УДК 517.98 Дата поступления 20 июня 2012 г.

Кусраев А. Г. Булевозначный подход к изучению инъективных банаховых решеток. II.—Владикавказ, 2012.—26 с.—(Препринт / ЮМИ ВНЦ РАН; № 1).

Статья является продолжением работы [Кусраев А. Г. Булевозначный подход к изучению инъективных банаховых решеток I.—Владикавказ, 2011.—28 с.—(Препринт / ЮМИ ВНЦ РАН; № 1)] и посвящена таким полезным конструкциям из теории банаховых решеток, как однородное функциональное исчисление, овыпукление, прямая сумма и тензорное произведение. Основная задача — из заданных инъективных банаховых решеток построить новую инъективную банахову решетку, используя указанные конструкции.

Работа выполнена при финансовой поддержке федеральной целевой программы «Научные и научно-педагогические кадры инновационной России» на 2009–2013 гг., заявка № 2012-1.2.1-12-000-1001-004.

Ключевые слова: AL-пространство, AM-пространство, инъективная банахова решетка, однородное функциональное исчисление, овыпукление, прямая сумма, тензорное произведение, булевозначное представление.

Библиогр. 54.

Received June 20, 2012

Kusraev A. G. Boolean-Valued Analysis Approach To Injective Banach Lattices. II.—Vladikavkaz, 2012.—26 p.—(Preprint / SMI VSC RAS; № 1).

This work is a continuation of [Kusraev A. G. Boolean-valued Analysis Approach to Injective Banach Latices. I.—Vladikavkaz, 2011.—28 p.—(Preprint / SMI VSC RAS; № 1)] and is devoted to some useful construction in Banach lattice theory: homogeneous functional calculus, convexification and concavification, direct sum and tensor product. We are mainly concerned with the problem how to produce new injective Banach lattice from the given ones using these constructions.

Supported by a grant from the Federal Grant-in-Aid Program «Human Capital for Science and Education in Innovative Russia» for 2009–2013; Application № 2012-1.2.1-12-000-1001-004.

Mathematics Subject Classification (2000): 46A40, 47B60, 12F20, 03C90, 03C98.

Keywords: AL-space, AM-space, injective Banach lattice, homogeneous functional calculus, convexification, concavification, direct sum, tensor product, Boolean-valued representation.

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BOOLEAN VALUED ANALYSIS APPROACH TO INJECTIVE BANACH LATTICES. II[1]

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[1]Supported by a grant from the Federal Grant-in-Aid Program «Human Capital for Science and Education in Innovative Russia» for 2009–2013; Application № 2012-1.2.1-12-000-1001-004.

1. Introduction

This work is a continuation of [32] and is devoted to some useful construction in Banach lattice theory: homogeneous functional calculus, convexification and concavification, direct sum and tensor product. We are mainly concerned with the problem how to produce new injective Banach lattice from the old ones using these constructions.

We attack the problem by means of the *Boolean-valued transfer principle* from AL-spaces to injective Banach lattices established in [32]: every injective Banach lattice embeds into an appropriate Boolean-valued model, becoming an AL-space. According to this fact and fundamental principles of Boolean-valued models, each theorem about the AL-space within Zermelo–Fraenkel set theory has an analog for the original injective Banach lattice interpreted as the Boolean-valued AL-space. Translation of theorems from AL-spaces to injective Banach lattices is carried out by appropriate general operations of Boolean-valued analysis. This approach stems from Gordon [19, 20] and Takeuti [49].

In Section 2 we recall some basic notions and facts about injective Banach lattices, Kaplansky–Hilbert modules, and Fremlin's tensor products of vector and Banach lattices. Section 3 deals with the interrelation between homogeneous functional calculi in a Boolean-valued Banach lattice and in its bounded descent. Section 4 demonstrates that Kaplansky–Hilbert lattices and injective Banach lattices may be produced from each other by means of the well known convexification and concavification procedures. Section 5 contains three lemmas which are basic for the understanding of the interplay between the vector-valued norm and the conventional one. In Section 6 we sketch a Boolean-valued version of the cocavification procedure and demonstrate that it enables us to produce new examples of injective Banach lattices. Section 7 deals with injective sum of injective Banach lattices and contains also a characterization of injective Banach lattices in terms of summable sequences. In Section 8 we construct the injective tensor product of two injective Banach lattice which, in a particular case of one-dimensional M-centers, is similar to the relationship between the L_1 space of a product measure and the L_1 spaces of its factors.

In this work we deal only with the isometric theory, i.e. with 1-injective Banach lattices. For λ-injective Banach lattices ($\lambda > 1$) see [37, 39].

For the theory of Banach lattices and positive operators we refer to the books [1, 2, 40]. The elementary theory of Boolean algebras can be found in [41, 47, 53]. The needed information on the theory of Boolean-valued models is briefly presented in [28, Chapter 9]; details may be found in [5, 33, 52].

In what follows X and Y denote Banach lattices. We denote by $\mathbb{P}(X)$ the Boolean algebra of all band projections in a vector lattice X. Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$, while $\Lambda := \Lambda(\mathbb{B})$ is a Dedekind complete AM-space with unit such that $\mathbb{B} = \mathbb{P}(\Lambda)$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$. Everywhere below $\mathrm{V}^{(\mathbb{B})}$ is the Boolean-valued model of set theory built from a complete Boolean algebra $\mathbb{B}$.

We let := denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ symbolize the naturals, the rationals, and the reals.

2. Prerequisites

Let Λ be a real Dedekind complete AM-space with unit $\mathbb{1}$ endowed with a unique f-algebra multiplication. Then $\bar{\Lambda}:=\Lambda\oplus i\Lambda$ is a commutative C^*-algebra often called a *Stone algebra*. We write $\Lambda=\Lambda(\mathbb{B})$ whenever $\mathbb{B}$ is a Boolean algebra of band projections (or positive idempotents) in Λ. The unexplained terms of use below can be found in [28] and [40].

Injective Banach Lattices. Recall some definitions and facts needed in the sequel.

Definition 2.1. A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0\subset Y$, and every positive linear operator $T_0:Y_0\to X$ there exists a positive linear extension $T:Y\to X$ with $\|T_0\|=\|T\|$. Equivalently, X is an injective Banach lattice if, whenever X is lattice isometrically imbedded into a Banach lattice Y, there exists a positive contractive projection from Y onto X.

Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [3, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms. More details see in Lotz [38], Cartwright [13], Haydon [23], Buskes [8], and Wickstead [54].

Definition 2.2. A band projection π in a Banach lattice X is called an *M-projection* if $\|x\|=\max\{\|\pi x\|,\|\pi^{\perp}x\|\}$ for all $x\in X$, where $\pi^{\perp}:=I_X-\pi$. The collection of all M-projections forms a subalgebra $\mathbb{M}(X)$ of the Boolean algebra of all band projections $\mathbb{P}(X)$ in X. The f-subalgebra of the center $\mathscr{Z}(X)$ generated by $\mathbb{M}(X)$ is called the *M-center* and denoted by $\mathscr{Z}_m(X)$.

Observe that $\mathbb{M}(X)$ is an order closed subalgebra of $\mathbb{P}(X)$ whenever X has the Fatou and Levi properties. The relations $\mathbb{B}\simeq\mathbb{M}(X)$ and $\Lambda(\mathbb{B})\simeq\mathscr{Z}_m(X)$ are equivalent. The notion of an M-projection plays a crucial role in the theory of injective Banach lattices. In a wider context of a general Banach space theory the concept see in [6] and [22].

Definition 2.3. Let X and Y be Banach lattices and $\mathbb{B}$ a Boolean algebra which is identified with a subalgebra of $\mathbb{P}(X)$ and a subalgebra of $\mathbb{P}(Y)$. An operator $T:X\to Y$ is called *$\mathbb{B}$-linear*, if it is linear and $b\circ T=T\circ b$ for all $b\in\mathbb{B}$. Say that X is *lattice $\mathbb{B}$-isometric* to Y and write $X\simeq_{\mathbb{B}}Y$ if there is a $\mathbb{B}$-linear lattice isometry from X onto Y.

Now we are able to state a Boolean valued transfer principle from AL-spaces to injective Banach lattices, see [32, Theorem 5.1 and Corollary 5.2].

Theorem 2.4. *The bounded descent $X:=\mathscr{X}{\Downarrow}$ of an AL-space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ is an injective Banach lattice with $\mathbb{B}\simeq\mathbb{M}(X)$ and $\Lambda(\mathbb{B})\simeq\mathscr{Z}_m(X)$. Conversely, if X is an injective Banach lattice and $\mathbb{B}=\mathbb{M}(X)$, then there exists a unique up to lattice isometry AL-space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ whose bounded descent is lattice $\mathbb{B}$-isometric to X.*

Corollary 2.5. *If Φ is a strictly positive Maharam operator with the Levi property taking values in a Dedekind complete AM-space Λ with unit and $|||x|||=\|\Phi(|x|)\|_\infty$ $(x\in L^1(\Phi))$, then $(L^1(\Phi),|||\cdot|||)$ is an injective Banach lattice and there is a Boolean isomorphism φ from $\mathbb{P}(\Lambda)$ onto $\mathbb{M}(L^1(\Phi))$ with $\pi\circ\Phi=\Phi\circ\varphi(\pi)$ $(\pi\in\mathbb{B})$.*

Conversely, any injective Banach lattice X is lattice $\mathbb{B}$-isometric to $(L^1(\Phi),|||\cdot|||)$ for some strictly positive Maharam operator Φ with the Levi property taking values in a Dedekind complete AM-space Λ with unit, where $\mathbb{B}=\mathbb{P}(\Lambda)\simeq\mathbb{M}(L^1(\Phi))$.

Corollary 2.6. *Let X be an injective Banach lattice, $\mathbb{B}=\mathbb{M}(X)$, and $\Lambda=\Lambda(\mathbb{B})$. Then Λ and $\mathscr{Z}_m(X)$ are f-algebra isomorphic. In particular, X can be endowed with the structure of lattice ordered Banach module over Λ.*

Kapansky–Hilbert Modules. Let X be a unitary $\bar{\Lambda}$-module.

Definition 2.7. The mapping $\langle\cdot\,|\,\cdot\rangle : X\times X\to\bar{\Lambda}$ is a $\bar{\Lambda}$-*valued inner product*, whenever for all $x,y,z\in X$ and $a\in\bar{\Lambda}$ the following are satisfied:
(1) $\langle x\,|\,x\rangle\geqslant 0$; $\langle x\,|\,x\rangle=0\iff x=0$;
(2) $\langle x\,|\,y\rangle=\langle y\,|\,x\rangle^*$;
(3) $\langle ax\,|\,y\rangle=a\langle x\,|\,y\rangle$;
(4) $\langle x+y\,|\,z\rangle=\langle x\,|\,z\rangle+\langle y\,|\,z\rangle$.
If X is complete with respect to the norm $\|x\|:=\sqrt{\|\langle x,x\rangle\|_\infty}$ ($x\in X$), it is called a C^*-*module* over $\bar{\Lambda}$.

Definition 2.8. A C^*-module X over $\bar{\Lambda}$ is a *Kaplansky–Hilbert module over* $\bar{\Lambda}=\bar{\Lambda}(\mathbb{B})$ if it enjoys the property: Given a norm-bounded family $(x_\xi)_{\xi\in\Xi}$ in X and a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$, there exists an element $x\in X$ such that $e_\xi x=e_\xi x_\xi$ for all $\xi\in\Xi$, see [28, Definition 7.4.5].

Consider a Kaplansky–Hilbert module X with a $\bar{\Lambda}$-valued inner product $\langle\cdot,\cdot\rangle$. The norm $\|x\|:=\sqrt{\|\langle x|x\rangle\|_\infty}$ ($x\in X$) and the Λ-valued norm $|x|:=\sqrt{\langle x|x\rangle}$ ($x\in X$) in X are related as $\|x\|=\big\||x|\big\|_\infty$ ($x\in X$). Moreover, two forms of the Cauchy–Bunyakovskiĭ inequality are fulfilled:

$$\langle x\,|\,y\rangle\leqslant|x|\cdot|y|,\quad \|\langle x\,|\,y\rangle\|_\infty\leqslant\|x\|\,\|y\|\quad (x,y\in X).$$

The following result due to M. Ozawa [43] (together with the other results from [42, 44]) tells us that the category of Kaplansky–Hilbert modules over $\bar{\Lambda}=\bar{\Lambda}(\mathbb{B})$ and bounded $\bar{\Lambda}$-linear operators is equivalent to the category of Hilbert spaces and bounded linear operators in $\mathrm{V}^{(\mathbb{B})}$. For a Banach space $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ the *descent* $\mathscr{X}{\downarrow}$ and the *bounded descent* $\mathscr{X}{\Downarrow}$ are defined as $\mathscr{X}{\downarrow}:=\big\{x\in\mathrm{V}^{(\mathbb{B})}:[\![x\in\mathscr{X}]\!]=\mathbb{1}\big\}$ and $\mathscr{X}{\Downarrow}:=\big\{x\in\mathscr{X}{\downarrow}:[\![\|x\|\leqslant C^\wedge]\!]=\mathbb{1}$ for some $C\in\mathbb{R}_+\big\}$. More details can be found in [28, Chapter 8].

Theorem 2.9. *The bounded descent of an arbitrary Hilbert space in $\mathrm{V}^{(\mathbb{B})}$ is a Kaplansky–Hilbert module over the Stone algebra $\bar{\Lambda}(\mathbb{B})$. Conversely, if X is a Kaplansky–Hilbert module over $\bar{\Lambda}(\mathbb{B})$, then there is a Hilbert space $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$ whose bounded descent $\mathscr{X}{\Downarrow}$ is unitarily equivalent with X. The space $\mathscr{X}$ is unique to within unitary equivalence inside $\mathrm{V}^{(\mathbb{B})}$.*

Remark 2.10. The concept of Kaplansky–Hilbert module was introduced by I. Kaplansky in [26] under the name AW^*-*module*. In the introduction he wrote:

"*… the new idea is to generalize Hilbert space by allowing the inner product to take values in a more general ring then the complex numbers. After the appropriate preliminary theory of these AW^*-modules has been developed, one can operate with a general AW^*-algebra of type I in almost the same manner as with the factor.*"

In other words, the central elements of an AW^*-algebra can be taken as complex numbers and one can work with with factors rather then with general AW^*-algebras.

Fremlin's tensor product of vector lattices. In his fundamental paper [15] Fremlin introduced for every two Archimedean vector lattices X and Y a new Archimedean vector lattice $X\bar{\otimes}Y$. In the sequel all vector lattices are assumed to be Archimedean.

Theorem 2.11. *Let X and Y be vector lattices. Then there exist a unique up to isomorphism vector lattice $X \bar{\otimes} Y$ and a bimorphism $\phi : X \times Y \to X \bar{\otimes} Y$ such that:*

(1) *whenever Z is a vector lattice and $\psi : X \times Y \to Z$ is a lattice bimorphism, there is a unique lattice homomorphism $T : X \bar{\otimes} Y \to Z$ with $T \circ \phi = \psi$;*

(2) *ϕ induces an embedding of the algebraic tensor product $X \otimes Y$ into $X \bar{\otimes} Y$;*

(3) *$X \otimes Y$ is dense in $X \bar{\otimes} Y$ in the sense that for every $v \in X \bar{\otimes} Y$ there exist $x_0 \in X$ and $y_0 \in Y$ such that for every $\varepsilon > 0$ there is an element $u \in X \otimes Y$ with $|v - u| \leqslant \varepsilon x_0 \otimes y_0$;*

(4) *if $0 < v \in X \bar{\otimes} Y$, then here exist $x \in X_+$ and $y \in Y_+$ with $0 < x \otimes y \leqslant v$.*

The lattice bimorphism ϕ is conventionally denoted by $\otimes$ and the algebraic tensor product $X \otimes Y$ is regarded as actually embedded into $X \bar{\otimes} Y$. If X_0 and Y_0 are vector sublattices in X and Y, respectively, then the tensor product $X_0 \bar{\otimes} Y_0$ is isomorphic to the vector sublattice in $X \bar{\otimes} Y$ generated by $X_0 \otimes Y_0$. Therefore, $X_0 \bar{\otimes} Y_0$ is regarded as a vector sublattice of $X \bar{\otimes} Y$, see [15, Corollaries 4.4 and 4.5]. D. Fremlin [15, Theorem 5.3] proved also the following important universal property of $X \bar{\otimes} Y$.

