat{V}$ of V is contained in K_0[8]. Then we have $\hat{U}_0 + \hat{V} \in \hat{\mathfrak{U}} \setminus \hat{\mathfrak{U}}_0$, since $0 < U_0 + V \le U$ and $\hat{U}_0 + \hat{V} > \hat{U}'$ for every $\hat{U}' \in \hat{\mathfrak{U}}_0$. But the last assertion contradicts to the maximality of $\hat{\mathfrak{U}}_0$. Consequently, $U = \sup \mathfrak{U}_0 = r(\sup \hat{\mathfrak{U}}_0) = r(\hat{U}_0) \in r(K)$. □

Definition. A band $K \in \hat{\mathfrak{E}}$ is said to be *saturated* if, for any positive operator $\hat{U}$ in K, all positive extensions to the whole of $\hat{E}$ of the restriction $r\hat{U}$ are likewise contained in K. Denote by $\hat{\mathfrak{E}}_0$ the set of all saturated bands in $\mathscr{L}(\hat{E},F)$.

Proposition 2. *The set $\hat{\mathfrak{E}}_0$ is a complete subalgebra of the Boolean algebra $\hat{\mathfrak{E}}$.*

Proof. Assume that K is a saturated band, $0 \le \hat{U} \in K^d$, and certain positive extension $\hat{V}$ of $r\hat{U}$ is not contained in K^d. Under this conditions the restriction V_1 of $\mathrm{Pr}_K \hat{V}$ satisfies the inequalities $0 < V_1 \le U = r\hat{U}$ and if $\hat{V}_1$ is its positive extension majorized by $\hat{U}$, then $\hat{V}_1 \in K^d$. At the same time $\hat{V}_1 \in K$, since K is saturated. This contradiction proves that K^d is also saturated. Moreover, it is evident that the intersection of an arbitrary family of saturated bands is a saturated band and thus $\hat{\mathfrak{E}}_0$ is a complete subalgebra of $\hat{\mathfrak{E}}$. □

Denote by the same letter r the mapping $K \mapsto r(K)$ $(K \in \hat{\mathfrak{E}})$ and let q stands for the mapping

$$L \mapsto \{\hat{U} \in \mathscr{L}^+(\hat{E},F) : \ r\hat{U} \in L\}^{dd} \quad (L \in \mathfrak{E}).$$

Theorem 1. *The mapping $r : \hat{\mathfrak{E}}_0 \to \mathfrak{E}$ is an isomorphism of Boolean algebras $\hat{\mathfrak{E}}_0$ and $\mathfrak{E}$; moreover, $r^{-1} = q$.*

Proof. If $K \in \hat{\mathfrak{E}}_0$, $\hat{U} \in \mathscr{L}^+(\hat{E},F)$, and $U = r(\hat{U})$, then $\hat{U}\, d\, K$ implies $U\, d\, r(K)$. Otherwise $U \wedge V = W > 0$ for some positive $V \in r(K)$ and, having chosen a positive extension $\hat{W}$ of W with $\hat{W} \le \hat{U}$, we would come to a contradictory relation $\hat{W} \in K$. Hence, it follows that $r(K)\, d\, r(K^d)$ for an arbitrary $K \in \hat{\mathfrak{E}}_0$. Furthermore, for the same U and $\hat{U}$ the relations $U\, d\, r(K^d)$ and $U\, d\, r(K)$ imply that $\hat{U}\, d\, K^d$ and $\hat{U}\, d\, K$ and hence $U = 0$. Therefore, we can conclude that $r(K)^d = r(K^d)$.

[6] Order dense ideal = Foundation

[7] Order ideal = Normal sublineal.

[8] Since $0 < U - U_0 \in r(K)^{dd}$, one can choose $0 < W \le U - U_0$ with a positive extension $\hat{W} \in K$; therefore, there exists $\hat{V} \in K_0$ with $0 < \hat{V} \le \hat{W}$ and thus $0 < V = r(\hat{V}) \le W \le U - U_0$.

Now, assume that K_1 and K_2 are saturated bands. The inclusion $r(K_1 \cap K_2) \subset r(K_1) \cap r(K_2)$ is obvious. To verify the converse inclusion observe that if a positive operator U is contained in $r(K_1) \cap r(K_2)$, then every its positive extension to the whole of $\hat{E}$ is contained in K_1 and in K_2 and hence in $K_1 \cap K_2$. Thus, we conclude that $r : \hat{\mathfrak{E}}_0 \to \mathfrak{E}$ is a Boolean homomorphism and it remains to observe that the equality $r(K) = \{0\}$ obviously implies $K = \{0\}$.

