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SOME PROBLEMS CONCERNING OPERATORS ON BANACH LATTICES

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Статья посвящена некоторым аспектам теории ограниченных линейных операторов в банаховых решетках. Основное внимание уделено взаимосвязи таких свойств, как регулярность, компактность, доминирование, а также порядковому строению различных пространств операторов. Обсуждается также булевозначный подход к этому кругу проблем.

Ключевые слова: Банахова решетка, AM-пространство, AL-пространство, инъективная банахова решетка, положительный оператор, регулярность, компактность, компактное доминирование, булевозначная модель, булевозначный принцип переноса, циклически компактный оператор, $\mathbb{B}$-суммирующий оператор.

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The paper deals with some aspects of bounded linear operators on Banach lattices. Under consideration are an interplay between regularity, compactness, and domination as well as the order structure of different spaces of operators. Boolean valued analysis framework for these problems is also discussed.

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Keywords: Banach lattice, AL-space, AM-space, injective Banach lattice, positive operator, compactness, regularity, compact domination property, Boolean-valued model, Boolean valued transfer principle, cyclically compact operator, cone $\mathbb{B}$-summing operator.

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SOME PROBLEMS CONCERNING OPERATORS ON BANACH LATTICES

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1. Definitions and Notation

These notes will present some open problems dealing with an interplay between regularity, compactness, and domination as well as the order structure of different classes of operators. Let us first recall that

Definition 1.1. If X and Y are vector lattices and $T : X \to Y$ is a linear operator then:

(1) T is *positive* (in symbols, $T \geqslant 0$) if $T(X_+) \subseteq Y_+$.

(2) If T has the property that for every $x \in X_+$ there is $y \in Y_+$ such that $T([-x,x]) \subset [-y,y]$ then T is termed *order bounded*.

(3) If there are positive operators $U, V : X \to Y$ such that $T = U - V$ then T is *regular*.

We denote the order bounded (resp. regular) operators from X into Y by $\mathscr{L}^b(X,Y)$ (resp. $\mathscr{L}^r(X,Y)$). We always consider these spaces with the order induced by the cone of positive operators: $T \geqslant S$ if and only if $T - S \geqslant 0$. We always have $\mathscr{L}^r(X,Y) \subseteq \mathscr{L}^b(X,Y)$ and the inclusion can be proper. If X and Y are Banach lattices then $\mathscr{L}(X,Y)$ will denote the bounded linear operators from X into Y. In this case, we always have $\mathscr{L}^b(X,Y) \subseteq \mathscr{L}(X,Y)$. In general not every bounded operator need be regular, or even order bounded.

Example 1.2. For $p \in [1,\infty)$ there is a bounded linear operator $T : L_p([0,1]) \to c_0$ which is not regular.

$\lhd$ Let (r_n) denote the sequence of Rademacher functions on $[0,1]$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Define an operator $T : L_p([0,1]) \to c_0$ by $Tx = \sum_{n=1}^{\infty} r_n(x)\,\mathbf{e}_n$, where $r_n(x) = \int_0^1 r_n(t)x(t)\,dt$ and $\mathbf{e}_n$ denotes the n'th standard basis vector in c_0. The Rademacher functions converge weak* to 0 when considered, as we do here, as elements of $L_p([0,1])^*$ so that $Tf \in c_0$. T is bounded as $\|Tx\| = \sup_{n=1}^{\infty} |r_n(x)| \leqslant \sup_{n=1}^{\infty} \|r_n\|_q \|x\|_p = \|x\|_p$. Note that $T(r_n) = \mathbf{e}_n$ for all $n \in \mathbb{N}$ and that $T(r_0) = 0$, because of orthogonality, so that $T(r_0 + r_n) = \mathbf{e}_n$. As $0 \leqslant r_0 + r_n \leqslant 2r_0$, if we had $U \geqslant T, 0$ then

$$U(2r_0) \geqslant U(r_0 + r_n) \geqslant T(r_0 + r_n) = \mathbf{e}_n$$

for all $n \in \mathbb{N}$, which is inconsistent with $U(2r_0)$ lying in c_0. It follows that T is not regular after all. $\rhd$

Two well known properties that vector lattices may or may not have are those of being Dedekind complete (when every non-empty set that is bounded above has a supremum) or Dedekind σ-complete (when non-empty *countable* sets that are bounded above have a supremum). We need notions in between these two extremes.