Theorem 2.12. *If Z is a relatively uniformly complete vector lattice, then for every positive bilinear operator $B : X \times Y \to Z$ there exists a unique positive linear operator $T : X \bar{\otimes} Y \to Z$ such that $B = T\otimes$.*

For Banach lattices X and Y Fremlin [16] defined the positive projective tensor norm $\|\cdot\|_{|\pi|}$ on $X \bar{\otimes} Y$ by putting

$$\|u\|_{|\pi|} := \inf\left\{ \sum_{k\leqslant n} \|x_k\|\|y_k\| : \ x_k \in X_+, \, y_k \in Y_+, \, k \leqslant n \in \mathbb{N}, \, |u| \leqslant \sum_{k\leqslant n} x_k \otimes y_k \right\}.$$

The *Fremlin projective tensor product* $X \hat{\otimes}_{|\pi|} Y$ ($\equiv \Lambda_1 \overset{\triangle}{\otimes} \Lambda_2$) of X and Y is the completion of $X \bar{\otimes} Y$ with respect to the norm $\|\cdot\|_{|\pi|}$.

Theorem 2.13. *For any Banach lattices X and Y the following hold:*

(1) *$X \hat{\otimes}_{|\pi|} Y$ is a Banach lattice and $\otimes : X \times Y \to X \hat{\otimes}_{|\pi|} Y$ is a lattice bimorphism.*

(2) *The positive cone $(X \hat{\otimes}_{|\pi|} Y)_+$ is the closure in $X \hat{\otimes}_{|\pi|} Y$ of the cone $\{x \otimes y : x \in X_+, \, y \in Y_+\}$.*

(3) *$\|x \otimes y\|_{|\pi|} = \|x\| \, \|y\|$ for all $x \in X$ and $y \in Y$.*

(4) *For any Banach lattice Z there is a one-to-one norm-preserving correspondence between continuous positive blinear operators $B : X \times Y \to Z$ and continuous positive operators $T : X \hat{\otimes}_{|\pi|} Y \to Z$ given by $B = T\otimes$.*

3. Homogeneous Functional Calculus

Everywhere below $\mathscr{X}$ is a Banach lattice in $\mathrm{V}^{(\mathbb{B})}$ and $X = \mathscr{X}{\Downarrow}$. In this section, we discuss the relation between homogeneous functional caculi in X and $\mathscr{X}$. To do this we need some auxiliary facts.

For a non-empty subset A of a vector lattice, denote by $\vee(A)$ (resp. $\wedge(A)$) the collection of all vectors that can be written as suprema (resp. infima) of finite subsets of A. Put $\wedge\vee(A) := \wedge(\vee(A))$ and $\vee\wedge(A) := \vee(\wedge(A))$. It always turns out that $\vee\wedge(A) = \wedge\vee(A)$. Denote by $\mathscr{P}_{\mathrm{fin}}(A)$ the set of all finite subsets of A.

Lemma 3.1. *For any nonempty set $A \subset X$ we have $[\![\wedge\vee(A{\uparrow}) = (\wedge\vee(A)){\uparrow}]\!] = \mathbb{1}$.*

$\lhd$ The relation $A \subset \wedge\vee(A)$ implies that $A{\uparrow} \subset \wedge\vee(A){\uparrow}$ and thus $\wedge\vee(A{\uparrow}) \subset \wedge\vee(A){\uparrow}$, since $\wedge\vee\,(A){\uparrow}$ is a sublattice. For the converse, take $u \in \wedge\vee\,(A)$ represented as $u = \bigwedge_{k\in n}\bigvee f(k)$ with $n \in \mathbb{N}$ and $f : n := \{0, 1, \dots, n-1\} \to \mathscr{P}_{\mathrm{fin}}(A)$. Making use of the relation $\mathscr{P}_{\mathrm{fin}}(A{\uparrow}) = \{\theta{\uparrow} : \theta \in \mathscr{P}_{\mathrm{fin}}(A)\}{\uparrow}$, define an internal function $g : n^{\wedge} \to \mathscr{P}_{\mathrm{fin}}(A{\uparrow})$ by $[\![g(k^{\wedge}) = f(k){\uparrow}]\!] = \mathbb{1}$ $(k \in n)$. It is easy to verify that $[\![u = \bigwedge_{k\in n^{\wedge}}\bigvee g(k)]\!] = \mathbb{1}$ and therefore $[\![u \in \wedge\vee(A{\uparrow})]\!] = \mathbb{1}$. Now, if $[\![x \in \wedge\vee(A){\uparrow}]\!] = \mathbb{1}$ then there exists a partition of unity (b_ξ) in $\mathbb{B}$ and a family (u_ξ) in $\wedge\vee\,(A)$ such that $b_\xi \leqslant [\![x = u_\xi]\!]$ for all ξ. Taking into account that u_ξ is in $\wedge\vee\,(A{\uparrow})$ we deduce $b_\xi \leqslant [\![x = u_\xi]\!] \wedge [\![u_\xi \in \wedge\vee\,(A{\uparrow})]\!] \leqslant [\![x \in \wedge\vee\,(A{\uparrow})\,]\!]$, whence $[\![x \in \wedge\vee\,(A{\uparrow})]\!] = \mathbb{1}$. $\rhd$

Denote by $\mathscr{H}(\mathbb{R}^N)$ the vector lattice of all continuous functions $\varphi : \mathbb{R}^N \to \mathbb{R}$ which are *positively homogeneous* ($\equiv \varphi(\lambda t) = \lambda\varphi(t)$ for $\lambda \geqslant 0$ and $t \in \mathbb{R}^N$). If $S := \{t = (t_1, \dots, t_N) \in \mathbb{R}^N : |t_1| + \dots + |t_N| = 1\}$, then the map $\varphi \mapsto \varphi|_S$ is a lattice isomorphism from $\mathscr{H}(\mathbb{R}^N)$ onto $C(S)$, the Banach lattice of continuous functions on S. Thus, $\mathscr{H}(\mathbb{R}^N)$ can be also considered as a Banach lattice with the induced order unit norm.

Observe, that $(\mathbb{R}^N)^{\wedge} = (\mathbb{R}^{\wedge})^{N^{\wedge}}$. If $\varphi \in \mathscr{H}(\mathbb{R}^N)$ then $[\![\,\varphi^{\wedge} : (\mathbb{R}^N)^{\wedge} \to \mathbb{R}^{\wedge}$ is a continuous function$\,]\!] = \mathbb{1}$ and $[\![$there exists a unique continuous function $\tilde{\varphi} \in \mathscr{H}(\mathscr{R}^{N^{\wedge}})$ such that $\tilde{\varphi}|_{(\mathbb{R}^N)^{\wedge}} = \varphi^{\wedge}\,]\!] = \mathbb{1}$, see [21, Lemma 16]. Evidently, the map $\tau : \varphi \mapsto \tilde{\varphi}$ is a lattice isomorphism from $\mathscr{H}(\mathbb{R}^N)$ into $\mathscr{H}(\mathscr{R}^{N^{\wedge}}){\downarrow}$. Let e_k stands for the kth coordinate function on $\mathbb{R}^N$, i.e. $e_k : (t_1, \dots, t_N) \mapsto t_k$. Clearly, $\tilde{e_k}$ is a kth coordinate function on $\mathscr{R}^{N^{\wedge}}$.

Lemma 3.2. *The following holds inside* $\mathrm{V}^{(\mathbb{B})}$*: the Banach lattice* $\mathscr{H}(\mathscr{R}^{N^{\wedge}})$ *is lattice isomorphic to the completion of the* $\mathscr{R}$*-normed vector lattice* $\mathscr{H}(\mathbb{R}^N)^{\wedge}$ *over* $\mathbb{R}^{\wedge}$.

$\lhd$ Recall that if $\mathbb{Q}$ is the field of rationals then $\mathbb{Q}^{\wedge}$ is the field of rationals in $\mathrm{V}^{(\mathbb{B})}$. Denote by $\mathbb{Q}(e_1, \dots, e_N)$ and $\mathbb{Q}\langle e_1, \dots, e_N\rangle$ the $\mathbb{Q}$-linear subspace and $\mathbb{Q}$-linear sublattice generated by $\{e_1, \dots, e_N\}$. Let $\mathbb{Q}^{\wedge}(\tilde{e}_1, \dots, \tilde{e}_N)$ and $\mathbb{Q}^{\wedge}\langle\tilde{e}_1, \dots, \tilde{e}_N\rangle$ be the corresponding internal objects in $\mathrm{V}^{(\mathbb{B})}$. If $A := \tau(\mathbb{Q}(e_1, \dots, e_N))$ and $B := \tau(\mathbb{Q}\langle e_1, \dots, e_N\rangle)$ then $B = \wedge\vee\,(A)$ (see [1, Lemma 5.63]) and $B{\uparrow} = \wedge\vee\,(A{\uparrow})$ by Lemma 3.1. Moreover, $A{\uparrow} = \mathbb{Q}^{\wedge}(\tilde{e}_1, \dots, \tilde{e}_N)$, since $\mathbb{Q}(e_1, \dots, e_N)$ is defined by a restricted formula. The last two observations imply $B{\uparrow} = \mathbb{Q}^{\wedge}\langle\tilde{e}_1, \dots, \tilde{e}_N\rangle$. It remains to note that $B{\uparrow}$ is uniformly dense in $\mathscr{H}(\mathscr{R}^{N^{\wedge}})$ and is lattice isometric to the sublattice $\mathbb{Q}\langle e_1, \dots, e_N\rangle^{\wedge}$ which is uniformly dense in $\mathscr{H}(\mathbb{R}^N)^{\wedge}$. $\rhd$

Theorem 3.3. *Let* $x_1, \dots, x_N \in X$, $\mathbf{x} := (x_1, \dots, x_N)$, *and* $\mathfrak{x}$ *be an element of* $\mathrm{V}^{(\mathbb{B})}$ *with* $[\![\,\mathfrak{x} = (x_1, \dots, x_{N^{\wedge}})^{\mathbb{B}}\,]\!] = \mathbb{1}$. *If* $\widehat{\mathbf{x}} : \mathscr{H}(\mathbb{R}^N) \to X$ *and* $\widehat{\mathfrak{x}} : \mathscr{H}(\mathscr{R}^{N^{\wedge}}) \to \mathscr{X}$ *are homogeneous functional calculi in* X *and* $\mathscr{X}$, *respectively, then* $\widehat{\mathfrak{x}}{\downarrow} \circ \tau = \widehat{\mathbf{x}}$.

$\lhd$ By Lemma 3.2 $\widehat{\mathfrak{x}}$ is a unique continuous extension of the map from $\tau\big(\mathscr{H}(\mathbb{R}^N)^{\wedge}\big)$ into $\mathscr{X}$ defined by $\widehat{\mathfrak{x}} : \tilde{\varphi} \mapsto \widehat{\mathbf{x}}(\varphi)$. Therefore, $[\![\,\widehat{\mathfrak{x}}(\tilde{\varphi}) = \widehat{\mathbf{x}}(\varphi)\,]\!] = \mathbb{1}$ for all $\varphi \in \mathscr{H}(\mathbb{R}^N)$. $\rhd$

Definition 3.4. A map $\varphi : \mathbb{R}^N \to \Lambda$ is said to be *ro-continuous* at $t \in \mathbb{R}^N$ if

$$\inf_{n\in\mathbb{N}} \sup\big\{|\varphi(s) - \varphi(t)| : \ s \in \mathbb{R}^N,\ \|s - t\| \leqslant 1/n\big\} = 0.$$

This is equivalent to saying that for every $0 < \varepsilon \in \mathbb{R}$ there exists a countable partition of unity (π_k) in $\mathbb{B}$ such that $\pi_k|\varphi(s) - \varphi(t)| \leqslant \varepsilon\mathbb{1}$ for all $k \in \mathbb{N}$ and $s \in \mathbb{R}^N$ with $\|s - t\| \leqslant 1/k$. Now, φ is called *ro-continuous* if it is *ro*-continuous at every $t \in \mathbb{R}^N$. Write $\mathscr{H}_{ro}(\mathbb{R}^N, \Lambda)$ for the set of all *ro*-continuous functions from $\mathbb{R}^N$ to Λ which are positively homogeneous ($\equiv \varphi(\lambda t) = \lambda\varphi(t)$ for $0 \leqslant \lambda \in \mathbb{R}$ and $t \in \mathbb{R}^N$). For any $\varphi \in \mathscr{H}(\mathbb{R}^N)$ and $\lambda \in \Lambda$ define $\varphi \otimes \lambda \in \mathscr{H}_{ro}(\mathbb{R}^N, \Lambda)$ by $\varphi \otimes \lambda : s \mapsto \varphi(s)\lambda$ $(s \in S)$.

Clearly, $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$ is a vector lattice and an f-algebra with unit $1_S\otimes\mathbb{1}$. Equip $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$ with a mixed norm by putting $|\varphi|:=\sup_{s\in S}|\varphi(s)|$ and $|\!|\!|\varphi|\!|\!|:=\||\varphi|\|_\infty$.

Lemma 3.5. *The map τ is a lattice $\mathbb{B}$-isometry from $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$ onto $\mathscr{H}(\mathscr{R}^{N^\wedge})\Downarrow$. In particular, $(\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda),|\!|\!|\cdot|\!|\!|)$ is a $\mathbb{B}$-cyclic Banach lattice.*

$\lhd$ By Lemma 3.2 and [32, Proposition 7.2] the $\mathbb{B}$-cyclic Banach lattices $\mathscr{H}(\mathscr{R}^{N^\wedge})\Downarrow$ and $C_\#(Q,\mathscr{H}(\mathbb{R}^N))$ are lattice $\mathbb{B}$-isometric, whenever Q is the Stone space of $\mathbb{B}$ and Λ is identified with $C(Q)$. Thus, we have only to show that $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$ and $C_\#(Q,\mathscr{H}(\mathbb{R}^N))$ are lattice $\mathbb{B}$-isometric. This is equivalent to $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$ and $C_\#(Q,\mathscr{H}(\mathbb{R}^N))$ being lattice isometric relative to Λ-valued norms. The algebraic tensor product $\mathscr{H}(\mathbb{R}^N)\otimes\Lambda$ can be identified with a dense subspace both in $C_\#(Q,\mathscr{H}(\mathbb{R}^N))$ and $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$. Moreover, the two induced Λ-valued norms on $\mathscr{H}(\mathbb{R}^N)\otimes\Lambda$ coincide. $\rhd$

For a subset M of a $\mathbb{B}$-cyclic Banach lattice X define $\mathbb{B}\langle M\rangle$ as the set of all $x\in X$ such that there are a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(x_\xi)_{\xi\in\Xi}$ in M with $\pi_\xi x=\pi_\xi x_\xi$ for all $\xi\in\Xi$. Denote $\mathbb{B}\langle x_1,\dots,x_N\rangle:=\mathbb{B}\langle\mathrm{lin}(x_1,\dots,x_N)\rangle$, where $\mathrm{lin}(x_1,\dots,x_N)$ is the linear hull of $\{x_1,\dots,x_N\}$.

Corollary 3.6. *Let X be a $\mathbb{B}$-cyclic Banach lattice. For any $\mathbf{x}:=(x_1,\dots,x_N)\in X^N$ there exists a unique lattice $\mathbb{B}$-homomorphism (denoted by the same symbol $\widehat{\mathbf{x}}$)*

$$\widehat{\mathbf{x}}:\varphi\mapsto\widehat{\mathbf{x}}(\varphi):=\widehat{\varphi}(x_1,\dots,x_N)\quad\big(\varphi\in\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)\big)$$

of $\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)$ into X with $\widehat{\mathbf{x}}(dt_k\otimes\lambda)=\lambda x_k$ for all $k:=1,\dots,N$ and $\lambda\in\Lambda$. Moreover, $\widehat{\mathbf{x}}\big(\mathscr{H}_{\tau o}(\mathbb{R}^N,\Lambda)\big)$ equals the norm closure of $\mathbb{B}\langle x_1,\dots,x_N\rangle$.