To check the equality $r^{-1} = q$ we have to verify that $q \circ r(K) = K$ for any $K \in \hat{\mathfrak{E}}_0$ and $r \circ q(L) = L$ for any $L \in \mathfrak{E}$. The first claim is obvious. At the same time if $L \in \mathfrak{E}$, then $r \circ q(L) \supset L$ by definition of r and q, while the relations $0 \le \hat{U} \in q(L)$ and $r(\hat{U}) \in L^d$ imply $U \in q(L)^d$, i. e. $U = 0$. Thus, $r \circ q(L) = L$ and the proof is complete. □

Proposition 3. *Two positive operators $U, V \in \mathscr{L}(E, F)$ are disjoint if and only if any their positive extensions $\hat{U}, \hat{V} \in \mathscr{L}(\hat{E}, F)$ to the whole of $\hat{E}$ are likewise disjoint.*

Proof. Denote $\{\hat{U}\} = \{\hat{U} \in \mathscr{L}^+(\hat{E}, F) : r(\hat{U}) = U\}$ and prove that $\{\hat{U}\}^{dd} = q(U^{dd})$. Let $\hat{V} \in \mathscr{L}^+(\hat{E}, F)$ and $V = r(\hat{V}) \le \alpha U$ for a positive real α. Put $W = \alpha U - V$ and extend W to a positive operator $\hat{W} \in \mathscr{L}^+(\hat{E}, F)$. Then $r(\hat{W} + \hat{V}) = U$ and $\hat{V} \le \hat{W} + \hat{V}$ and consequently $\hat{V} \in \{\hat{U}\}$. This implies that $\{\hat{U}\}^{dd}$ contains the set $\{\hat{V} : (\exists\, 0 < \alpha \in \mathbb{R})\, r(|\hat{V}|) \le \alpha U\}$, an order dense ideal in $q(U^{dd})$, and therefore $\{\hat{U}\}^{dd} \supset q(U^{dd})$. The converse inclusion is obvious. Now, if $\{\hat{U}\}$ and $\{\hat{V}\}$ are disjoint, then U and V are likewise disjoint by Theorem 1. □

2. Simultaneous extension operator

Let $U \in \mathscr{L}^+(E, F)$ and $\hat{U} \in \mathscr{L}^+(\hat{E}, F)$ is a positive extension of U. Take U and $\hat{U}$ as weak order units in the respective bands U^{dd} and $\hat{U}^{dd}$ and denote by $\mathfrak{E}(U)$ and $\mathfrak{E}(\hat{U})$ the Boolean algebras of all components[9] of U and $\hat{U}$, respectively. Since $q(K) \cap \hat{U}^{dd} \ne \{0\}$ for any band $\{0\} \ne K \subset U^{dd}$, the mapping $K \mapsto q(K) \cap \hat{U}^{dd}$ is a Boolean isomorphism. Therefore, the mapping defined by

$$\varphi e = \mathrm{Pr}_{q(e^{dd})} \hat{U} \quad (e \in \mathfrak{E}(U)) \tag{1}$$

is likewise a Boolean isomorphism of $\mathfrak{E}(U)$ onto a complete subalgebra of the Boolean algebra $\mathfrak{E}(\hat{U})$.

Lemma 4. *If a positive operator $e \in U^{dd}$ is a component of U, then φe is its extension. In symbols, $r \circ \varphi = \mathrm{id}_{\mathfrak{E}(U)}$.*

Proof. The required assertion is equivalent to the following: $\mathrm{Pr}_K U = r(\mathrm{Pr}_{q(K)} \hat{U})$ for every band $K \subset U^{dd}$. To prove this it suffices to observe that

$$\{U' \in K : 0 \le U' \le U\} = \{r(\hat{U}') : \hat{U}' \in q(K),\, 0 \le \hat{U}' \le \hat{U}\}.$$

Taking this into consideration we deduce

$$\mathrm{Pr}_K U = \sup\{U' \in K : 0 \le U' \le U\}$$
$$= \sup\{r(\hat{U}') : \hat{U}' \in q(K),\, 0 \le \hat{U}' \le \hat{U}\} = r(\mathrm{Pr}_{q(K)} \hat{U}). \ \square$$