Definition 1.3. Let α be an infinite cardinal. A vector lattice X is *Dedekind α-complete* (respectively *Dedekind $<\alpha$-complete*) if every nonempty subset of X, of cardinality at most (respectively less than) α and which is bounded above, must have a supremum.

For Banach lattices we have choices of topology that we can use to characterize density conditions. In vector lattices we have the following notion:

Definition 1.4. Let X be a vector lattice. A sequence (x_n) in X is *relatively uniformly convergent* (*uniformly convergent* for short) to $x \in X$ if there is $x_0 \in X$ such that for all reals $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with the property that $|x_n - x| \leqslant \varepsilon x_0$ for all $n \geqslant n_0$. Equivalently there is a sequence of reals $\varepsilon_n \downarrow 0$ such that $|x_n - x| \leqslant \varepsilon_n x_0$. A set $D \subset X$ is *relatively uniformly dense* in X if every $x \in X$ is the relative uniform limit of a sequence of elements of D.

Definition 1.5. A sequence (x_n) in a vector lattice E is said to be *uniformly Cauchy* whenever there exists some $0 \leqslant u \in E_+$ such that for each $0 < \varepsilon \in \mathbb{R}$ we have $|x_n - x_m| \leqslant \varepsilon u$ for all $n, m \in \mathbb{N}$ sufficiently large. A vector lattice space is called *uniformly complete* whenever every uniformly Cauchy sequence is relatively uniformly convergent.

A vector lattice is uniformly complete if and only if $\sup\{\sum_{k=1}^{n} x_k : n \in \mathbb{N}\}$ exists for every uniformly bounded sequence (x_n) in E_+, see [55, 1.1.7 (v)]. (A sequence (x_n) in E_+ is called *uniformly bounded* if $x \leqslant a_n e$ for some $e \in E_+$ and $(a_n) \in \ell_1$.)

Recall also some basic definitions concerning Banach lattices.

Definition 1.6. A Banach lattice X is said to have: the *property* (P) if there exists a positive contractive projection in X'' onto X [55, p. 47]; the *Levi property* (or a *Levi norm*) if $0 \leqslant x_\alpha \uparrow$ and $\|x_\alpha\| \leqslant 1$ imply that $\sup_\alpha x_\alpha$ exists in X [3, Definition 7 (2)]; the *Fatou property* (or a *Fatou norm*) if $0 \leqslant x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$ [3, Definition 7 (3)]. A Banach lattice with the Levi (Fatou) property is also called *order semicontinuous* (resp. *monotonically complete*) [55].

Let $\mathbb{B}(X)$ and $\mathbb{P}(X)$ stand respectively for the complete Boolean algebras of all bands and all band projections in a vector lattice X. Throughout the sequel $\mathbb{B}$ is a complete Boolean algebra with unit $\mathbb{1}$ and zero $\mathbb{O}$, while $\Lambda := \Lambda(\mathbb{B})$ is a Dedekind complete AM-space with unit such that $\mathbb{B} \simeq \mathbb{P}(\Lambda)$; in this event $\mathbb{B}$ and $\mathbb{P}(\Lambda)$ are identified with $\mathbb{1}$ taken as the unit element both in $\mathbb{B}$ and $\mathbb{P}(\Lambda)$. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi \in \Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{O}$ whenever $\xi \neq \eta$.

For the theory of Banach lattices and positive operators we refer to the books Abramovich and Aliprantis [2], Aliprantis and Burkinshaw [7], and Meyer-Nieberg [55]. The needed information on the theory of Boolean-valued models is briefly presented in Kusraev [38, Chapter 9] and Kusraev and Kutateladze [47, Chapter 1]; details may be found in Bell [12], Kusraev and Kutateladze [45], Takeuti and Zaring [67]. We let $:=$ denote the assignment by definition, while $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ symbolize the naturals, the rationals, and the reals.