$\lhd$ By the Boolean-valued transfer principle and maximum principle there is a unique lattice homomorphism $\widehat{\mathfrak{x}}$ inside $\mathrm{V}^{(\mathbb{B})}$ from $\mathscr{H}(\mathscr{R}^{N^\wedge})$ to $\mathscr{X}^{N^\wedge}$ with $\widehat{\mathfrak{x}}(\tilde{e}_k)=x_k$ $(k:=1,\dots,N)$, see [10] and [36]. Therefore, the required statement is immediate from Theorem 3.3 and Lemma 3.5. $\rhd$

Definition 3.7. For a positive invertible $\alpha\in\Lambda$ and arbitrary $s,t\in\mathbb{R}$ we denote $t^\alpha:=\mathrm{sgn}(t)|t|^\alpha$ and $\sigma_\alpha(s,t):=(s^{1/\alpha}+t^{1/\alpha})^\alpha$, where $1/\alpha:=\alpha^{-1}$. Clearly, $t^\alpha\in\Lambda$ and $\sigma_\alpha\in\mathscr{H}_o(\mathbb{R}^2,\Lambda)$, so that $\sigma_\alpha(x,y)$ is well defined in X for every $x,y\in X$ by Corollary 3.6. In a vector lattice X, we introduce new vector operations $\oplus$ and $*$, while the original ordering $\leqslant$ remain unchanged:

$$x\oplus y:=\sigma_\alpha(x,y):=(x^{1/\alpha}+y^{1/\alpha})^\alpha,\quad t*x:=t^\alpha x\quad(x,y\in X;\ t\in\mathbb{R}).$$

Then $X^{(\alpha)}:=(X,\oplus,*,\leqslant)$ is again a vector lattice called an *α-convexification* of X. Note, that $1/\alpha$-covexification is also called an *α-concavification*, cf. [36, pp. 53, 54].

Corollary 3.8. *Let α be a positive invertible element in $\Lambda=\mathscr{R}\Downarrow$. Then for a Banach lattice $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ we have*

$$(\mathscr{X}^{(\alpha)})\Downarrow=(\mathscr{X}\Downarrow)^{(\alpha)}.$$

$\lhd$ Denote $\mathbf{x}=(x_1,x_2)$, $\mathfrak{x}:=(x_1,x_2)^{\mathbb{B}}$ and observe that $(x_1,x_2)\mapsto\widehat{\mathbf{x}}(\sigma_\alpha)$ and $(x_1,x_2)\mapsto\widehat{\mathfrak{x}}(\sigma_\alpha)$ are the operations of addition in $X:=\mathscr{X}\Downarrow$ and $\mathscr{X}$, respectively. The addition in $X^{(\alpha)}$ is the bounded descent of the addition in $\mathscr{X}^{(\alpha)}$, since $\widehat{\mathfrak{x}}{\downarrow}(\tilde{\sigma_\alpha})=\widehat{\mathbf{x}}(\sigma_\alpha)$ by Theorem 3.3. Similar assertion about multiplication is evident. $\rhd$

Define also a homogeneous function $\|\cdot\|_{\langle\alpha\rangle}:X^{(\alpha)}\to\mathbb{R}$ by

$$\|x\|_{\langle\alpha\rangle}:=\inf\{0<\lambda\in\mathbb{R}:\ \|\lambda^{-\alpha}x\|\leqslant 1\}\quad(x\in X).\tag{1}$$

Evidently, if $\alpha=\nu\mathbb{1}$, then $\|x\|_{\langle\alpha\rangle}:=\|x\|^{1/\nu}$.

Theorem 3.9. *Let $(X, |||\cdot|||)$ be a $\mathbb{B}$-cyclic Banach lattice and $\mathbb{1} \leqslant \alpha \in \Lambda$. Then $(X^{(\alpha)}, |||\cdot|||_{\langle\alpha\rangle})$ is also a $\mathbb{B}$-cyclic Banach lattice and there exists a (non-linear) order isomorphism $\iota_\alpha : X \to X^{(\alpha)}$ such that*

$$\iota_\alpha(\lambda^\alpha x) = \lambda * \iota_\alpha(x), \quad \iota_\alpha((x^{1/\alpha} + y^{1/\alpha})^\alpha) = \iota_\alpha(x) \oplus \iota_\alpha(y)$$

for all $x, y \in X$ and $\lambda \in \mathbb{R}$. Moreover, $|||x||| = 1$ if and only if $|||\iota_\alpha(x)|||_{\langle\alpha\rangle} = 1$.

$\lhd$ We can assume without loss of generality that $X = \mathscr{X}{\Downarrow}$ and $|||\cdot||| = |||{\lvert\cdot\rvert}|||_\infty$, where $(\mathscr{X}, \|\cdot\|)$ is a Banach lattice in $\mathrm{V}^{(\mathbb{B})}$ and $\lvert\cdot\rvert$ is the descent of $\|\cdot\|$. Then $X^{(\alpha)} = (\mathscr{X}^{(\alpha)}){\Downarrow}$ by Corollary 3.8. Write $\|\cdot\|_{(\alpha)}$ and $|||\cdot|||_{(\alpha)}$ for the norms in $\mathscr{X}^{(\alpha)}$ (internal) and the mixed norm in $(\mathscr{X}^{(\alpha)}){\Downarrow}$ (external), respectively. Then, taking into account that α is an internal positive real, we have $[\![\|x\|_{(\alpha)} = \|x\|^{1/\alpha}\,(x \in \mathscr{X})]\!] = \mathbb{1}$ and hence $|||x|||_{(\alpha)} = \big\|\lvert x\rvert^{1/\alpha}\big\|_\infty$ for all $x \in X$. It follows that $|||x|||_{(\alpha)} \leqslant 1$ if and only if $\lvert x\rvert^{1/\alpha} \leqslant \mathbb{1}$ if and only if $\lvert x\rvert \leqslant \mathbb{1}$. At the same time, according to (1), $|||x|||_{\langle\alpha\rangle} \leqslant 1$ if and only if for every $\varepsilon > 0$ there is $\lambda_\varepsilon \leqslant (1+\varepsilon)$ with $|||\lambda_\varepsilon^{-\alpha} x||| \leqslant 1$. Because $|||\lambda^{-\alpha}x||| = |||\lvert\lambda^{-\alpha}x\rvert|||_\infty = \|\lambda^{-\alpha}\lvert x\rvert\|_\infty$, we conclude that $|||\lambda_\varepsilon^{-\alpha}x||| \leqslant 1$ if and only if $\lvert x\rvert \leqslant \lambda_\varepsilon^\alpha$. Now, it is clear that if $|||x|||_{\langle\alpha\rangle} \leqslant 1$, then $\lvert x\rvert \leqslant \lambda_\varepsilon^\alpha \leqslant (1+\varepsilon)^\alpha \leqslant \mathbb{1} + \varepsilon\alpha$ for any $\varepsilon > 0$, so that $\lvert x\rvert \leqslant \mathbb{1}$.

We have proved thus that the bounded descent of $(\mathscr{X}^{(\alpha)}, \|\cdot\|_{(\alpha)})$ is lattice $\mathbb{B}$-isometric to $(X^{(\alpha)}, |||\cdot|||_{\langle\alpha\rangle})$. Because $(\mathscr{X}^{(\alpha)}, \|\cdot\|_{(\alpha)})$ is a Banach lattice inside $\mathrm{V}^{(\mathbb{B})}$, we conclude that $(X^{(\alpha)}, |||\cdot|||_{\langle\alpha\rangle})$ is a $\mathbb{B}$-cyclic Banach lattice. Since the underlying sets of X and $X^{(\alpha)}$ coincide, we can define ι_α as the identity map. The required properties of ι_α are obvious. $\rhd$

Remark 3.10. In various situations it is desirable to have an extended version of the homogeneous functional calculus. Apart from the standard homogeneous functional calculus [10, 36, 48] there are at least four instances: 1) A generalized functional calculus on a Banach lattice with the center $\mathscr{Z}(X)$ was developed in [24] for $\mathscr{H}(\mathbb{R}^N, \mathscr{Z}(X))$, the f-algebra of positively homogeneous continuous $\mathscr{Z}(X)$-valued functions on $\mathbb{R}^N$. 2) Corollary 3.6 above treats the case of $\mathscr{H}_{\tau o}(\mathbb{R}^N, \Lambda)$ with a special f-subalgebra $\Lambda \subset \mathscr{Z}(X)$. 3) It is shown in [29, 30] that the function of elements of a relatively uniformly complete vector lattice can naturally be defined if the continuous positively homogeneous function is defined on some conic set in $\mathbb{R}^N$. 4) In [35] the homogeneous functional calculus from 3) is extended so that it becomes the continuous functional calculus. A unified treatment of these different approaches is desirable.

4. Kaplansky–Hilbert Lattices

In this section we show that injective Banach lattices and Kaplansky–Hilbert lattices are closely related and one can be transformed into another by means of the procedures of 2-convexification and 1/2-concavification. This surprising fact is almost trivial inside an appropriate Boolean valued model.

Definition 4.1. A real Banach lattice X is said to be a *Kaplansky–Hilbert lattice* over Λ whenever $X \oplus iX$ is a Kaplansky–Hilbert module over $\bar{\Lambda}$ with respect to the norm $\|x+iy\| := \sqrt{\|x\|^2 + \|y\|^2}$ $(x, y \in X)$. A Kaplansky–Hilbert lattice over $\Lambda = \mathbb{R}$ is called a *Hilbert lattice,* see [40].

It can be easily seen that a real Banach lattice X is a Kaplansky–Hilbert lattice if and only if the following hold:

(1) X is a unitary Λ-module;

(2) There is a Λ-valued inner product $\langle\cdot\,|\,\cdot\rangle$ (i. e. $\langle\cdot\,|\,\cdot\rangle$ satisfy 2.6 (1–4) with $\lambda=\lambda^*$ for all $\lambda\in\Lambda$) in X such that $\|x\|:=\sqrt{\|\langle x\,|\,x\rangle\|_\infty}$ $(x\in X)$;

(3) Given a norm-bounded family $(x_\xi)_{\xi\in\Xi}$ in X and a partition of unity $(e_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$, there exists an element $x\in X$ such that $e_\xi x=e_\xi x_\xi$ for all $\xi\in\Xi$.

Definition 4.2. We need one more useful concept introduced by G. Buskes and A. van Rooij [12], see also [9]. Let X be a vector lattice. The pair $(X^\odot,\odot)$ is called a *square* of E if the following conditions are fulfilled:

(1) $X^\odot$ is a vector lattice;

(2) $\odot: X\times X\to X^\odot$ is a symmetric lattice bimorphism;

(3) for any vector lattice Y and every symmetric lattice bimorphism $\varphi: X{\times}X\to Y$ there exists a unique lattice homomorphism $S: X^\odot\to Y$ such that $S\circ\odot=\varphi$.

Theorem 4.3. *An arbitrary Archimedean vector lattice X has a unique* (*up to a lattice isomorphism*) *square $(X^\odot,\odot)$. If X is uniformly complete then $X^\odot=X^{(1/2)}$ and $x\odot y:=(xy)^{1/2}$ for all $x,y\in X$. If X is a q-convex Banach lattice for some $q\geqslant 2$, then $X^\odot$ equipped with the norm $\|x\odot x\|^\odot:=\|x\odot x\|_{(1/2)}:=\|x\|^2$ is also a Banach lattice.*

$\lhd$ The existence of $X^\odot$ was established in [12]. For the identity $X^\odot=X^{(1/2)}$ see [48, Proposition 4.8 (ii)]. The last statement can be found in [36, p. 53]. $\rhd$

Lemma 4.4. *Let $\mathscr{X}^\odot$ stand for the square of $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$. Then*

$$\big(\mathscr{X}^\odot\big)\Downarrow=\big(\mathscr{X}\Downarrow\big)^\odot.$$

$\lhd$ Apply Corollary 3.8 with $\alpha:=1/2$. $\rhd$

Theorem 4.5. *Let X be a Banach lattice and Λ a Dedekind complete AM-space with unit. Then X is a Kaplansky–Hilbert lattice over Λ if and only if the square $X^\odot$ is an injective Banach lattice with $\Lambda\simeq\mathscr{Z}_m(X^\odot)$. In this case the map $\iota: x\mapsto x\odot|x|$ is an isometric order isomorphism from X onto $X^\odot$.*

$\lhd$ Assume that X is a Kaplansky–Hilbert lattice over Λ. By Theorem 2.8 there is a real Hilbert space $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$ such that X and $\mathscr{X}\Downarrow$ are unitary equivalent real Kaplansky–Hilbert modules over Λ. In view of [32, Theorem 4.1] $\mathscr{X}$ is also a Banach lattice inside $\mathrm{V}^{(\mathbb{B})}$. Thus, from [40, Corollary 2.7.5] we deduce $[\![\mathscr{X}$ is lattice isometric to $L^2(\mu)$ for some measure space $(\Omega,\Sigma,\mu)]\!]=\mathbb{1}$. Taking into consideration the relation $(L^2(\mu))^\odot=L^1(\mu)$ we conclude that $[\![\mathscr{X}^\odot$ is lattice isometric to $L^1(\mu)]\!]=\mathbb{1}$. By Theorem 2.4 $\mathscr{X}^\odot\Downarrow$ is an injective Banach lattice with $\Lambda\simeq\mathscr{Z}_m(X^\odot)$ and it remains to apply Lemma 4.4.

Conversely, suppose that $X^\odot$ is an injective Banach lattice Then, in view of Theorem 2.4, $X^\odot$ is lattice $\mathbb{B}$-isometric to $\mathscr{Y}\Downarrow$ for some $\mathscr{Y}\in\mathrm{V}^{(\mathbb{B})}$ with $[\![\mathscr{Y}=L^1(\mu)=(L^2(\mu))^\odot]\!]=\mathbb{1}$. Using again Lemma 4.4 we deduce $X=(X^\odot)^{(2)}\simeq_{\mathbb{B}}(\mathscr{Y}\Downarrow)^{(2)}=((L^2(\mu))^\odot\Downarrow)^{(2)}\simeq_{\mathbb{B}}L^2(\mu)\Downarrow$. By Theorem 2.8 and [32, Theorem 4.1] X is a Kaplansky–Hilbert lattice over Λ. $\rhd$

Remark 4.6. Theorem 4.5 says that Kaplansky–Hilbert lattices and injective Banach lattices are related as L^2 and L^1. Ivanov [25] proved that if $X:=L^2([0,1])$ (and hence $X^\odot=L^1([0,1])$), then the bijection $\iota: x\mapsto x\odot|x|$ is also a (non-linear) homeomorphism.

Definition 4.7. Denote by $\surd$ the inverse of ι, i. e. $\surd(x\odot|x|)=x$ and $\surd(y)\odot|\surd(y)|=y$ for all $x\in X$ and $y\in X^\odot$. Then $\|y\|^\odot=\|\surd(y)\|^2$ and

$\|x\| = \sqrt{\|x \odot |x|\|^{\odot}}$. The maps ι and $\surd$ were named in [25] the *alternating square* and the *alternating square root,* respectively.

Definition 4.8. Let X and Y be vector lattices. A positive operator $T : X \to Y$ (resp. a positive bilinear operator $T : X \times X \to Y$) is said to have the *Levi property* if $\mathrm{Im}(T)^{\perp} = \{0\}$ and $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that (Tx_α) (resp. $(T(x_\alpha, x_\alpha))$) is order bounded in Y, see [32, Definition 2.5].