Lemma 5. *There exists an order continuous lattice isomorphism[10] of U^{dd} into $\hat{U}^{dd}$ such that $r \circ s = \mathrm{id}_{U^{dd}}$.*

Proof. First we prove that the spectral integral

$$sV = \int_{-\infty}^{\infty} \lambda \, d\varphi(e_\lambda^V), \tag{2}$$

exists for each $V \in U^{dd}$, where $\left(e_\lambda^V\right)_{\lambda \in \mathbb{R}}$ is the spectral system[11] (or spectral function) of V and φ is an order bounded measure on $\mathfrak{E}(U)$, with values in the Dedekind complete vector lattice $\hat{U}^{dd}$,

[9] Component (of U) = Unit element (with respect to U).
[10] Lattice isomorphism = Algebraic and structure isomorphism.
[11] with respect to U, i. e. $e_\lambda^V := \mathrm{Pr}_{\left((\lambda U - V)^+\right)^{dd}} U$.

given by (1). It suffices to ensure that at least one of the integral sums exists (see [2; Ch. VIII, § 1.21]). Take a partition of the real axis

$$-\infty \leftarrow \cdots < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \to +\infty$$

with $\delta = \sup_n(\lambda_n - \lambda_{n-1}) < +\infty$. Then the series $\sum\limits_{-\infty}^{\infty} l_n\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)$ where $\lambda_{n-1} \le l_n \le \lambda_n$, is convergent and consequently the series $\sum\limits_{-\infty}^{\infty} |l_n|\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)$ is likewise convergent. This means that the sequence of partial sums $S_k = \sum\limits_{n=-k}^{k} |l_n|\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)$ is increasing, order bounded, and $S = \sup_k S_k$ is the sum of the latter series. Denote $\hat{S}_k = \sum\limits_{n=-k}^{k} |l_n|\varphi\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)$. Since the sequence $(\hat{S}_k)$ is increasing and $r(\hat{S}_k) = S_k$ for all $k \in \mathbb{N}$, we can apply Lemma 2 and conclude that the sequence $(\hat{S}_k)$ is bounded above, i. e. the series $\sum\limits_{-\infty}^{\infty} |l_n|\varphi\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)$ is convergent, and with it the series $\sum\limits_{-\infty}^{\infty} l_n\varphi\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)$. Thus, the spectral integral (2) exists and the operator s defined by (2) is order continuous (see [1; Ch. VIII, § 10]). Moreover,

$$r \circ s(V) = r\left(\lim_{\delta\to 0}\sum_{-\infty}^{\infty} l_n\varphi\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big)\right) = \lim_{\delta\to 0}\sum_{-\infty}^{\infty} l_n r \circ \varphi\big(e^V_{\lambda_n} - e^V_{\lambda_{n-1}}\big) = V.$$

Clearly, $se = \varphi(e) \in \mathfrak{E}(\hat{U})$ for every $e \in \mathfrak{E}(U)$, which imply by [2; Ch. VII, Theorem 3.13] that s is multiplicative and hence preserves the lattice operations, see [2; Ch. VII, Theorem 3.23]. □

Theorem 2. *Let $\hat{E}$ be an Archimedean vector lattice, let E be its majorizing sublattice, and let F be a Dedekind complete vector lattice. Then there exists an order continuous lattice isomorphism s from $\mathscr{L}(E,F)$ into $\mathscr{L}(\hat{E},F)$ such that $r \circ s = \mathrm{id}_{\mathscr{L}(E,F)}$.*

Proof. Let $(U_\xi)_{\xi\in\Xi}$ be a total family of pairwise disjoint positive operators in $\mathscr{L}(E,F)$ and for every $\xi \in \Xi$ choose a positive extension $\hat{U}_\xi$ of U_ξ to the whole of $\hat{E}$. For an arbitrary $V \in \mathscr{L}^+(E,F)$ put $V_\xi = \mathrm{Pr}_{U_\xi^{dd}}V$ and define $sV = \sup_{\xi\in\Xi} s_\xi V_\xi$, where s_ξ is the order continuous lattice isomorphism corresponding to the pair $(U_\xi, \hat{U}_\xi)$ by Lemma 5. Clearly, Lemma 2 implies that sV is correctly defined for every operator $V \in \mathscr{L}^+(E,F)$. It is easily seen that s is additive and positively homogeneous and can be extended to $\mathscr{L}(E,F)$ by differences. Making use of the properties of s_ξ we can also ensure that s is order continuous and preserves the lattice operations. Moreover, for $V \in \mathscr{L}^+(E,F)$ and $x \in E^+$ we have