2. Lattices of Regular Operators

The earliest results in this area date back to Kantorovich's 1936 paper [35]. Proofs of this may be found in any text book on the subject.

Theorem 2.1. *If X and Y are Archimedean vector lattices and Y is Dedekind complete then $\mathscr{L}^r(X,Y) = \mathscr{L}^b(X,Y)$ is a Dedekind complete vector lattice.*

Not only did he show that it is a lattice, he gave a formula for the lattice operations. If $T \in \mathscr{L}^r(X,Y)$ and $x \in X_+$ then $T^+(x) = \sup T([0,x])$, $T^-(x) = \sup T([-x,0])$, $|T|(x) = \sup T([-x,x])$. More generally, if $S,T \in \mathscr{L}^r(X,Y)$ then

$$(S \vee T)(x) = \sup\{S(y) + T(x) : y,z \in X_+ \text{ and } x = y+z\}$$

and

$$(S \wedge T)(x) = \inf\{S(y) + T(x) : y,z \in X_+ \text{ and } x = y+z\}.$$

These are known as the *Freudenthal–Kantorovich–Riesz* formulae.

There are several cases known where the conclusion of this theorem holds but with a stronger condition on X and a weaker condition on Y. It isn't possible to just weaken the condition on Y as Kantorovich's result is the best possible. This is due to Abramovich and Gejler, [4].

Theorem 2.2. *The following conditions on a vector lattice Y are equivalent:*

(1) *Y is Dedekind complete.*

(2) *For every vector lattice X, every order bounded operator from X into Y is regular and $\mathscr{L}^r(X,Y)$ is a Dedekind complete vector lattice.*

(3) *For every vector lattice X every order bounded operator from X into Y is regular and $\mathscr{L}^r(X,Y)$ is a vector lattice.*

If we restrict the size of *order intervals* in X then we can weaken the assumption on Y. For the case that $\alpha = \aleph_0$ it was proved in [5] that (1) implies (2), whilst (2) implies (1) is in [4].

Theorem 2.3. *Let α be an infinite cardinal and Y a vector lattice, then the following are equivalent:*

(1) *Y is Dedekind α-complete.*

(2) *If X is a vector lattice in which every order interval has a relatively uniformly dense subset of cardinality at most α, then $\mathscr{L}^b(X,Y) = \mathscr{L}^r(X,Y)$ is a Dedekind α-complete vector lattice in which all the lattice operations are given by the Freudenthal–Kantorovich–Riesz formulae.*

Kantorovich's original result, along with Abramovich and Gejler's converse, characterizes the best possible range spaces. At the opposite extreme is the following result due to van Rooij ([60] or [61]) which characterizes the best possible domains.

Theorem 2.4. *The following conditions on a vector lattice X are equivalent:*

(1) *Every principal ideal in X is finite dimensional.*

(2) *For every vector lattice Y, $\mathscr{L}^b(X,Y)$ is a vector lattice.*

Furthermore the lattice operations will be given by the Freudenthal–Kantorovich–Riesz formulae and in fact in this case *every* linear operator from X into Y is regular.

There are other results which are similar. In all known cases we either have a lattice in which all lattice operations are given by the Freudenthal–Kantorovich–Riesz formulae, or we don't have a lattice at all. A major, and difficult, open problem in this area is:

PROBLEM 2.5: Is it true that if X and Y are Archimedean vector lattices such that $\mathscr{L}^r(X,Y)$ is a vector lattice then the lattice operations in $\mathscr{L}^r(X,Y)$ all satisfy the Freudenthal–Kantorovich–Riesz formulae?

In fact, ever since the beginnings of this subject it has been a significant question as to whether or not if X and Y are Archimedean vector lattices and $S,T \in \mathscr{L}^r(X,Y)$ have a supremum then it is given by the Freudenthal–Kantorovich–Riesz formula, although the question has only been explicitly posed fairly recently. A recent, as yet unpublished, result of Michael Elliott has answered this question. Namely, he has constructed an operator on $L_1([0,1])$ into a certain $C(K)$ space (constructed as a closed unital sublattice of ℓ_∞) which has a modulus which is not given by the Freudenthal–Kantorovich–Riesz formula. It is noteworthy that these spaces are actually Banach lattices.