Definition 4.9. A positive bilinear operator B from $X \times X$ into Y is said to be *orthosymmetric* if $|x| \wedge |y| = 0$ implies $B(x, y) = 0$ for arbitrary $x, y \in X$, see [11]. Say that B possesses the *Maharam property* if $B([0, x] \times [0, y]) = [0, B(x, y)]$ for all $x, y \in X_+$. A positive order continuous bilinear operator with the Maharam property is called a *bilinear Maharam operator*, see [31].

Corollary 4.10. *A Banach lattice X is injective if and only if its M-center $\mathscr{Z}_m(X)$ is Dedekind complete and $X^{(2)}$ is a real Kaplansky–Hilbert lattice over $\mathscr{Z}_m(X)$.*

$\lhd$ By Theorem 4.5 $X^{(2)}$ is a Kaplansky–Hilbert lattice over $\mathscr{Z}_m(X)$ if and only if $(X^{(2)})^{\odot}$ is an injective Banach lattice. It remains to observe that $(X^{(2)})^{\odot} = (X^{(2)})^{(1/2)} = X^{(1)} = X$. $\rhd$

Corollary 4.11. *A Banach lattice X with the Dedekind complete M-center $\Lambda = \mathscr{Z}_m(X)$ is a Kaplansky–Hilbert lattice over Λ if and only if there exists a linear Maharam operator $\Phi : X^{\odot} \to \Lambda$ with the Levi property such that $\|x\| = \sqrt{\|\Phi(x \odot x)\|_\infty}$ $(x \in X)$.*

$\lhd$ This is immediate from Theorem 4.5 and [32, Theorem 5.1 (3)]. A standard proof (i. e. without involving $\mathrm{V}^{(\mathbb{B})}$) can be given by using [9, Theorem 3.1] and [31, Proposition 4.4]. $\rhd$

Corollary 4.12. *A Banach lattice X is injective if and only if $\Lambda = \mathscr{Z}_m(X)$ is Dedekind complete and there exists a bilinear orthosymmetric Maharam operator $\langle \cdot, \cdot \rangle : X^{(2)} \times X^{(2)} \to \Lambda$ with the Levi property such that $\|x\| = \|\langle \surd(x), \surd(x) \rangle\|_\infty^2$ $(x \in X)$.*

$\lhd$ Apply Corollary 4.9 with $X := X^{(2)}$, observe that the bilinear operator $\langle x, y \rangle := \Phi(x \odot y)$ is Maharam if and only if so is the linear operator $\Phi : X = (X^{(2)})^{\odot} \to \Lambda$ (see [31, Proposition 4.4]), and take into account [32, Theorem 5.1] with the identity $\langle \surd(x), \surd(x) \rangle = \Phi(|x|)$. $\rhd$

5. The Three Basic Lemmas

This section contains three relatively simple lemmas which are basic for the understanding of the interplay between the vector norm (with values in the M-center) and the mixed (scalar) norm of an injective Banach lattice.

Denote by $\mathrm{Prt}_\sigma := \mathrm{Prt}_\sigma(\mathbb{B})$ and $\mathscr{P}_{\mathrm{fin}}(X)$ the set of all countable partitions of unity in $\mathbb{B}$ and the collection of all finite subsets of X, respectively. Let X be a $\mathbb{B}$-cyclic Banach lattice. Recall that $\|x\| = \big\| |x| \big\|_\infty$ $(x \in X)$, where $(X, |\cdot|)$ is a Banach–Kantorovich space with Λ-valued norm, $\Lambda = \Lambda(\mathbb{B}) = \mathscr{Z}_m(X)$.

Lemma 5.1. *Let $1 \leqslant p \leqslant \infty$. For any finite collection $x_1, \dots, x_N \in X$ we have*

$$\bigg\| \bigg(\sum_{i=1}^{N} |x_i|^p \bigg)^{\frac{1}{p}} \bigg\|_\infty = \inf_{(\pi_k) \in \mathrm{Prt}_\sigma} \sup_{k \in \mathbb{N}} \bigg(\sum_{i=1}^{N} \|\pi_k x_i\|^p \bigg)^{\frac{1}{p}}. \tag{2}$$

If $1 \leqslant p < \infty$ *then the following equality is also true:*

$$\Big\| \sum_{i=1}^{N} |x_i|^p \Big\|_\infty = \inf_{(\pi_k)\in \mathrm{Prt}_\sigma} \sup_{k\in\mathbb{N}} \sum_{i=1}^{N} \|\pi_k x_i\|^p. \tag{3}$$

$\lhd$ We restrict to the first equality. The second equality can be obtained by raising the first one to the power p. Observe that in the case $p = \infty$ the needed equation is immediate from the Fatou property of the norm $\|\cdot\|_\infty$:

$$\begin{aligned}\sup_{k\in\mathbb{N}} \Big(\sum_{n=1}^{N} \|\pi_k x_n\|^p \Big)^{\frac{1}{p}} &= \sup_{k\in\mathbb{N}} \big\| |\pi_k x_1| \big\|_\infty \vee \dots \vee \big\| |\pi_k x_1| \big\|_\infty \\ &= \sup_{k\in\mathbb{N}} \big\| \pi_k(|x_1| \vee \dots \vee |x_1|) \big\|_\infty = \Big\| \bigvee_{k\in\mathbb{N}} \pi_k(|x_1| \vee \dots \vee |x_1|) \Big\|_\infty \\ &= \big\| |x_1| \vee \dots \vee |x_N| \big\|_\infty = \Big\| \Big(\sum_{i=1}^{N} |x_i|^p \Big)^{\frac{1}{p}} \Big\|_\infty .\end{aligned}$$

Assume now that $p < \infty$. For a fixed finite collection $x_1, \dots, x_N \in X$ and for an arbitrary countable partition of unity (π_k) in $\mathbb{B}$ we deduce

$$\begin{aligned}\Big(\sum_{n=1}^{N} |x_n|^p \Big)^{\frac{1}{p}} &= \bigvee_{k=1}^{\infty} \Big(\sum_{n=1}^{N} |\pi_k x_n|^p \Big)^{\frac{1}{p}} \\ &= \bigvee_{k=1}^{\infty} \pi_k(\mathbb{1}) \Big(\sum_{n=1}^{N} \|\pi_k x_n\|^p \Big)^{\frac{1}{p}} \leqslant \mathbb{1} \cdot \sup_{k\in\mathbb{N}} \Big(\sum_{n=1}^{N} \|\pi_k x_n\|^p \Big)^{\frac{1}{p}},\end{aligned}$$

whence the inequality $\leqslant$ holds. To prove the reverse inequality, take an arbitrary $0 < \varepsilon \in \mathbb{R}$ and choose a countable partition of unity (π_k) such that

$$\|\pi_k x_i\| \pi_k \mathbb{1} \leqslant \pi_k(|x_i| + \varepsilon\mathbb{1}) \quad (k \in \mathbb{N},\ n := 1, \dots, N). \tag{4}$$

Forming p-sums over $n \leqslant N$ and taking the supremum over $k \in \mathbb{N}$ in (4) we obtain

$$\begin{aligned}\sup_{k\in\mathbb{N}} \Big(\sum_{i=1}^{N} \|\pi_k x_i\|^p \Big)^{\frac{1}{p}} &= \Big\| \bigvee_{k=1}^{\infty} \Big(\sum_{i=1}^{N} \|\pi_k x_i\|^p \pi_k \mathbb{1} \Big)^{\frac{1}{p}} \Big\|_\infty \\ &\leqslant \sup_{k\in\mathbb{N}} \Big\| \Big(\sum_{i=1}^{N} \pi_k(|x_i| + \varepsilon\mathbb{1})^p \Big)^{\frac{1}{p}} \Big\|_\infty \\ &\leqslant \Big\| \Big(\sum_{i=1}^{N} |x_i|^p \Big)^{\frac{1}{p}} \Big\|_\infty + N^{\frac{1}{p}} \varepsilon.\end{aligned}$$

Since $0 < \varepsilon$ is arbitrary, we arrive at the required equality. $\rhd$

Lemma 5.2. *Given* $1 \leqslant p < \infty$ *and* $x \in X$, *denote*

$$A(x) := \inf \Big\{ \sum_{i=1}^{n} |u_i|^p : \ |x| \leqslant \Big(\sum_{i=1}^{n} u_i^p \Big)^{\frac{1}{p}},\ \{u_1, \dots, u_n\} \subset X_+,\ n \in \mathbb{N} \Big\}.$$

The following representation holds:

$$\|A(x)\|_\infty = \inf \left\{ \sup_{k\in\mathbb{N}} \sum_{i=1}^{n_k} \|\pi_k u_{i,k}\|^p : \ |x| \leqslant \left(\sum_{i=1}^{n_k} u_{i,k}^p \right)^{\frac{1}{p}}, \right.$$
$$\left. u_{i,k} \in X_+, \ i := 1, \dots, n_k \in \mathbb{N}, \ k \in \mathbb{N}, \ (\pi_k) \in \mathrm{Prt}_\sigma(\mathbb{B}) \right\}. \quad (5)$$

$\lhd$ Denote by $|||x|||$ the right-hand side of (5). For an arbitrary countable partition of unity (π_k) in $\mathbb{B}$ and for a sequence of finite collection $\{u_{1,k} \dots, u_{n_k,k}\}$ with $|x| \leqslant \left(\sum_{i=1}^{n_k} u_{i,k}^p\right)^{\frac{1}{p}}$ $(k \in \mathbb{N})$ in X we have $A(x) \leqslant \bigvee_{k=1}^{\infty} \sum_{i=1}^{n_k} \pi_k |v_{i,k}|^p$. Taking norms and making use of Lemma 5.1 we deduce

$$\|A(x)\|_\infty \leqslant \sup_{k\in\mathbb{N}} \left\| \sum_{i=1}^{n_k} |\pi_k v_{i,k}|^p \right\|_\infty = \sup_{k\in\mathbb{N}} \inf_{(\rho_j)\in\mathrm{Prt}_\sigma} \sup_{j\in\mathbb{N}} \sum_{i=1}^{n_k} \|\rho_j \pi_k v_{i,k}\|^p$$
$$\leqslant \inf_{(\rho_j)\in\mathrm{Prt}_\sigma} \sup_{j,k\in\mathbb{N}} \sum_{i=1}^{n_k} \|\rho_j \pi_k v_{i,k}\|^p \leqslant \sup_{j,k\in\mathbb{N}} \sum_{i=1}^{n_k} \|\pi_j \pi_k v_{i,k}\|^p = \sup_{k\in\mathbb{N}} \sum_{i=1}^{n_k} \|\pi_k v_{i,k}\|^p.$$

Taking the infimum on the right over all (π_k) and $(v_{i,k})$, we obtain $\|A(x)\|_\infty \leqslant |||x|||$.

Conversely, for $\varepsilon > 0$ there is a partition of unity (π_k) in $\mathbb{B}$ such that for any $k \in \mathbb{N}$ there exists a finite collection $v_{1,k}, \dots, v_{n_k,k} \in X_+$ with $\sum_{i=1}^{n_k} v_{i,k} \geqslant |x|$ and

$$\pi_k \sum_{i=1}^{n_k} |v_{i,k}|^p \leqslant \pi_k(A(x) + \varepsilon \mathbb{1}).$$

Summing over $k \in \mathbb{N}$ and taking into account the relation $\sum_{k\in\mathbb{N}} \pi_k A(x) = A(x)$, we get

$$\bigvee_{k=1}^{\infty} \sum_{i=1}^{n_k} \pi_k |v_{i,k}|^p \leqslant A(x) + \varepsilon \mathbb{1}.$$

Taking norms and making use of the fact that $\|\cdot\|_\infty$ is a Levi norm we arrive at the inequality

$$\sup_{k\in\mathbb{N}} \left\| \sum_{i=1}^{n_k} |\pi_k v_{i,k}|^p \right\|_\infty \leqslant \|A(x)\|_\infty + \varepsilon.$$

By Lemma 5.1 we can rewrite the last inequality as $\sup_{k\in\mathbb{N}} a_k \leqslant \|A(x)\|_\infty + \varepsilon$, where

$$a_k := \inf \left\{ \sup_{j\in\mathbb{N}} \sum_{i=1}^{n_k} \|\rho_j \pi_k v_{i,k}\|^p : \ (\rho_j) \in \mathrm{Prt}_\sigma(\mathbb{B}) \right\}.$$

For every $k \in \mathbb{N}$ we can choose a partition of unity $(\rho_{j,k})_{j\in\mathbb{N}}$ such that

$$\sup_{j\in\mathbb{N}} \sum_{i=1}^{n_k} \|\rho_{j,k} \pi_k v_{i,k}\|^p \leqslant a_k + \varepsilon.$$

Moreover, we may assume that $(\rho_{j,k})_{j\in\mathbb{N}}$ is a refinement of $(\pi_k)_{k\in\mathbb{N}}$. Denote $\tau_{j,k} := \rho_{j,k}\pi_k$ and $u_{i,j,k} := v_{i,j}$ whenever $\rho_{j,k}\pi_k = \rho_{j,k}$. Then

$$|||x||| \leqslant \sup_{j,k\in\mathbb{N}} \sum_{i=1}^{n_k} \|\tau_{j,k} u_{i,j,k}\|^p = \sup_{k\in\mathbb{N}} \sup_{j\in\mathbb{N}} \sum_{i=1}^{n_k} \|\rho_{j,k} \pi_k v_{i,k}\|^p \leqslant \sup_{k\in\mathbb{N}} a_k + \varepsilon \leqslant \|A(x)\|_\infty + 2\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $|||x||| = \|A(x)\|_\infty$. $\rhd$

Definition 5.3. Let Q_1 and Q_2 be extremally disconnected compact Hausdorff spaces, $\Lambda_1 = C(Q_1)$, $\Lambda_2 = C(Q_2)$, and let Λ be the Dedekind completion of $C(Q_1 \times Q_2)$. If $u_i \in \Lambda_i$ then $u_1 \otimes u_2 \in \Lambda$ is defined as $u_1 \otimes u_2 : (q_1, q_2) \mapsto u_1(q_1)u_2(q_2)$. Consider injective Banach lattices X_1 and X_2 whose norms are expressible in the form $\|x\|_1 = \|\lvert x\rvert_1\|_\infty$ and $\|x\|_2 = \|\lvert x\rvert_2\|_\infty$, where $\lvert\cdot\rvert_i$ is a Λ_i-valued norm on X_i. For any $x \in X_1 \bar{\otimes} X_2$ define $B(x) \in \Lambda$ by

$$B(x) := \inf \Bigg\{ \sum_{i=1}^{n} \lvert u_i\rvert \otimes \lvert v_i\rvert : \ 0 \leqslant u_i \in X_1, \ 0 \leqslant v_i \in X_2, \\ i = 1, \dots, n \in \mathbb{N}, \ |x| \leqslant \sum_{i=1}^{n} u_i \otimes v_i \Bigg\}.$$

Lemma 5.4. *For any $x \in X_1 \bar{\otimes} X_2$ the representation holds:*

$$\|B(x)\|_\infty = \inf \Bigg\{ \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k u_{i,k}\| \cdot \|\rho_k v_{i,k}\| : \ (\pi_k) \in \mathrm{Prt}_\sigma(\mathbb{B}_1), \ (\rho_k) \in \mathrm{Prt}_\sigma(\mathbb{B}_2), \\ 0 \leqslant u_{i,k} \in X_1, \ 0 \leqslant v_{i,k} \in X_2 \ (i \leqslant n), |x| \leqslant \sum_{i=1}^{n} u_{i,k} \otimes v_{i,k} \ (k \in \mathbb{N}) \Bigg\}.$$

$\lhd$ The proof goes along similar lines to the proof of Lemma 5.2 making use of Lemma 5.1. $\rhd$

6. Concavification

Definition 6.1. Take a positive invertible $\alpha \in \Lambda$. An α-concavification $X_{(\alpha)}$ may be defined as in Definition 3.7. In a uniformly complete vector lattice X, introduce new vector operations $\oplus$ and $*$, while the original ordering $\leqslant$ remain unchanged:

$$x \oplus y := \sigma_{\alpha^{-1}}(x, y) := (x^\alpha + y^\alpha)^{1/\alpha}, \quad t * x := t^{1/\alpha} x \quad (x, y \in X; \ t \in \mathbb{R}).$$

Then $X_{(\alpha)} := (X, \oplus, *, \leqslant)$ is again a vector lattice called an *α-concavification* of X.