$$(r \circ s)(V)x = r\left(\sup\nolimits_{\xi\in\Xi} s_\xi(V_\xi)\right)x = \Big(\sum\nolimits_{\xi\in\Xi} r \circ s_\xi V_\xi\Big)x = \Big(\sum\nolimits_{\xi\in\Xi} V_\xi\Big)x = Vx$$

and the proof is complete. □

Remark 1. The “simultaneous extension operator” s is multiplicative[12]. Indeed, if $V \ge 0$, $W \ge 0$ and the product $V \cdot W$ exists in $\mathscr{L}(E,F)$, then putting $V_\xi = \mathrm{Pr}_{U_\xi^{dd}}V$ and $W_\xi = \mathrm{Pr}_{U_\xi^{dd}}W$ we observe that $V_\xi \cdot W_\xi$ exists for all $\xi \in \Xi$ and $V \cdot W = \sup_\xi V_\xi \cdot W_\xi$. Therefore, $s(V \cdot W) = s(\sup_\xi V_\xi \cdot W_\xi) = s(\sup_\xi V_\xi \cdot \sup_\xi W_\xi) = \sup_\xi s(V_\xi) \cdot \sup_\xi s(W_\xi) = sV \cdot sW$.

Remark 2. The set $\mathrm{im}(s) = \{sU : U \in \mathscr{L}(E,F)\}$ is an order closed sublattice in $\mathscr{L}(\hat{E},F)$ and there exist an order continuous positive projection onto it, namely $P = s \circ r$. Indeed, $P^2 = s \circ (r \circ s) \circ r = s \circ r = P$. The projection P is majorized by the identity operator (i. e. P is a band projection) if and only if every regular operator admit a unique regular extension to all of $\hat{E}$. To ensure this, we need only to observe that if a positive operator U has two comparable positive extensions, say $\hat{U}_1$ and $\hat{U}_1$ with $\hat{U}_1 \le \hat{U}_1$, then the relations $r(\hat{U}_2 - \hat{U}_1) = 0$ and $\hat{U}_2 - \hat{U}_1 \ge 0$ imply that $\hat{U}_2 - \hat{U}_1 = 0$, i.e. $\hat{U}_1 = \hat{U}_2$.

Remark 3. If $\hat{E}$ is the Dedekind completion of E, then we can choose s so that sV is normal for any normal operator V.

[12] A partial multiplication in $\mathscr{L}(E,F)$ is defined by taking U_ξ as an order unit in $\{U_\xi\}^{dd}$.

Remark 4. If E is an arbitrary vector sublattice in $\hat{E}$, then Theorem 2 remain valid provided that every positive operator from E_1 to F admit a positive extension to the whole of $\hat{E}$, where E_1 denotes the order ideal in $\hat{E}$ generated by E. It is well known that for a simultaneous extension operator from E_1 to $\hat{E}$ can be taken the *minimal extension* m defined by

$$(mU)x = \sup\{Ux' : 0 \leq x' \leq x,\, x' \in E_1\} \quad (x \in \hat{E}^+,\ U \in \mathscr{L}^+(E_1, F)).$$

Now, if s_1 stands for the simultaneous extension from E to E_1, then $s = s_1 \circ m$ is the desired simultaneous extension from E to $\hat{E}$.

3. Extension of bilinear operators

Now we turn our attention to the problem of extension of bilinear operators. Let E_1 and E_2 be vector lattices and F be a Dedekind complete vector lattice. A bilinear operator $b : E_1 \times E_2 \to F$ is called *positive* if $b(x, y) \geq 0$ for all $x \in E_1^+$ and $y \in E_2^+$, and *regular* if it can be represented as the difference of two positive bilinear operators. Denote by $\mathscr{B}(E_1, E_2; F)$ the set of all regular bilinear operators from $E_1 \times E_2$ to F.