If we assume rather more, by working with Banach lattices rather than vector lattices, then we can say rather more.

At one extreme we have:

Theorem 2.6 (van Rooij, [60]). *The following conditions on a Banach lattice X are equivalent:*

(1) *X is atomic with an order continuous norm.*

(2) *For every Banach lattice Y, $\mathscr{L}^b(X,Y)=\mathscr{L}^r(X,Y)$ is a Banach lattice under the regular norm in which the lattice operations are given by the Freudenthal–Kantorovich–Riesz formulae.*

(3) *For every compact Hausdorff space K, $\mathscr{L}^r\big(X,C(K)\big)$ is a vector lattice.*

Whilst at the other extreme is:

Theorem 2.7. *The following conditions on a Banach lattice Y are equivalent:*

(1) *Y is Dedekind complete.*

(2) *For every Banach lattice X, $\mathscr{L}^b(X,Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Freudenthal–Kantorovich–Riesz formulae.*

(3) *For every Banach lattice X, $\mathscr{L}^r(X,Y)$ is a vector lattice.*

What we would really like is a condition on the *pair* (X,Y) which tells us whether or not $\mathscr{L}^r(X,Y)$ is a lattice. Whilst we are still some way from answering this question completely, we can answer it given one of a number of natural restrictions on X. For example:

Theorem 2.8. *If X is a* separable *Banach lattice and Y is any Banach lattice the following are equivalent:*

(1) *Either X is atomic with an order continuous norm or Y is Dedekind σ-complete.*

(2) *$\mathscr{L}^b(X,Y)$ is a Dedekind σ-complete Banach lattice under the regular norm in which all lattice operations satisfy the Freudenthal–Kantorovich–Riesz formulae.*

(3) *$\mathscr{L}^r(X,Y)$ is a Banach lattice under the regular norm.*

(4) *$\mathscr{L}^r(X,Y)$ is a vector lattice.*

An interesting consequence applies when $X = Y$.

Corollary 2.9. *If X is a separable Banach lattice then $\mathscr{L}^r(X)$ is a vector lattice if and only if X is Dedekind complete.*

PROBLEM 2.10: The next result depends on density character only of order intervals in X rather than of X itself. For example, no matter how large the index set I, order intervals in $\ell_1(I)$ are separable even though $\ell_1(I)$ itself has density character card(I). Does the preceding theorem remain true under the weaker hypothesis that every order interval in X is separable?

The reason that separability of X suffices here is not directly connected with the separability of order intervals in X. There is an important condition that van Rooij identified.

DEFINITION 2.11. A Banach lattice has *property* $(\star)$ if, for every sequence (f_n) in X^*_+ which converges $\sigma(X^*, X)$ to $f \in X^*_+$ as $n \to \infty$, we have $|f_n - f| \to 0$ for $\sigma(X^*, X)$ as $n \to \infty$.

There are two other definitions that are at first sight slightly weaker and which might seem easier to verify in practise.

DEFINITION 2.12. A Banach lattice has *property* $(\star\star)$ if, for every sequence (f_n) in X^* such that $f_n \to f$ and with $|f_n| \to h \in X^*$ for $\sigma(X^*, X)$ as $n \to \infty$, we have $h = |f|$.

DEFINITION 2.13. A Banach lattice X has *property* $(\star\star\star)$ if, for every sequence (f_n) in X^* such that $f_n \to 0$ and $|f_n| \to h \in X^*$ for $\sigma(X^*, X)$ as $n \to \infty$, we have $h = 0$.

The only obvious examples of Banach lattices having any of these properties are those which are atomic with an order continuous norm. So far, these are the only examples!