As was mentioned in Definition 3.7, $X_{(\alpha)} = X^{(1/\alpha)}$. But we no longer have a guarantee that $X_{(\alpha)}$ is normable for an arbitrary Banach lattice X even if $1 \leqslant \alpha = p \in \mathbb{R}$. The function $x \mapsto \|x\|^p$ is not a norm in general. Nevertheless, we can define a lattice seminorm $\|\cdot\|_{(p)}$ on $X_{(p)}$ as

$$\|x\|_{(p)} := \inf \left\{ \sum_{i=1}^{n} \|u_i\|^p : \ |x| \leqslant u_1 \oplus \dots \oplus u_n, \ u_i \in X_+, \ n \in \mathbb{N} \right\}.$$

The seminorm $\|\cdot\|_{(p)}$ does not have to be a norm, and if it is a norm, it need not be complete (see [7, Examples 18 and 26]). Denote by $(X_{[p]}, \|\cdot\|_{[p]})$ the norm completion of the normed lattice $X_{(p)}/\ker\|\cdot\|_{(p)}$.

Assume now that X is a $\mathbb{B}$-cyclic Banach lattice and $\Lambda = \Lambda(\mathbb{B})$. Then there is a Λ-valued norm $\lvert\cdot\rvert$ on X such that $\|x\| = \|\lvert x\rvert\|_\infty$ $(x \in X)$. Define $\lvert x\rvert_{(\alpha)} : X \to \Lambda$ as

$$\lvert x\rvert_{(\alpha)} := \inf \left\{ \sum_{i=1}^{n} \lvert x\rvert^{\alpha} : \ |x| \leqslant u_1 \oplus \dots \oplus u_n, \ u_i \in X_+, \ n \in \mathbb{N} \right\}$$

and put $\|x\|_{\langle\alpha\rangle} = \big\|\lvert x\rvert_{(\alpha)}\big\|_\infty$ $(x \in X)$. Then $\|\cdot\|_{\langle\alpha\rangle}$ is a lattice seminorm on $X_{\langle\alpha\rangle}$. Again, the seminorm $\|\cdot\|_{\langle\alpha\rangle}$ is not generally a norm and need not be complete. Write $(X_{\langle\alpha\rangle}, \|\cdot\|_{\langle\alpha\rangle})$ for the completion of $X/\ker(\|\cdot\|_{\langle\alpha\rangle})$ with respect to the norm $\|\cdot\|_{\langle\alpha\rangle}$.

Definition 6.2. The Banach lattice $\mathscr{X}_{[\alpha]}$ is called the $[\alpha]$-*concavification* of a Banach lattice $\mathscr{X}$ and the $\mathbb{B}$-cyclyc Banach lattice $X_{\langle\alpha\rangle}$ is called the $\langle\alpha\rangle$-*concavification* of a $\mathbb{B}$-cyclic Banach lattice X.

Proposition 6.3. *Let $\alpha \in \Lambda = \mathscr{R}{\Downarrow}$ and $\mathbb{1} \leqslant \alpha$. For any Banach lattice $(\mathscr{X}, \|\cdot\|)$ inside $\mathrm{V}^{(\mathbb{B})}$ the bounded descent of the $[\alpha]$-concavification of $\mathscr{X}$ equals the $\langle\alpha\rangle$-concavification of the bounded descent of $\mathscr{X}$; in symbols, $(\mathscr{X}_{[\alpha]}){\Downarrow} = (\mathscr{X}{\Downarrow})_{\langle\alpha\rangle}$.*

$\lhd$ Put $X := \mathscr{X}{\Downarrow}$ and $\lvert\cdot\rvert := \|\cdot\|{\Downarrow}$. Note that $[\![1^\wedge \leqslant \alpha \in \mathscr{R}]\!] = \mathbb{1}$ Take $x \in X$ and denote by $\mathscr{A}(x) \in \mathrm{V}^{(\mathbb{B})}$ the internal set consisting of all finite sums $\sum_{i\leqslant n}\|x_i\|^\alpha$ with $n \in \mathbb{N}^\wedge$, $\{x_1,\dots,x_n\} \subset \mathscr{X}_+$, and $|x| \leqslant x_1 \oplus \dots \oplus x_n$. Denote by $A(x)$ the set of all finite sums $\sum_{i\leqslant n}\lvert x_i\rvert^\alpha$ with $n \in \mathbb{N}$, $\{x_1,\dots,x_n\} \subset X_+$, and $|x| \leqslant x_1 \oplus \dots \oplus x_n$. Observe that $\lvert x\rvert_{(\alpha)} = \inf A(x)$ and $[\![\|x\|_{(\alpha)} = \inf \mathscr{A}(x)]\!] = \mathbb{1}$. It can be easily checked that $[\![\inf A(x)$ is a greatest lower bound of $\mathscr{A}(x)]\!] = \mathbb{1}$, so that $[\![\|x\|_{(\alpha)} = \lvert x\rvert_{(\alpha)}]\!] = \mathbb{1}$ $(x \in X)$ or, equivalently, $\|\cdot\|_{(\alpha)}{\Downarrow} = \lvert\cdot\rvert_{(\alpha)}$. Write $\mathscr{X}_0$ for the set $\ker\|\cdot\|_{(\alpha)}$ and put $X_0 := \mathscr{X}_0{\Downarrow}$.

Clearly, $X_0 := \{x \in X : \lvert x\rvert_{(\alpha)} = 0\} = \{x \in X : \|x\|_{\langle\alpha\rangle} = 0\}$ and X_0 is an order ideal. Moreover, $\mathbb{B}\langle X_0\rangle = X_0$ and thus X/X_0 is a normed $\mathbb{B}$-lattice and Λ-normed lattice simultaneously. Let $(X_{[\![\alpha]\!]}, \lvert\cdot\rvert_{(\alpha)})$ stands for the completion of X/X_0 with respect to the norm $\lvert x\rvert_{(\alpha)}$. Then $(X_{[\![\alpha]\!]}, \lvert\cdot\rvert_{(\alpha)})$ is linearly isometric (in the sense of Λ-valued norms) to $(\mathscr{X}_{[\alpha]}, \|\cdot\|_{[\alpha]}){\Downarrow}$. Now, it remains to observe that $X_{\langle\alpha\rangle} = X_{[\![\alpha]\!]}$ is a $\mathbb{B}$-cyclic Banach lattice and $\|x\|_{\langle\alpha\rangle} = \big\|\lvert x\rvert_{(\alpha)}\big\|_\infty$ $(x \in X_{\langle\alpha\rangle})$. $\rhd$

Remark 6.4. The metric properties of the internal Banach lattice $(\mathscr{X}_{[\alpha]}, \|\cdot\|_{[\alpha]})$ can be immediately transferred into the language of the lattice $(X_{[\![\alpha]\!]}, \lvert\cdot\rvert_{(\alpha)})$ by means of the descent procedure. But the further translation into the language of the Banach lattice $(X_{\langle\alpha\rangle}, \|\cdot\|_{\langle\alpha\rangle})$ is not so easy. Therefore, we restrict ourselves to the case $1 \leqslant \alpha := p \in \mathbb{R}$.

Definition 6.5. A $\mathbb{B}$-cyclic Banach lattice X is said to be $(\mathbb{B}, p)$-*concave* if there exists a constant $M > 0$ such that for every $x_1,\dots,x_n \in X_+$ and $\varepsilon > 0$ there exists a countable partition of unity (π_n) in $\mathbb{B}$ with

$$\bigg\|\bigg(\sum_{i=1}^n x_i^p\bigg)^{\frac{1}{p}}\bigg\| \geqslant \frac{1}{M+\varepsilon}\bigg(\sum_{i=1}^n \|\pi_k x_i\|^p\bigg)^{\frac{1}{p}}$$

for all $k \in \mathbb{N}$. Equivalently, X is $(\mathbb{B}, p)$-concave if there is a constant $M > 0$ such that

$$\bigg\|\bigg(\sum_{i=1}^n x_i^p\bigg)^{\frac{1}{p}}\bigg\| \geqslant \frac{1}{M} \inf_{(\pi_k)\in \mathrm{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \bigg(\sum_{i=1}^n \|\pi_k x_i\|^p\bigg)^{\frac{1}{p}}$$

whenever $x_1,\dots,x_n \in X_+$.

Definition 6.6. Similarly, X is called $(\mathbb{B}, p)$-convex if there is a constant $M > 0$ such that for every finite collection $x_1,\dots,x_n \in X_+$, countable partition of unity (π_n) in $\mathbb{B}$, and $0 < \varepsilon \in \mathbb{R}$ there exists $k \in \mathbb{N}$ with

$$\bigg\|\bigg(\sum_{i=1}^n x_i^p\bigg)^{\frac{1}{p}}\bigg\| \leqslant (M+\varepsilon)\bigg(\sum_{i=1}^n \|\pi_k x_i\|^p\bigg)^{\frac{1}{p}}.$$

Equivalently, X is $(\mathbb{B},p)$-convex if there is a constant $M>0$ such that

$$\Big\|\Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}}\Big\| \leqslant M \inf_{(\pi_k)\in\mathrm{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \Big(\sum_{i=1}^{n}\|\pi_k x_i\|^p\Big)^{\frac{1}{p}}$$

whenever $x_1,\dots,x_n\in X_+$.

Definition 6.7. A $\mathbb{B}$-cyclic Banach lattice X satisfies a lower $(\mathbb{B},p)$-estimate with constant $M>0$ whenever for any finite collection of pair-wise disjoint elements $x_1,\dots,x_n\in X$ and every $\varepsilon>0$ there exists a countable partition of unity (π_n) in $\mathbb{B}$ with

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \geqslant \frac{1}{M+\varepsilon}\Big(\sum_{i=1}^{n}\|\pi_k x_i\|^p\Big)^{\frac{1}{p}}$$

for all $k\in\mathbb{N}$. Equivalently, X satisfies a lower $(\mathbb{B},p)$-estimate with constant $M>0$ whenever for any finite collection of pair-wise disjoint elements $x_1,\dots,x_n\in X$ we have

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \geqslant \frac{1}{M} \inf_{(\pi_k)\in\mathrm{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \Big(\sum_{i=1}^{n}\|\pi_k x_i\|^p\Big)^{\frac{1}{p}}.$$

Definition 6.8. A $\mathbb{B}$-cyclic Banach lattice X satisfies an upper $(\mathbb{B},p)$-estimate with constant $M>0$ whenever for any finite collection of pair-wise disjoint elements $x_1,\dots,x_n\in X$, a countable partition of unity (π_n) in $\mathbb{B}$, and $\varepsilon>0$ there exists $k\in\mathbb{N}$ with

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \leqslant (M+\varepsilon)\Big(\sum_{i=1}^{n}\|\pi_k x_i\|^p\Big)^{\frac{1}{p}}.$$

Equivalently, X satisfies a lower $(\mathbb{B},p)$-estimate with constant $M>0$ whenever for any finite collection of pair-wise disjoint elements $x_1,\dots,x_n\in X$ we have

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \leqslant M \inf_{(\pi_k)\in\mathrm{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \Big(\sum_{i=1}^{n}\|\pi_k x_i\|^p\Big)^{\frac{1}{p}}.$$

Proposition 6.9. *Suppose that X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X}\in\mathrm{V}^{(\mathbb{B})}$ is its Boolean valued representation. The following assertions hold:*

(1) $[\![\,\mathscr{X}$ *is* $p^\wedge$*-convex*$]\!]=\mathbb{1} \iff$ *there exists a countable partition of unity* $(\pi_k)_{k\in\mathbb{N}}$ *in* $\mathbb{B}$ *such that* $\pi_k X$ *is* $(\pi_k\mathbb{B},p)$*-convex for all* $k\in\mathbb{N}$.

(2) $[\![\,\mathscr{X}$ *is* $p^\wedge$*-concave*$]\!]=\mathbb{1} \iff$ *there exists a countable partition of unity* $(\pi_k)_{k\in\mathbb{N}}$ *in* $\mathbb{B}$ *such that* $\pi_k X$ *is* $(\pi_k\mathbb{B},p)$*-concave for all* $k\in\mathbb{N}$.

(3) $[\![\,\mathscr{X}$ *satisfies a lower* $p^\wedge$*-estimate with constant* $M^\wedge\,]\!]=\mathbb{1} \iff X$ *satisfies a lower* $(\mathbb{B},p)$*-estimate with constant* M.

(4) $[\![\,\mathscr{X}$ *satisfies an upper* $p^\wedge$*-estimate with constant* $M^\wedge\,]\!]=\mathbb{1} \iff X$ *satisfies an upper* $(\mathbb{B},p)$*-estimate with constant* M.

$\lhd$ We can assume that $X=\mathscr{X}{\Downarrow}$ for some Banach lattice $\mathscr{X}$ in $\mathrm{V}^{(\mathbb{B})}$. By Lemma 5.1 the left hand-side of the equivalence 6.9 (1) is true if and only if there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ such that

$$\pi_k\Big|\Big(\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}}\Big| \leqslant k\pi_k\Big(\sum_{i=1}^{n}|x_i|^p\Big)^{\frac{1}{p}}$$

for all $k \in \mathbb{N}$ and finite collections $x_1, \dots, x_n \in X_+$. The latter is equivalent to $p^\wedge$-convexity of $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$. Similarly, by Lemma 5.1 X satisfy a lower $(\mathbb{B}, p)$-estimate with constant M if and only if

$$\Big|\sum_{i=1}^{n} x_i\Big| \geqslant \frac{1}{M}\bigg(\sum_{i=1}^{n} |x_i|^p\bigg)^{\frac{1}{p}}$$

for any finite collections of pair-wise disjoint $x_1, \dots, x_n \in X$. This is equivalent to saying that $\mathscr{X}$ satisfy a lower $p^\wedge$-estimate with constant $M^\wedge$ inside $\mathrm{V}^{(\mathbb{B})}$. We have thus proved 6.9 (1) and 6.9 (3). The proofs of 6.9 (2) and 6.9 (4) are similar. $\rhd$

Proposition 6.10. *Let X be a $\mathbb{B}$-cyclic Banach lattice and there is a countable partition of unity $(\pi_k)_{k\in\mathbb{B}}$ in $\mathbb{B}$ such that $\pi_k X$ is $(\pi_k\mathbb{B}, p)$-convex for every $k \in \mathbb{N}$. Then $X_{\langle p\rangle}$ is a $\mathbb{B}$-cyclic Banach lattice.*

$\lhd$ Let $X = \mathscr{X}{\Downarrow}$ for some Banach lattice $\mathscr{X}$ inside $\mathrm{V}^{(\mathbb{B})}$. If X satisfies the hypotheses then $[\![\mathscr{X}$ is $p^\wedge$-convex$]\!] = \mathbb{1}$ by Proposition 6.9. It follows that $[\![\mathscr{X}_{(p^\wedge)}$ is a Banach lattice$]\!] = \mathbb{1}$, see [37, pp. 53, 54]. By Proposition 6.3 $X_{\langle p\rangle}$ coincides with $\mathscr{X}_{(p^\wedge)}{\Downarrow}$, while the latter is a $\mathbb{B}$-cyclic Banach lattice in view of [32, Theoren 4.1]. $\rhd$

Theorem 6.11. *Let X be a $\mathbb{B}$-cyclic Banach lattice satisfying a lower $(\mathbb{B}, p)$-estimate with constant M. Then $X_{\langle p\rangle}$ is lattice $\mathbb{B}$-isomorphic to an injective Banach lattice with a Boolean algebra of M-projections isomorphic to $\mathbb{B}$. Moreover, if $M = 1$ then $X_{\langle p\rangle}$ is an injective Banach lattice with $\mathbb{M}(X_{\langle p\rangle}) \simeq \mathbb{B}$.*

$\lhd$ We can assume without loss of generality that $X = \mathscr{X}{\Downarrow}$. If X satisfies a lower $(\mathbb{B}, p)$-estimate with constant M, then $[\![\,\mathscr{X}$ satisfies a lower $p^\wedge$-estimate with constant $M^\wedge\,]\!] = \mathbb{1}$. By [7, Theorem 10] and Boolean-valued transfer principle there exist $\mathscr{L}, u \in \mathrm{V}^{(\mathbb{B})}$ such that $[\![\mathscr{L}$ is an AL-space$]\!] = \mathbb{1}$ and $[\![u : \mathscr{X}_{[p^\wedge]} \to \mathscr{L}$ is a lattice isomorphism of $\mathscr{X}_{[p^\wedge]}$ onto $\mathscr{L}]\!] = \mathbb{1}$. Moreover, $\|u\|, \|u^{-1}\| \leqslant \max\{1, 1/M^{p^\wedge}\}$. It follows that $U := u{\Downarrow}$ is a lattice $\mathbb{B}$-isomorphism of $(\mathscr{X}_{[p^\wedge]}){\Downarrow}$ onto $L := \mathscr{L}{\Downarrow}$ and $\|U\|, \|U^{-1}\| \leqslant \max\{1, 1/M^p\}$. It remains to observe that L is an injective Banach lattice with $\mathbb{M}(L) \simeq \mathbb{B}$ by Theorem 2.4 and $(\mathscr{X}_{[p^\wedge]}){\Downarrow}$ is lattice $\mathbb{B}$-isomorphic to $X_{\langle p\rangle}$ by Proposition 6.3. $\rhd$

Remark 6.12. (1) Theorem 6.11, for the particular case of $\mathbb{B} = \{\mathbb{O}, \mathbb{1}\}$, asserts that if X is a Banach lattice satisfying a lower p-estimate with constant M, then $X_{\langle p\rangle} = X_{[p]}$ is lattice isomorphic to an AL-space and X is an AL-space if, in addition, $M = 1$. This is theorem 10 in [7]. At the same time Theorem 6.11 itself is nothing but a Boolean-valued interpretation of [7, Theorem 10].