Let E_1 and E_2 be some majorizing sublattices of the vector lattices $\hat{E}_1$ and $\hat{E}_2$, respectively. The restriction of a regular operator $\hat{b} : \hat{E}_1 \times \hat{E}_2 \to F$ to $E_1 \times E_2$ is also denoted by $r\hat{b}$. It is easily deduced from Theorem 2 that a positive bilinear operator $b \in \mathscr{B}(E_1, E_2; F)$ admits a positive bilinear extension $\hat{b}$ to the whole of $\hat{E}_1 \times \hat{E}_2$. But a stronger assertion holds.

Theorem 3. *Let E_1 and E_2 be majorizing sublattices of vector lattices $\hat{E}_1$ and $\hat{E}_2$ respectively, and let F be a Dedekind complete vector lattice. Then there exists an order continuous lattice isomorphism s acting from $\mathscr{B}(E_1, E_2; F)$ to $\mathscr{B}(\hat{E}_1, \hat{E}_2; F)$ such that $r \circ s = \mathrm{id}_{\mathscr{B}(E_1,E_2;F)}$.*

Proof. Since the vector lattices $\mathscr{B}(E_1, E_2; F)$ and $\mathscr{L}(E_1, \mathscr{L}(E_2, F))$ as well as the vector lattices $\mathscr{B}(\hat{E}_1, E_2; F)$ and $\mathscr{L}(\hat{E}_1, \mathscr{L}(E_2, F))$ are pairwise isomorphic, the simultaneous extension operator $s_1 : \mathscr{L}(E_1, \mathscr{L}(E_2, F)) \to \mathscr{L}(\hat{E}_1, \mathscr{L}(E_2, F))$, which exists by Theorem 2, defines a simultaneous extension s_1' of regular bilinear operators from $E_1 \times E_2$ to $\hat{E}_1 \times E_2$. Further, making use of the isomorphism of another pairs of vector lattices, namely $\mathscr{B}(\hat{E}_1, E_2; F)$ and $\mathscr{L}(E_2, \mathscr{L}(\hat{E}_1, F))$ as well as $\mathscr{B}(\hat{E}_1, \hat{E}_2; F)$ and $\mathscr{L}(\hat{E}_1, \mathscr{L}(\hat{E}_2, F))$, a simultaneous extension s_2' of regular bilinear operators from $\hat{E}_1 \times E_2$ to $\hat{E}_1 \times \hat{E}_2$ can be defined in a similar way. Clearly, $s = s_2' \circ s_1'$ is the desired operator. □

Lemma 6. *Let $b, d \in \mathscr{B}(E_1, E_2; F)$, $0 \leq b \leq d$, and let $\hat{d}$ be an arbitrary positive bilinear extension of d to the whole of $\hat{E}_1 \times \hat{E}_2$. Then there exists a bilinear extension $\hat{b}$ of b to $\hat{E}_1 \times \hat{E}_2$ such that $0 \leq \hat{b} \leq \hat{d}$.*

Proof. The proof can be obtained by using the existence of the tensor product $E_1 \bar{\otimes} E_2$ of any Archimedean vector lattices E_1 and E_2 and taking into consideration the fact that $E_1 \bar{\otimes} E_2$ is a majorizing sublattice of $\hat{E}_1 \bar{\otimes} \hat{E}_2$ (see [3; Theorems 4.4 and 5.3]). □

Definition. A bilinear operator $b \in \mathscr{B}(E_1, E_2; F)$ is called *order continuous* if for every $x' \in E_1$ and $x'' \in E_2$ we have $b(u_\alpha, x'') \xrightarrow{(o)} b(u, x'')$ and $b(x', v_\alpha) \xrightarrow{(o)} b(x', v)$, whenever $u_\alpha \xrightarrow{(o)} u$ in E_1 and $v_\alpha \xrightarrow{(o)} v$ in E_2.

Theorem 4. *Let F be a Dedekind complete vector lattice and let $\hat{E}_1$ and $\hat{E}_2$ be the Dedekind completions of vector lattices E_1 and E_2, respectively. Then every order continuous regular bilinear operator $b : E_1 \times E_2 \to F$ admits a unique order continuous regular bilinear extension to $\hat{E}_1 \times \hat{E}_2$.*

Proof. This follows immediately from Theorem 2 and Remark 3. □

Definition. A bilinear operator $b \in \mathscr{B}(E_1, E_2; F)$ is called 1) *abnormal* if it vanishes on some order dense ideal in the vector lattice $E_1 \times E_2$, 2) *normal* if it is disjoint from all abnormal operators, and 3) *antinormal* if it is disjoint from all normal bilinear operators.

Denote by $\mathscr{B}_n(E_1, E_2; F)$ and $\mathscr{B}_{ant}(E_1, E_2; F)$ the sets of al normal and antinormal bilinear operators and let $\mathscr{L}_n(E, F)$ stands for the set of all normal bilinear operators from $\mathscr{L}(E, F)$.

Proposition 4. *The set of all order continuous regular bilinear operators is a band in $\mathscr{B}(E_1, E_2; F)$ which coincides with the band $\mathscr{B}_n(E_1, E_2; F)$.*

Proof. Just as in the proof of Theorem 3 we shall employ the isomorphism between vector lattices $\mathscr{B}(E_1, E_2; F)$ and $\mathscr{L}(E_1, \mathscr{L}(E_2, F))$. Denote by τ this isomorphism. Take an order continuous bilinear operator $b \in \mathscr{B}(E_1, E_2; F)$. For every $x \in E_1^+$ we have

$$(\tau b^+)x = (\tau b)^+ x = \sup\{(\tau b)x' : \ 0 \leq x' \leq x, \, x' \in E_1\}.$$

Since $(\tau b)x' \in \mathscr{L}_n(E_2, F)$ for all $x' \in E_1$, it follows that $(\tau b^+)x \in \mathscr{L}_n(E_2, F)$. This means that $b^+(x, \cdot)$ is order continuous for every $x \in E_1$. Similar reasoning shows that $b^+(\cdot, y)$ is also order continuous for any $y \in E_2$. Suppose that $d \in \mathscr{B}(E_1, E_2; F)$ is abnormal and positive and put $a = b \wedge d$. Then a is also abnormal, since $a \leq d$ as well as a is order continuous, since $a \leq d$. Consequently $a = 0$.

Conversely, assume that $b \in \mathscr{B}_n(E_1, E_2; F)$. Since an order dense ideal in $E_1 \times E_2$ is the Cartesian product of order dense ideals, $\tau^{-1}(U)$ is an abnormal bilinear operator for every abnormal $U \in \mathscr{L}(E_1, \mathscr{L}(E_2, F))$ and thus b and $\tau^{-1}U$ are disjoint. Consequently τb and U are also disjoint and we come to the relation $\tau b \in \mathscr{L}_n(E_1, \mathscr{L}(E_2, F))$ which implies that $b(x, \cdot)$ is order continuous for any $x \in E_1$. Similar reasoning ensures that $b^+(\cdot, y)$ is also order continuous for any $y \in E_2$. □

Now we are able to state a result similar to the theorem on homomorphism between the classes of regular operators on vector lattices and their Dedekind completions due to A. I. Veksler [4].

Theorem 5. *Let r be the restriction operator sending each $b \in \mathscr{B}(\hat{E}_1, \hat{E}_2; F)$ to its restriction onto $E_1 \times E_2$. Then the following assertions hold:*

(1) *r is a strongly isotonic algebraic homomorphism;*

(2) *the inverse image of $\mathscr{B}_{ant}(E_1, E_2; F)$ coincide with $\mathscr{B}_{ant}(\hat{E}_1, \hat{E}_2; F)$;*

(3) *the restriction of r to $\mathscr{B}_n(\hat{E}_1, \hat{E}_2; F)$ is an algebraic and lattice isomorphism onto $\mathscr{B}_n(E_1, E_2; F)$;*

(4) *for any $b \in \mathscr{B}_n(E_1, E_2; F)$ the set $r^{-1}(b)$ is of the form $\hat{b} + \mathscr{E}$ where $\hat{b} \in \mathscr{B}_n(\hat{E}_1, \hat{E}_2; F)$, $r(\hat{b}) = b$, and $\mathscr{E}$ is a set of antinormal operators vanishing on $E_1 \times E_2$.*

Proof. This follows immediately from Theorem 4, Proposition 4, and Lemma 6. □

4. Representation of regular functionals

In this section G denotes a universally complete vector lattice[13] with the principal order ideal M generated by the order unit[14]. Let E and F be arbitrary vector sublattices of G. In [5] the space of regular functionals on an arbitrary order ideal in G was embedded into the universally complete vector lattice $\mathfrak{M}(\widetilde{M})$[*)] so that functionals disjoint in a generalized sense are transformed into functionals disjoint in the conventional sense. We are going to obtain similar results for an arbitrary sublattices of G by applying the results of Section 1.