(2) Various p-characteristics of spaces and operators are an important tool in modern Banach space theory, see [14, 36]. Definitions 6.5–6.8 should be considered as an attempt to produce a useful combination of p-characteristics and M-structure in Banach lattices. Further progress depends partly on advanced functional calculus, see Remark 3.10.

7. Sums

In this section we will construct the injective sum of injective Banach lattices and prove a characterization of injective Banach lattices in terms of summable sequences.

Lemma 7.1. *For a family $(x_\alpha)_{\alpha\in\mathrm{A}}$ in X with o-summable $(|x_\alpha|)_{\alpha\in\mathrm{A}}$ we have*

$$\sup_{\theta\in\mathscr{P}_{\mathrm{fin}}(\mathrm{A})}\ \inf_{(\pi_k)\in\mathrm{Prt}_\sigma}\ \sup_{k\in\mathbb{N}}\ \sum_{\alpha\in\theta}\|\pi_k x_\alpha\| = \bigg\|\, o\text{-}\sum_{\alpha\in\mathrm{A}}|x_\alpha|\,\bigg\|_\infty. \qquad (6)$$

Moreover, $(|x_\alpha|)_{\alpha\in\mathrm{A}}$ is o-summable if and only if the left hand-side in (6) is finite.

$\lhd$ According to Lemma 5.1, for every $\theta \in \mathscr{P}_{\rm fin}(\mathrm{A})$ we can write down

$$\Big\| \sum_{x\in\theta} |x| \Big\|_\infty = \inf_{(\pi_k)\in \mathrm{Prt}_\sigma} \sup_{k\in\mathbb{N}} \sum_{x\in\theta} \|\pi_k x\|. \tag{7}$$

It follows that the family $(|x_\alpha|)_{\alpha\in\mathrm{A}}$ is order summable if and only if the numerical family $(\|\sum_{\alpha\in\theta}|x_\alpha|\|)_{\alpha\in\mathrm{A}}$ is bounded, since $\|\cdot\|_\infty$ is a Levi norm. Taking the supremum in (7) over all finite subsets $\theta \subset \mathrm{A}$ and making use of the fact that $\|\cdot\|_\infty$ is a Fatou norm we arrive at (6). $\rhd$

Theorem 7.2. *Let $(X_\alpha)_{\alpha\in\mathrm{A}}$ be a family of injective Banach lattices. Assume that there is a complete Boolean algebra $\mathbb{B}$ and a family $(b_\alpha)_{\alpha\in\mathrm{A}}$ in $\mathbb{B}$ with $\bigvee_{\alpha\in\mathrm{A}} b_\alpha = \mathbb{1}$ and $\mathbb{M}(X_\alpha) \simeq \mathbb{B}_\alpha = [\mathbb{O}, b_\alpha]$ for all $\alpha \in \mathrm{A}$. Then there exists a unique up to a lattice $\mathbb{B}$-isometry injective Banach lattice X such that the following hold:*

(1) $\mathbb{B} \simeq \mathbb{M}(X)$.

(2) *For any $\alpha \in \mathrm{A}$ there is a lattice $\mathbb{B}_\alpha$-isometry $\iota_\alpha : X_\alpha \to X$.*

(3) $\big(\iota_\alpha(X_\alpha)\big)_{\alpha\in\mathrm{A}}$ *is a family of pair-wise disjoint bands in X.*

(4) $\mathbb{B}\big\langle \bigoplus_{\alpha\in\mathrm{A}} \iota_\alpha(X_\alpha)\big\rangle$ *is norm dense in X.*

$\lhd$ We may assume without loss of generality that $\Lambda_\alpha := \Lambda_\alpha(\mathbb{B}_\alpha)$ and b_α is a band projection in Λ with $\Lambda_\alpha = b_\alpha(\Lambda)$. According to [32, Theorem 5.1] there exists a strictly positive Maharam operator $\Phi_\alpha : X_\alpha \to \Lambda_\alpha$ with the Levi property such that $\|x\|_\alpha = \Phi_\alpha(|x|)$ $(x \in X_\alpha)$. Define an order ideal $X \subset \prod_{\alpha\in\mathrm{A}}$ and an operator $\Phi : X \to \Lambda$ as

$$X := \Big\{ \mathbf{x} = (x_\alpha)_{\alpha\in\mathrm{A}} \in \prod_{\alpha\in\mathrm{A}} X_\alpha : \ o\text{-}\sum_{\alpha\in\mathrm{A}} \Phi_\alpha(|x_\alpha|) \in \Lambda \Big\},$$

$$\Phi(\mathbf{x}) =: o\text{-}\sum_{\alpha\in\mathrm{A}} \Phi_\alpha(x_\alpha) \quad (\mathbf{x} = (x_\alpha)_{\alpha\in\mathrm{A}} \in X).$$

It is easy to check that Φ is a strictly positive Maharam operator with the Levi property. Thus, the vector lattice X, equipped with the norm $\|\mathbf{x}\|_{\rm ins} := \Phi(|\mathbf{x}|)$ $(\mathbf{x} \in X)$, is an injective Banach lattice and $\mathbb{B} \simeq \mathbb{M}(X)$ by [32, Theorem 5.1].

Given $\alpha \in \mathrm{A}$, denote by $\iota_\alpha(x_\alpha)$ the family $(x_\beta)_{\beta\in\mathrm{A}}$ with $x_\beta = 0$ for $\alpha \neq \beta \in \mathrm{A}$. Then ι_α is a lattice isometry from X_α into X. Moreover, the sets $\iota_\alpha(X_\alpha)$ are pairwise disjoint bands in $\bigoplus_{\alpha\in\mathrm{A}} X_\alpha$, which is an order ideal in X. Take $\mathbf{x} \in X$ and a finite subset $\theta \subset \mathrm{A}$ and put

$$\mathbf{x}_\theta = \sum_{\alpha\in\theta} \iota_\alpha(x_\alpha), \quad \lambda_\theta := \Phi(|\mathbf{x} - \mathbf{x}_\theta|) = \Phi\bigg(\Big| \sum_{\alpha\in\mathrm{A}\setminus\theta} \iota_\alpha(x_\alpha)\Big|\bigg).$$

Clearly, $(\lambda_\theta)_{\theta\in\mathscr{P}_{\rm fin}(\mathrm{A})}$ is a downward directed net in Λ and $\inf_\theta \lambda_\theta = 0$. Therefore, given $0 < \varepsilon \in \mathbb{R}$, there exists a partition of unity (π_θ) in $\mathbb{B}$ such that

$$\pi_\theta \lambda_\theta \leqslant \varepsilon \mathbb{1} \quad \big(\theta \in \mathscr{P}_{\rm fin}(\mathrm{A})\big).$$

Since X is injective, there exists a unique $\mathbf{x}_\varepsilon \in X$ with $\pi_\theta \mathbf{x}_\varepsilon = \pi_\theta \mathbf{x}_\theta$ for all $\theta \in \mathscr{P}_{\rm fin}(\mathrm{A})$. Now we can deduce

$$\pi_\theta \Phi(|\mathbf{x} - \mathbf{x}_\varepsilon|) = \Phi(|\pi_\theta \mathbf{x} - \pi_\theta \mathbf{x}_\varepsilon|) = \pi_\theta \Phi(|\mathbf{x} - \mathbf{x}_\theta|) = \pi_\theta \lambda_\theta \leqslant \varepsilon \mathbb{1}.$$

It follows that $\Phi(|\mathbf{x} - \mathbf{x}_\varepsilon|) \leqslant \varepsilon \mathbb{1}$, whence $\|\mathbf{x} - \mathbf{x}_\varepsilon\| = \|\Phi(|\mathbf{x} - \mathbf{x}_\varepsilon|)\|_\infty \leqslant \varepsilon$. $\rhd$

Definition 7.3. Denote $\sum_{\alpha\in\mathrm{A}}^{\mathrm{inj}} X_\alpha := X$. The Banach lattice $(X, \|\cdot\|_{\mathrm{ins}})$ is called an *injective sum* of injective Banach lattices.

Corollary 7.4. *The norm $\|\cdot\|_{\mathrm{ins}}$ of the injective sum of a family (X_α) of injective Banach lattices has the following representations:*

$$\|\mathbf{x}\|_{\mathrm{ins}} = \sup_{\theta\in\mathscr{P}_{\mathrm{fin}}(\mathrm{A})} \inf_{(\pi_k)\in\mathrm{Prt}_\sigma} \sup_{k\in\mathbb{N}} \sum_{\alpha\in\theta} \|\pi_k x_\alpha\|$$

$$= \Big\| o\text{-}\sum_{\alpha\in\mathrm{A}} \Phi_\alpha(|x_\alpha|)\Big\|_\infty \quad \Big(\mathbf{x} = (x_\alpha)_{\alpha\in\mathrm{A}} \in \sum_{\alpha\in\mathrm{A}}^{\mathrm{inj}} X_\alpha\Big).$$

In particular, the injective sum of a family $(X_\alpha)_{\alpha\in\mathrm{A}}$ may be defined as

$$\sum_{\alpha\in\mathrm{A}}^{\mathrm{inj}} X_\alpha := \Big\{\mathbf{x} \in \prod_{\alpha\in\mathrm{A}} X_\alpha : \ \|\mathbf{x}\|_{\mathrm{ins}} < \infty\Big\}.$$

◁ Immediate from Lemma 5.2. ▷

Corollary 7.5. *If X is an injective Banach lattice and $(\pi_\alpha)_{\alpha\in\mathrm{A}}$ is a partition of unity in $\mathbb{M}(X)$, then $X = \sum_{\alpha\in\mathrm{A}}^{\mathrm{inj}} \pi_\alpha X$.*

Definition 7.6. A sequence (x_n) in X is said to be $\mathbb{B}$-*summable* if there is $x \in X$ such that for every $0 < \varepsilon \in \mathbb{R}$ there exists a countable partition of unity (π_n) in $\mathbb{B}$ with $\big\|\pi_n\big(x - \sum_{k=1}^N x_k\big)\big\| \leqslant \varepsilon$ for all $n \in \mathbb{N}$ and $N \geqslant n$. In this event x is called the $\mathbb{B}$-sum of (x_n). A sequence (x_n) is called *absolutely* $\mathbb{B}$-*summable* if

$$\sup_{N\in\mathbb{N}} \inf_{(\pi_k)\in\mathrm{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \sum_{n=1}^N \|\pi_k x_n\| < +\infty.$$

Theorem 7.7. *For a $\mathbb{B}$-cyclic Banach lattice X the following are equivalent:*

(1) *There is a countable partition of unity $(\pi_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $\pi_n X$ is lattice $\pi_n\mathbb{B}$-isomorphic to an injective Banach lattice for every $n \in \mathbb{N}$.*

(2) *Every positive $\mathbb{B}$-summable sequence in X is absolutely $\mathbb{B}$-summable.*

(3) *Every positive $\mathbb{B}$-summable sequence of pairwise disjoint elements in X is absolutely $\mathbb{B}$-summable.*

◁ Let X be a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X}$ a Banach lattice inside $\mathrm{V}^{(\mathbb{B})}$, the Boolean valued representation of X, see [32, Theorem 4.1 and Definition 4.2]. By Lemma 7.1 a sequence (x_n) in X is absolutely $\mathbb{B}$-summable if an only if $o\text{-}\sum_{n\in\mathbb{N}}|x_n|$ exists in Λ. But the latter is equivalent to saying that $[\![$the sequence (x_n) is absolutely summable in $\mathscr{X}]\!] = \mathbb{1}$. It is easily seen that $\mathbb{B}$-sum of (x_n) exists and equals x if and only if $o\text{-lim}_{n\to\infty}|x - \sum_{k=1}^n x_k| = 0$ or, equivalently, $\lim_{n\to\infty}\big\|x - \sum_{k=1}^n x_k\big\| = 0$. Consequently, (2) is equivalent to saying that $[\![$every positive summable sequence in $\mathscr{X}$ is absolutely summable$]\!] = \mathbb{1}$. The Boolean-valued transfer principle enables us to apply inside $\mathrm{V}^{(\mathbb{B})}$ the Schlotterbeck's characterization of AL-spaces, so that by the Maximum Principle there exist $\mathscr{L}, u \in \mathrm{V}^{(\mathbb{B})}$ such that $[\![\mathscr{L}$ is an AL-space$]\!] = \mathbb{1}$ and $[\![u : \mathscr{X} \to \mathscr{L}$ is a lattice isomorphism of $\mathscr{X}$ onto $\mathscr{L}]\!] = \mathbb{1}$. Assume, in addition, that $[\![\|u\|, \|u^{-1}\| \leqslant n^\wedge]\!] = \mathbb{1}$ for some $n \in \mathbb{N}$. Then $U := u{\Downarrow}$ is a lattice $\mathbb{B}$-isomorphism of $\mathscr{X}{\Downarrow}$ onto $L := \mathscr{L}{\Downarrow}$ and $\|U\|, \|U^{-1}\| \leqslant n$. To handle the general case observe that the sentence $(\exists\, n \in \mathbb{N})\|U\|, \|U^{-1}\| \leqslant n$ is true, so that by The Transfer Principle

$$\mathbb{1} = [\![(\exists\, n \in \mathbb{N}^\wedge)\|u\|, \|u^{-1}\| \leqslant n]\!] = \bigvee_{n\in\mathbb{N}} [\![\|u\|, \|u^{-1}\| \leqslant n^\wedge]\!].$$

It follows that there is a partition of unity π_n in $\mathbb{B}$ such that $\pi_n \leqslant [\![\|u\|, \|u^{-1}\| \leqslant n^\wedge]\!]$. Denote $\mathbb{B}_n := \pi_n\mathbb{B} := [\mathbb{O}, \pi_n]$, $\mathscr{X}_n := \pi_n\mathscr{X}$, $\mathscr{L}_n := \pi_n\mathscr{L}$, $u_n := \pi_n u$ and note that $\mathrm{V}^{(\mathbb{B}_n)} \models$ "u_n is a lattice isomorphism from $\mathscr{X}_n$ onto $\mathscr{L}_n$". It follows that $U_n := u_n{\Downarrow}$ is a lattice $\mathbb{B}_n$-isomorphism from $(\mathscr{X}_n{\Downarrow}$ onto $L_n := \mathscr{L}_n{\Downarrow}$ and $\|U_n\|, \|U_n^{-1}\| \leqslant n$. $\rhd$

Remark 7.8. Taking $\mathbb{B}$ to be the two-element Boolean algebra $\{\mathbb{O}, \mathbb{1}\}$ in Theorem 7.7, one obtains the characterization of AL-spaces due to Schlotterbeck: *a Banach lattice X is lattice isomorphic to an AL-space if and only if every positive summable sequence in X is absolutely summable*, see [45, Theorem 2.7].