First of all we introduce the notion of generalized disjointness of regular functionals defined on different sublattices $E \subset G$ and $F \subset G$ of the same universally complete vector lattice G.

Definition. Let $f \in \widetilde{E}$ and $g \in \widetilde{F}$[15]. We say that f and g are *disjoint* (and write $f \, \delta \, g$) if arbitrary positive extensions of $|f|$ and $|g|$ respectively to $I(E)$ and $I(F)$ are disjoint in the sense of [5]. (From here on $I(E)$ denotes the order ideal in G generated by E.)

It follows from Proposition 5 and [5; Lemma 9] that if $E = F$, then the generalized disjointness is equivalent to the conventional one.

Lemma 7. *Let G_1 and G_2 be universally complete vector lattices with order units $\mathbb{1}_1$ and $\mathbb{1}_2$, respectively. Let T_1 be an order dense ideal in G_1, T_2 an order ideal in G_2, and φ a Boolean isomorphism of the Boolean algebra $\mathfrak{E}(T_1)$[16] onto a complete subalgebra of the Boolean algebra $\mathfrak{E}(T_2)$. Then there exists a unique pair (S, V), where V is an order closed vector sublattice of G_2 and S is an isomorphism of G_1 onto V, satisfying the following conditions:*

(1) $S(\mathbb{1}_1) = \mathrm{Pr}_{V^{dd}} \mathbb{1}_2$;

[13] with a fixed order unit, say $\mathbb{1}$. Universally complete vector lattice = Extended K-space.
[14] Principal ideal generated by the order unit = Subspace of bounded elements.
[*)] $\mathfrak{M}(\widetilde{M})$ denotes the universal completion (= maximal extension) of the Dedekind complete vector lattice $\widetilde{M}$.
[15] $\widetilde{E}$ denotes the set of all regular functionals on E.
[16] $\mathfrak{E}(T)$ denotes the Boolean algebra of all bands of the vector lattice T.

(2) *for any band H in G_1 the bands generated in G_2 by $S(H)$ and $\varphi(H \cap T_1)$ coincide.*

Proof. Extend φ to an isomorphism $\bar{\varphi}$ between the bases of G_1 and the universal completion of T_2. Denote by μ the corresponding isomorphism between complete Boolean algebras $\mathfrak{B}_1 = \{\mathrm{Pr}_L \mathbb{1}_1 : L \in \mathfrak{E}(G_1)\}$ and $\mathfrak{B}_2 = \{\mathrm{Pr}_K \mathbb{1}_2 : K \in \bar{\varphi}(\mathfrak{E}(G_2))\}$. Let V denotes the sublattice in G_2 consisting of the elements whose spectral functions take their values in the Boolean algebra $\mathfrak{B}_2$. Define an isomorphism between G_1 and V by

$$Sz = \int_{-\infty}^{\infty} \lambda \, d\mu(e_\lambda^z)$$

where $(e_\lambda^z)_{\lambda \in \mathbb{R}}$ is a spectral function of z. Clearly, S satisfy (1) and (2). □

Now, if we insert q for φ and replace T_1 and T_2 by $\widetilde{E}$ and $\widetilde{I(E)}$ respectively, then we obtain the following.

Corollary. *Assume that some order units $\mathbb{1}_1$ and $\mathbb{1}_2$ are fixed in the universally complete vector lattices $\mathfrak{M}\big(\widetilde{I(E)}\big)$ and $\mathfrak{M}(\widetilde{E})$, respectively. Then there exists a unique pair (S_E, V_E) such that V_E is an order closed sublattice of $\mathfrak{M}\big(\widetilde{I(E)}\big)$, S_E is a lattice isomorphism of $\mathfrak{M}(\widetilde{E})$ onto V_E, and the following hold:*

(1) $S_E(\mathbb{1}_1) = \mathrm{Pr}_{V_E^{dd}} \mathbb{1}_2$;

(2) *for any band H in $\mathfrak{M}(\widetilde{E})$ the bands generated in $\mathfrak{M}\big(\widetilde{I(E)}\big)$ by $S_E(H)$ and $q(H \cap \widetilde{E})$ coincide.*

Lemma 8. *For a positive functional $f \in \widetilde{E}$ the bands generated in $\mathfrak{M}\big(\widetilde{I(E)}\big)$ by $S_E(f)$ and by the set of all positive extensions of f to the whole of $I(E)$ coincide.*