8. Tensor Products

Given injective Banach lattices X_1 and X_2, there exists an injective Banach lattice $X_1\hat{\otimes}_{\delta|\pi|}X_2$, the *mixed tensor product* of X_1 and X_2, and the canonical bilinear map $\otimes$ from $X_1 \times X_2$ to $X_1\hat{\otimes}_{\delta|\pi|}X_2$ such that the structure of $\mathbb{B}$-cyclic Banach lattice of $X_1\hat{\otimes}_{\delta|\pi|}X_2$ is uniquely determined up to lattice $\mathbb{B}$-isometry by X_1, X_2, and $\otimes$.

Definition 8.1. Let Λ_1 and Λ_2 be Dedekind complete AM-spaces with units and $\mathbb{B}_1$ and $\mathbb{B}_2$ complete Boolean algebras. Define a *Dedekind complete projective tensor product* $\Lambda_1\hat{\otimes}_{\delta|\pi|}\Lambda_2$ of Λ_1 and Λ_2 as the Dedekind completion of the AM-space $\Lambda_1\hat{\otimes}_{|\pi|}\Lambda_2$ ($\equiv$ $\Lambda_1 \overset{\triangle}{\otimes} \Lambda_2$) endowed with the order unit norm, see [16, Corollary 3C]. Write $\mathbb{B} := \mathbb{B}_1 \hat{\otimes} \mathbb{B}_2$ for the Dedekind completion of the free product $\mathbb{B}_0 := \mathbb{B}_1 \otimes \mathbb{B}_2$ of Boolean algebras $\mathbb{B}_1$ and $\mathbb{B}_2$, see [27, §11]. Let $Q^\bullet$ and $\mathrm{Clop}(Q)$ stand for the absolute of Q and the Boolean algebra of clopen subsets of Q, respecively.

Recall that the *absolute* $Q^\bullet$ of a compact Hausdorff space Q is characterized by the following universal mapping property: $Q^\bullet$ is an extremally disconnected and there is an irreducible mapping $p : Q^\bullet \to Q$ such that for each continuous mapping f from a compact Hausdorff P onto Q there is a continuous mapping $g : Q^\bullet \to P$ with $p = f \circ g$. The Gleason theorem says that $Q^\bullet$ exists and is unique up to homeomorphism for every compact Hausdorff Q, see [4, Theorem 5.8].

Proposition 8.2. *The Boolean algebras $\mathbb{P}(\Lambda_1 \hat{\otimes}_{\delta|\pi|} \Lambda_2)$ and $\mathbb{P}(\Lambda_1) \hat{\otimes} \mathbb{P}(\Lambda_2)$ are isomorphic.*

$\lhd$ Let Q_i be the Stone space of $\mathbb{P}(\Lambda_i)$, so that Λ_i is lattice isometric to $C(Q_i)$, the space of all continuous functions on Q_i. Then $\Lambda_1\hat{\otimes}_{|\pi|}\Lambda_2$ is lattice isometric to $C(Q_1 \times Q_2)$, see [15, Corollary 3E]. Observe that if $C(Q)^\delta$ stands for the Dedekind completion of $C(Q)$ then the Boolean algebras $\mathbb{P}(C(Q)^\delta)$ and $\mathrm{Clop}(Q^\bullet)$ are isomorphic. Therefore, the Boolean algebras $\mathbb{P}(\Lambda_1\hat{\otimes}_{\delta|\pi|}\Lambda_2)$ and $\mathrm{Clop}((Q_1 \times Q_2)^\bullet)$ are also isomorphic. It remains to observe that $\mathbb{P}(\Lambda_1) \otimes \mathbb{P}(\Lambda_2) \simeq \mathrm{Clop}(Q_1 \times Q_2)$ and thus $\mathbb{P}(\Lambda_1) \hat{\otimes} \mathbb{P}(\Lambda_2) \simeq \mathrm{Clop}((Q_1 \times Q_2)^\bullet)$. $\rhd$

Definition 8.3. A subset $X_0 \subset X$ is said to be $\mathbb{M}(X)$-*dense* in X whenever for every $x \in X$ and $0 < \varepsilon \in \mathbb{R}$ there are $x_\varepsilon \in X$, a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{M}(X)$, and a family $(x_\xi)_{\xi\in\Xi}$ in X_0 such that $\|x - x_\varepsilon\| \leqslant \varepsilon$ and $\pi_\xi x_\varepsilon = \pi_\xi x_\xi$ $(\xi \in \Xi)$. Equivalently, X_0 is $\mathbb{B}$-dense in X with $\mathbb{B} = \mathbb{M}(X)$ if $\mathbb{B}\langle X_0\rangle$ is norm dense in X.

Recall that $X_0^{\downarrow}$ is the collection of all elements $x \in X$ representable as $x = \inf(A)$, where A is a downward directed subset of X_0. The set $X_0^{\uparrow}$ is defined similarly on using upward-directed sets. We also put $X_0^{\downarrow\uparrow} := (X_0^{\downarrow})^{\uparrow}$.

Theorem 8.4. *Let X_1 and X_2 be injective Banach lattices. Then there exist a unique up to isomorphism injective Banach lattice $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ and a lattice bimorphism $\bar{\otimes} : X_1 \times X_2 \to X_1 \hat{\otimes}_{\delta|\pi|} X_2$ such that the following hold:*

(1) $\bar{\otimes}$ *induces an embedding* ϕ *of the Fremlin tensor product* $X_1 \bar{\otimes} X_2$ *into* $X_1 \hat{\otimes}_{\delta|\pi|} X_2$.

(2) *There is a Boolean isomorphism* $\jmath$ *from* $\mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$ *onto* $\mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$ *with* $\jmath(\pi_1 \otimes \pi_2)(x_1 \bar{\otimes} x_2) = \pi_1 x_1 \bar{\otimes} \pi_2 x_2$ *for all* $\pi_i \in \mathbb{M}(X_i)$ *and* $x_i \in X_i$ $(i = 1, 2)$.

(3) $\|x_1 \otimes x_2\|_{\delta|\pi|} = \|x_1\| \cdot \|x_2\|$ *for all* $x_1 \in X_1$ *and* $x_2 \in X_2$.

(4) $X_1 \bar{\otimes} X_2$ *is* $\mathbb{B}$*-dense in* $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ *with* $\mathbb{B} = \mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$.

(5) $X_1 \hat{\otimes}_{\delta|\pi|} X_2 = X_0^{\downarrow\uparrow}$, *where* X_0 *comprises all finite sums* $\sum_{k=1}^{n} \pi_k \phi(u_k)$ *with* $\pi_k \in \mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$ *and* $u_k \in X_1 \bar{\otimes} X_2$ $(k = 1, \dots, n \in \mathbb{N})$.

$\lhd$ Take injective Banach lattices $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$. By [32, Corollary 5.2] there are strictly positive Maharam operators $\Phi_i : X_i \to \Lambda_i$ with the Levi property such that $\|x\|_i = \Phi_i(|x|)$ $(x \in X_i,\ i = 1, 2)$. Put $\Lambda := \Lambda_1 \hat{\otimes}_{\delta|\pi|} \Lambda$ and consider a positive bilinear operator $B : X_1 \times X_2 \to \Lambda$ defined as $B : (x_1, x_2) \mapsto \Phi_1(x_1) \otimes \Phi_2(x_2)$. Making use of the linearization via Fremlin's tensor product (see [15, Theorem 5.3]), we can find a positive operator $\Phi : X_1 \bar{\otimes} X_2 \to \Lambda$ such that $B = \Phi\otimes$. Clearly, Φ is strictly positive, since for any $0 \neq u \in X_1 \bar{\otimes} X_2$ there exist $0 < x_1 \in X_1$ and $0 < x_2 \in X_2$ with $x_1 \otimes x_2 \leqslant |u|$ (see [15, Theorem 4.2 (iv)]), so that $\Phi(|u|) \geqslant \Phi_1(x_1) \otimes \Phi_2(x_2) > 0$. Define $X_1 \hat{\otimes}_{\delta|\pi|} X_2 := L_1(\tilde{\Phi})$ and $\|u\|_{\delta|\pi|} := \tilde{\Phi}(|u|)$, where $\tilde{\Phi} : X_1 \hat{\otimes}_{\delta|\pi|} X_2 \to \Lambda$ is the Maharam extension of Φ, see [32, Theorem 5.6]. Then $(X_1 \hat{\otimes}_{\delta|\pi|} X_2, \|\cdot\|_{\delta|\pi|})$ is an injective Banach lattice and $\mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2) \simeq \mathbb{P}(\Lambda)$ in virtue of [32, Corollary 5.2]. Observe also that $\|x_1 \otimes x_2\|_{\delta|\pi|} = \Phi(|x_1 \otimes x_2|) = B(|x_1|, |x_2|) = \|x_1\| \cdot \|x_2\|$ for all $x_1 \in X_1$ and $x_2 \in X_2$ and 8.4 (4, 5) follow from [32, Theorem 5.6 (3, 4)].

According to Proposition 8.2 we can identify $\mathbb{P}(\Lambda)$ with $\mathbb{P}(\Lambda_1) \hat{\otimes} \mathbb{P}(\Lambda_2)$. In virtue of Theorem 2.6 there are Boolean isomorphisms h_i from $\mathbb{P}(\Lambda_i)$ onto $\mathbb{M}(X_i)$ with $\pi_i \Phi_i(x) = \Phi_i(h_i(\pi_i)x)$ for all $\pi_i \in \Lambda_i$ and $x \in X_i$ $(i = 1, 2)$ and h from $\mathbb{P}(\Lambda)$ onto $\mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$ with $\pi\tilde{\Phi}(u) = \tilde{\Phi}(h(\pi)u)$ for all $\pi \in \Lambda$ and $u \in X_1 \bar{\otimes} X_2$. There is a unique Boolean isomorphism $h_1 \otimes h_2$ from $\mathbb{P}(\Lambda_1) \hat{\otimes} \mathbb{P}(\Lambda_2)$ onto $\mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$ such that $(h_1 \otimes h_2)(\pi_1 \otimes \pi_2) = h_1(\pi_1) \otimes h_2(\pi_2)$ for all $\pi_1 \in \mathbb{P}(\Lambda_1)$ and $\pi_2 \in \mathbb{P}(\Lambda_2)$. Clearly, $\jmath := h \circ (h_1 \otimes h_2)^{-1}$ is a Boolean isomorphism from $\mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$ onto $\mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$. Moreover, in view of [32, Theorem 5.6] there is a lattice isomorphism ι from $X_1 \bar{\otimes} X_2$ into $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ and an f-algebra isomorphism h from Λ to $\mathscr{Z}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$ such that $\pi\Phi(u) = \tilde{\Phi}(h(\pi)\iota(u))$ for all $u \in X_1 \bar{\otimes} X_2$ and $\pi \in \Lambda$. Denote $\bar{\otimes} := \iota\otimes$, where $\otimes$ is the canonical bilinear map from $X_1 \times X_2$ to $X_1 \bar{\otimes} X_2$, and let ϕ is a unique lattice isomorphism from $X_1 \bar{\otimes} X_2$ into $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ with $\phi\bar{\otimes} = \iota\otimes$, see [15, Theorem 5.3]. Given $\rho_i \in \mathbb{M}(X_i)$ $(i := 1, 2)$, the map $\iota \circ (\rho_1 \otimes \rho_2) \circ \iota^{-1}$ is an M-projection in $\iota(X_1 \bar{\otimes} X_2)$ and has a unique extension to an M-projection $\jmath_0(\rho_1 \otimes \rho_2)$ in $X_1 \hat{\otimes}_{\delta|\pi|} X_2$. The map $\jmath_0$ has a unique extension to a Boolean isomorphism from $\mathbb{M}(X_1) \otimes \mathbb{M}(X_2)$ into $\mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$. It can be easily verified that $\jmath_0(\rho_1 \otimes \rho_2)(x_1 \bar{\otimes} x_2) = \rho(x_1) \bar{\otimes} \rho_2(x_2)$. It remains to prove that $\jmath$ is an extension of $\jmath_0$. Take arbitrary $x_i \in X_i$, $\pi_i \in \mathbb{P}(\Lambda_i)$ and put $\rho_i := h_i(\pi_i)$ $(i = 1, 2)$ and $\pi := \pi_1 \otimes \pi_2$. Then

$$
\begin{aligned}
\pi\tilde{\Phi}(x_1 \bar{\otimes} x_2) =& \pi(\Phi_1(x_1) \otimes \Phi_2(x_2)) = (\pi_1 \Phi_1(x_1)) \otimes (\pi_2 \Phi_2(x_2)) \\
=& \Phi_1(\rho_1(x_1)) \otimes \Phi(\rho_2(x_2)) = \tilde{\Phi}(\rho_1(x_1) \bar{\otimes} \rho_2(x_2)) \\
=& \tilde{\Phi}(\jmath_0(\rho_1 \otimes \rho_2)(x_1 \bar{\otimes} x_2)) = h^{-1}(\jmath_0(\rho_1 \otimes \rho_2))\tilde{\Phi}(x_1 \bar{\otimes} x_2) \\
=& h^{-1}(\jmath_0((h_1 \otimes h_2)(\pi)))\tilde{\Phi}(x_1 \bar{\otimes} x_2).
\end{aligned}
$$

Thus, $h(\pi) = \jmath_0 \circ (h_1 \otimes h_2)(\pi)$, so that $\jmath_0$ coincides with the restriction of $\jmath$ onto $\mathbb{M}(X_1) \otimes \mathbb{M}(X_2)$. The uniqueness assertion follows from Theorem 8.10 below. ▷

Remark 8.5. The lattice bimorphism ϕ is conventionally denoted by $\otimes$. Identifying $\mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$ and $\mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$ we can rewrite the identity in 8.4 (2) as $(\pi_1 \otimes \pi_2)(x_1 \otimes x_2) = (\pi_1 x_1) \otimes (\pi_2 x_2)$ for all $\pi_i \in \mathbb{M}(X_i)$ and $x_i \in X_i$ $(i = 1, 2)$. Moreover, one can endow $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ with the structure of a lattice ordered module over Λ identifying Λ with $\Lambda_1 \hat{\otimes}_{\delta|\pi|} \Lambda_2$. As is seen from the proof of Theorem 8.4, there is more to say about canonical bimorphism: If $\lambda_i \in \Lambda_i$ and $x_i \in X_i$ $(i = 1, 2)$ then $(\lambda_1 \otimes \lambda_2)(x_1 \otimes x_2) = (\lambda_1 x_1) \otimes (\lambda_2 x_2)$.