Proof. Follows immediately from the above Corollary and Proposition 3. □

Theorem 6. *Assume that some order units $\mathbb{1}_1$ and $\mathbb{1}_2$ are fixed in the universally complete vector lattices $\mathfrak{M}\big(\widetilde{I(E)}\big)$ and $\mathfrak{M}(\widetilde{M})$, respectively. Then there exists a unique pair $(\bar{R}_E, \bar{V}_E)$ such that $\bar{V}_E$ is an order closed sublattice of $\mathfrak{M}(\widetilde{M})$, $\bar{R}_E$ is a lattice isomorphism of $\mathfrak{M}(\widetilde{E})$ onto $\bar{V}_E$, and the following hold:*

(1) $\bar{R}_E(\mathbb{1}_1) = \mathrm{Pr}_{\bar{V}_E^{dd}} \mathbb{1}_2$;

(2) *for any $f \in \widetilde{E}$ and $g \in \widetilde{M}$ the relations $f \, \delta \, g$ and $\bar{R}_E(f) \, d \, g$ are equivalent.*

Proof. Denote by (R_E, V_E) the canonical representation of $\widetilde{I(E)}$[17] and put $\bar{R}_E = R_E \circ S_E$, $\bar{V}_E = \bar{R}_E(V_E)$. Then the pair $(\bar{R}_E, \bar{V}_E)$ obey the required conditions. Indeed, (1) follows obviously from the above corollary[18] and it suffice to prove (2) only for positive functionals. Thus, take positive functionals $f \in \widetilde{E}$ and $g \in \widetilde{M}$. Denote by H the band in $\mathfrak{M}(\widetilde{E})$ generated by f and put $H_1 = H \cap \widetilde{E}$. By definition $f \, \delta \, g$ means that all positive extensions of f to $I(E)$ are disjoint from g in the sense of [5], i. e. $f \, \mathrm{D} \, g$[19]. By Proposition 3 this is equivalent to $q(H_1) \, \mathrm{D} \, g$ which means, by Theorem 3.1 of [5], that the set $R_E(q(H))$ is disjoint from g. This latter, combined with the corollary to Lemma 7, gives $R_E(S_E(H)) \, d \, g$, i. e. $\bar{R}_E f \, d \, g$.

Uniqueness is easily deduced from Lemma 7 in the same way as in [5; Theorem 3.1]. We need only to observe that $\bar{R}_E$ corresponds by Lemma 7 to the isomorphism $B \circ q$[20] of Boolean algebras $\mathfrak{E}(\widetilde{E})$ and $\mathfrak{E}(\bar{V}_E)$. □

Theorem 7. *If $f \in \widetilde{E}$ and $g \in \widetilde{F}$, then the relations $f \, \delta \, g$ and $\bar{R}_E(f) \ d \ \bar{R}_F(g)$ are equivalent.*

Proof. Without loss of generality we may assume that f and g are positive. Let $\mathscr{E}(f)$ and $\mathscr{E}(g)$ stand for the sets of all positive extensions of f to the whole of $I(E)$ and of g to the whole of $I(F)$, respectively. Then by Proposition 3, Lemma 8, and Theorem 3.3 from [5] we have

$$\big(R_E(S_E f)\big)^{dd} = R_E\big(\mathscr{E}(f)^{dd}\big) \, d \, R_F\big(\mathscr{E}(g)^{dd}\big) = \big(R_F(S_F g)^{dd}\big).$$

Thus, $\bar{R}_E f \, d \, \bar{R}_F g$ and the proof is complete. □

[17] See [5; Theorem 3.1].

[18] Combine Lemma 7 (1), [5; Theorem 3.1 (2)], and the above definitions of $\bar{V}_E$ and $\bar{R}_E$.

[19] If $f \in \widetilde{E}$ and $g \in \widetilde{F}$, then $f \, \mathrm{D} \, g$ means that $f_{(u)} \, d \, g_{(v)}$ in $\widetilde{M}$ for all $u \in E^+$ and $v \in F^+$, where $f_{(u)} \in \widetilde{M}$ is correctly defined by $f_{(u)}(x) = f(xu)$ $(x \in M)$, since G is endowed with the product that makes G an f-algebra having $\mathbb{1}$ as its unit element and $E \cdot M \subset E$, see [5; 3.2 and 3.3].

[20] For the definition of B see [5; Lemmas 10 and 12].

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