Lemma 8.6. *For any $x \in X_1 \bar{\otimes} X_2$ the representation holds:*

$$\lvert x\rvert = \inf\left\{ \sum_{i=1}^{n} \lvert u_i\rvert \otimes \lvert v_i\rvert : \ 0 \leqslant u_i \in X_1,\ 0 \leqslant v_i \in X_2\ (i \leqslant n),\ |x| \leqslant \sum_{i=1}^{n} u_i \otimes v_i \right\}.$$

◁ Denote by $\lvert x\rvert_{|\pi|}$ the right-hand side of the required representation. It can be easily seen that $\lvert\cdot\rvert_{|\pi|} : X_1 \bar{\otimes} X_2 \to \Lambda$ is a Λ-valued seminorm. If $|x| \leqslant \sum_{i=1}^{n} u_i \otimes v_i$ for some $0 \leqslant u_i \in X_1$ and $0 \leqslant v_i \in X_2$ $(i \leqslant n)$, then

$$\lvert x\rvert \leqslant \Phi\left(\sum_{i=1}^{n} u_i \otimes v_i \right) = \sum_{i=1}^{n} \Phi_1(u_i) \otimes \Phi_2(v_i) = \sum_{i=1}^{n} \lvert u_i\rvert \otimes \lvert v_i\rvert,$$

so that we get $\lvert x\rvert \leqslant \lvert x\rvert_{|\pi|}$. Conversely, given $x_0 \in X_1 \bar{\otimes} X_2$, by Hahn–Banach–Kantorovich Theorem we can pick a linear operator $S : X_1 \bar{\otimes} X_2 \to \Lambda$ such that $S(|x_0|) = \lvert x_0\rvert_{|\pi|}$ and $S(x) \leqslant \lvert x\rvert_{|\pi|}$ $(x \in X_1 \bar{\otimes} X_2)$. Clearly, S is positive. For any positive $x_1 \in X_1$ and $x_2 \in X_2$ we have $S(x_1 \otimes x_2) \leqslant \lvert x_1 \otimes x_2\rvert_{|\pi|} \leqslant \lvert x_1\rvert \otimes \lvert x_2\rvert = \Phi(x_1 \otimes x_2)$ and thus $S\otimes \leqslant \Phi\otimes$. Consequently, $S \leqslant \Phi$ by [15, Theorem 5.3], so that $\lvert x_0\rvert = \Phi(|x_0|) \geqslant S(|x_0|) = \lvert x_0\rvert_{|\pi|}$. ▷

Proposition 8.7. *For any $x \in X_1 \bar{\otimes} X_2$ the representation holds:*

$$\|x\|_{\delta|\pi|} = \inf\Bigg\{ \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k u_{i,k}\| \cdot \|\rho_k v_{i,k}\| : \ (\pi_k) \in \mathrm{Prt}_\sigma(\mathbb{B}_1),\ (\rho_k) \in \mathrm{Prt}_\sigma(\mathbb{B}_2),$$
$$0 \leqslant u_{i,k} \in X_1,\ 0 \leqslant v_{i,k} \in X_2\ (i \leqslant n),\ |x| \leqslant \sum_{i=1}^{n} u_{i,k} \otimes v_{i,k}\ (k \in \mathbb{N}) \Bigg\}.$$

◁ This is immediate from Lemmas 5.4 and 8.6. ▷

Lemma 8.8. *If $M \subset X_1 \bar{\otimes} X_2$ and M_0 is a norm dense subset of M, then $\mathbb{B}_0\langle M_0\rangle$ is norm dense in $\mathbb{B}\langle M\rangle$. In particular, $\mathbb{B}_0\langle X_1 \otimes X_2\rangle$ is norm dense in $X_1 \hat{\otimes}_{\delta|\pi|} X_2$.*

◁ For $x \in \mathbb{B}\langle M\rangle$ choose a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a norm bounded family $(x_\xi)_{\xi\in\Xi}$ in M such with $x = \sum_{\xi\in\Xi} \pi_\xi x_\xi$. There is no loss of generality in supposing that $\pi_\xi \in \mathbb{B}_0 = \mathbb{B}_1 \otimes \mathbb{B}_2$ for all $\xi \in \Xi$. Indeed, because $\mathbb{B}_0$ is an order dense subalgebra in $\mathbb{B}$, for every $\xi \in \Xi$ one can pick a family $(\pi_{\xi,\eta})_{\eta\in \mathrm{H}(\xi)}$ of pair-wise disjoint projections in $\mathbb{B}_0$ with $\pi_\xi = \bigvee_{\eta\in\mathrm{H}(\xi)} \pi_{\xi,\eta}$. Putting $\Gamma := \bigcup_{\xi\in\Xi} \{\xi\} \times \mathrm{H}(\xi)$, $\rho_\gamma := \pi_{\xi,\eta}$ and $u_\gamma := x_\xi$, whenever $\gamma = (\xi, \eta)$, we see that $(\rho_\gamma)_{\gamma\in\Gamma}$ is a partition of unity in $\mathbb{B}_0$, $(u_\gamma)_{\gamma\in\Gamma}$ is a norm bounded family in M and $x = \sum_{\gamma\in\Gamma} \rho_\gamma u_\gamma$.

Take an arbitrary $\varepsilon > 0$. By hypothesis, for every $\xi \in \Xi$ there is a $u_\xi \in M_0$ such that $\|x_\xi - u_\xi\| < \varepsilon$. Since the family (u_ξ) is norm bounded, we can define

$u_\varepsilon \in \mathbb{B}_0\langle M_0\rangle$ by putting $u_\varepsilon := \sum_{\xi\in\Xi}\pi_\xi u_\xi$. Making use of the equivalence of the relations $\|x_\xi - u_\xi\| \leqslant \varepsilon$ and $|x_\xi - u_\xi| \leqslant \varepsilon\mathbb{1}$, we deduce

$$\|x - u_\varepsilon\| = \||x - u_\varepsilon|\|_\infty = \Big\|\sum_{\xi\in\Xi}\pi_\xi|x_\xi - u_\xi|\Big\|_\infty = \Big\|\bigvee_{\xi\in\Xi}\pi_\xi|x_\xi - u_\xi|\Big\|_\infty$$
$$= \sup_{\xi\in\Xi}\|\pi_\xi|x_\xi - u_\xi|\|_\infty \leqslant \sup_{\xi\in\Xi}\|\pi_\xi(\varepsilon\mathbb{1})\|_\infty = \varepsilon$$

Thus, $\|x - u_\varepsilon\| \leqslant \varepsilon$ and, as $\varepsilon > 0$ is arbitrary, the first part of the lemma is proved. The second one follows from Theorem 8.4 (4) by putting $M_0 := X_1 \otimes X_2$, $M := X_1 \bar{\otimes} X_2$, since $X_1 \otimes X_2$ is dense in $X_1 \bar{\otimes} X_2$ by [15, Theorem 4.2 (iii)]. $\rhd$

Corollary 8.9. *The positive cone in $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ is the closure of $\mathbb{B}_0\langle P\rangle$, where P is the convex cone in $X_1 \otimes X_2$ generated by $\{x_1 \otimes x_2 :\ 0 \leqslant x_i \in X_i\ (i = 1, 2)\}$.*

$\lhd$ The Banach lattice $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ can be considered as the completion of the normed lattice $\mathbb{B}\langle X_1 \bar{\otimes} X_2\rangle$. Consequently, the positive cone $\mathbb{B}\langle X_1 \bar{\otimes} X_2\rangle_+$ of $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ is the closure of the positive cone of $\mathbb{B}\langle X_1 \bar{\otimes} X_2\rangle$. But the cone $\mathbb{B}\langle X_1 \bar{\otimes} X_2\rangle_+$ is easily seen to coincide with the cone $\mathbb{B}\langle (X_1 \bar{\otimes} X_2)_+\rangle$. It remains to observe that P is dense in $(X_1 \bar{\otimes} X_2)_+$ by [16, Corollary 1B (b)] and thus $\mathbb{B}_0\langle P\rangle$ is dense in $\mathbb{B}\langle (X_1 \bar{\otimes} X_2)_+\rangle$ by Lemma 8.8. $\rhd$

Theorem 8.10. *Let X_1 and X_2 be injective Banach lattices and $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ their injective tensor product. Assume that Y is a $\mathbb{B}$-cyclic Banach space with $\mathbb{B} = \mathbb{M}(X_1 \hat{\otimes}_{\delta|\pi|} X_2)$ and $B : X_1 \times X_2 \to Y$ is a bounded bilinear operator such that $(\pi_1 \otimes \pi_2) \circ B(x_1, x_2) = B(\pi_1 x_1, \pi_2 x_2)$ for all $\pi_i \in \mathbb{M}(X_i)$ and $x_i \in X_i$ $(i = 1, 2)$. Then there exists a unique bounded $\mathbb{B}$-linear operator $T : X_1 \hat{\otimes}_{\delta|\pi|} X_2 \to Y$ such that $T = B \circ \otimes$ and $\|T\| = \|B\|$.*

$\lhd$ First, it is easy to see that $|B(x_1, x_2)| \leqslant \|B\||x_1| \otimes |x_2|$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Indeed, if $|B(\bar{x}_1, \bar{x}_2)| \nleqslant \|B\||\bar{x}_1| \otimes |\bar{x}_2|$ for some $\bar{x}_1 \in X_1$ and $\bar{x}_2 \in X_2$ then, making use of the fact that $\mathbb{B}_1 \otimes \mathbb{B}_2$ is order dense in $\mathbb{B}$, we can pick nonzero $\pi_i \in \mathbb{B}_i$ $(i = 1, 2)$ and $\varepsilon > 0$ such that $\pi|B(\bar{x}_1, \bar{x}_2)| \geqslant \pi(\|B\||\bar{x}_1| \otimes |\bar{x}_2| + \varepsilon\mathbb{1})$, where $\pi := \pi_1 \otimes \pi_2$. It follows that $|B(\pi_1\bar{x}_1, \pi_2\bar{x}_2)| \geqslant \|B\||\pi_1\bar{x}_1| \otimes |\pi_2\bar{x}_2| + \varepsilon\pi\mathbb{1}$, so that

$$\|B(\pi_1\bar{x}_1, \pi_2\bar{x}_2)\| = \||B(\pi_1\bar{x}_1, \pi_2\bar{x}_2)|\|_\infty \geqslant \|\,\|B\||\pi_1\bar{x}_1| \otimes |\pi_2\bar{x}_2| + \pi\varepsilon\mathbb{1}\|_\infty$$
$$= \|B\| \cdot \||\pi_1\bar{x}_1|\|_\infty \||\pi_2\bar{x}_2|\|_\infty + \varepsilon = \|B\| \cdot \|\pi_1\bar{x}_1\| \cdot \|\pi_2\bar{x}_2\| + \varepsilon.$$

But this contradicts the boundedness of B.

Write T_0 for the linear operator from $X_1 \otimes X_2$ to Y such that $B = T_0 \circ \otimes$ and prove that $|T_0 x| \leqslant |x|$ and $\pi T_0 x = T_0(\pi x)$ for all $x \in X_1 \otimes X_2$ and $\pi \in \mathbb{B}_1 \otimes \mathbb{B}_2$. If $x = \sum_{i=1}^n u_i \otimes v_i$ for some $u_1, \dots, u_n \in X_1$ and $v_1, \dots, v_n \in X_2$, then

$$|T_0 x| \leqslant \sum_{i=1}^n |B(u_i, v_i)| \leqslant \|B\| \sum_{i=1}^n |u_i| \otimes |v_i|,$$

so that $|T_0 x| \leqslant \|B\||x|$ by Lemma 8.6. Moreover, for $\pi := \rho \otimes \sigma$ with $\rho \in \mathbb{B}_1$ and $\sigma \in \mathbb{B}_2$ we have

$$\pi T_0 x = \sum_{i=1}^n \pi B(u_i, v_i) = \sum_{i=1}^n B(\rho u_i, \sigma v_i)$$
$$= T_0\Big(\sum_{i=1}^n \rho u_i \otimes \sigma v_i\Big) = T_0\Big(\sum_{i=1}^n \pi(u_i \otimes v_i)\Big) = T_0(\pi x).$$

An arbitrary projection $\pi \in \mathbb{B}_1 \otimes \mathbb{B}_2$ can be written in the form $\pi = \sum_{i=1}^{n} \rho_i \otimes \sigma_i$ with $\rho_i \in \mathbb{B}_1$ and $\sigma_i \in \mathbb{B}_2$ $(i = 1, 2)$, so that

$$\pi T_0 x = \sum_{i=1}^{n} (\rho_i \otimes \sigma_i)\, T_0 x = T_0\bigg(\sum_{i=1}^{n} (\rho_i \otimes \sigma_i) x\bigg) = T_0(\pi x).$$

Now, extend the operator T_0 to an operator T_1 from $\mathbb{B}_0\langle X_1 \otimes X_2\rangle$ to Y by putting $T_1 x := \sum \pi_\xi T x_\xi$, whenever $x = \sum \pi_\xi x_\xi$ for a partition of unity (π_ξ) in $\mathbb{B}_0$ and a norm bounded family (x_ξ) in $X_1 \otimes X_2$. It can be easily verified that this definition is sound. Moreover, T_1 is a linear operator with the same properties: $|T_1 x| \leqslant \|B\| |x|$ and $\pi T x = T(\pi x)$ for all $x \in X_1 \otimes X_2$ and $\pi \in \mathbb{B}_0$. In particular, T_1 is norm bounded and $\|T_1\| \leqslant \|B\|$, since $\|T_1 x\| = \||T_1 x|\|_\infty \leqslant \|B\| \||x|\|_\infty = \|B\| \|x\|$.

Finally, extend T_1 to an operator $T : X_1 \hat{\otimes}_{\delta|\pi|} X_2 \to Y$ by norm continuity making use of Lemma 8.8. Then T is a linear operator and $\|T_1\| = \|T\|$. Moreover, $|T_1 x| \leqslant \|B\| |x|$ for all $x \in X_1 \hat{\otimes}_{\delta|\pi|} X_2$, since $|\cdot| : X_1 \hat{\otimes}_{\delta|\pi|} X_2 \to \Lambda$ is a norm continuous operator. If $\pi \in \mathbb{B}$ and $x \in X_1 \hat{\otimes}_{\delta|\pi|} X_2$ then $|\pi T(\pi^\perp x)| \leqslant \|B\| \pi |\pi^\perp x| = 0$. It follows that $\pi T(I - \pi) = 0$ and $\pi T = \pi T \pi$. Similarly, $(I - \pi) T \pi = 0$ and $T\pi = \pi T \pi$, so that T is $\mathbb{B}$-linear.

Assume that a norm bounded $\mathbb{B}$-linear operator $\bar{T} : X_1 \hat{\otimes}_{\delta|\pi|} X_2 \to Y$ satisfies the condition $B = \bar{T}\otimes$. Then T and $\bar{T}$ agree on the dense subspace $\mathbb{B}_0\langle X_1 \times X_2\rangle$ of $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ and hence $T = \bar{T}$. ▷

Proposition 8.11. *Let Y is a $\mathbb{B}$-cyclic Banach lattice, while B and T are the same as in Theorem* 8.10. *Then B is positive if and only if so is T.*

Corollary 8.12. *The tensor product $X_1 \hat{\otimes}_{\delta|\pi|} X_2$ of injective Banach lattices X_1 and X_2 is unique up to lattice $\mathbb{B}$-isometry with $\mathbb{B} = \mathbb{M}(X_1) \hat{\otimes} \mathbb{M}(X_2)$.*

Remark 8.13. In the particular case of $\mathbb{B} = \mathbb{B}_1 = \mathbb{B}_2 = \{\mathbb{O}, \mathbb{1}\}$ we obtain from Theorem 8.4 relationship between the L_1 space of a product measure and the L_1 spaces of its factors. Theorem 8.10 shows that the canonical map from $L_1(\mu) \times L_1(\nu)$ to $L_1(\mu \times \nu)$ is universal for continuous bilinear maps from $L_1(\mu) \times L_1(\nu)$ to Banach spaces and determines the product $L_1(\mu \times \nu)$ up to isomorphism as Banach lattice, see [17, Theorems 253F and 253G].

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ИНЪЕКТИВНЫХ БАНАХОВЫХ РЕШЕТОК. II

Ответственный за выпуск
Кибизова В. В.

Подписано в печать 20.08.2012.
Формат $60 \times 84\ {}^1\!/_8$. Усл. п. л. 3,02. Тираж 75 экз.