Theorem 2.14 (Chen and Wickstead)**.** *Let X be a Banach lattice and consider the following conditions.*

(1) *X is atomic with an order continuous norm.*

(2) *The lattice operations in X^* are sequentially $\sigma(X^*, X)$ continuous.*

(3) *X has property $(\star)$.*

(4) *X has property $(\star\star)$.*

(5) *X has property $(\star\star\star)$.*

It is always true that rm(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). If X is Dedekind σ-complete or is separable or $X = C_0(\Sigma)$, where Σ is a locally compact Hausdorff space, then all five conditions are equivalent.

PROBLEM 2.15: Are these five conditions equivalent in general? Either an affirmative or a negative answer would be useful and interesting.

A positive solution would automatically mean a positive answer to Problem 2.10.

We can similarly obtain a complete answer if we assume instead that X has an order continuous norm. This result depends on several, as yet, unpublished results from Michael Elliott's Ph. D. thesis, [24].

Theorem 2.16. *Let X be a Banach lattice with an order continuous norm and Y be any Banach lattice. The following are equivalent:*

(1) *Either X is atomic with an order continuous norm or Y is Dedekind α-complete where α is the smallest cardinal that is greater than the density character of every order interval in X.*

(2) *$\mathscr{L}^b(X,Y)$ is a Banach lattice under the regular norm in which all lattice operations satisfy the Freudenthal–Kantorovich–Riesz formulae.*

(3) *$\mathscr{L}^r(X,Y)$ is a vector lattice.*

You might hope to simplify the statement of this theorem by changing the order theoretic restriction on Y to being that it is β-complete where β is the smallest cardinal that if greater than or equal to the density character of every order interval in X. It turns out that such a simplification is possible if and only if the set theoretic hypothesis that there is no weakly inaccessible cardinal is true!

Problem 2.17: It is still an open question exactly when the space $\mathscr{L}^r(X,Y)$ is a vector lattice. Our results so far suggest that this might be the case if and only if either X is atomic with an order continuous norm or if Y is Dedekind α-complete where α is the smallest cardinal that is greater than the density character of every order interval in X. Prove or disprove! Notice that a consequence of this would be that all lattice operations would be given by the Freudenthal–Kantorovich–Riesz formulae.

3. Compactness and Regularity

We look first at the question of when all compact operators are regular. However, we need to be clear about what we mean. Just like bounded operators in general, compact operators need not be regular. More interestingly a compact operator T may be regular, so that we can write $T = U - V$ where U and V are positive operators but without being able to choose U and V as also being compact. Indeed, Krengel [37] showed that even in as nice a case as $X = Y = \ell_2$, when the space of regular operators is a lattice, it is possible for T to be compact and have a modulus which is not compact. It then follows from the Dodds–Fremlin theorem that it is impossible to write T as the difference of two positive compact operators.

Definition 3.1. If X and Y are Banach lattices then $\mathscr{K}^r(X,Y)$ will denote the linear span of $\mathscr{K}(X,Y)_+$, the cone of compact positive operators from X to Y.

So Krengel's example shows that $\mathscr{K}^r(X,Y) \neq \mathscr{L}^r(X,Y) \cap \mathscr{K}(X,Y)$ in general.

We have two questions that we can ask now. When is $\mathscr{K}(X,Y) \subseteq \mathscr{L}^r(X,Y)$ and when is $\mathscr{K}(X,Y) = \mathscr{K}^r(X,Y)$? If we seek only a condition on either X or Y for a guaranteed positive answer for all choices of the other space then we have complete answers to both our possible questions.

Theorem 3.2 (Krengel, [37]; Cartwright and Lotz, [15])**.** *The following conditions on a Banach lattice X are equivalent:*

(1) *X is isomorphic to an AL-space.*

(2) *For every Banach lattice Y, every compact operator from X into Y is regular.*

(3) *For every Banach lattice Y, $\mathscr{K}(X,Y)$ is a vector lattice.*

Theorem 3.3 (Krengel, [37]; Cartwright and Lotz, [15])**.** *The following conditions on a Banach lattice Y are equivalent:*

(1) *Y is isomorphic to an AM-space.*

(2) *For every Banach lattice X, every compact operator from X into Y is regular.*

(3) *For every Banach lattice X, $\mathscr{K}(X,Y)$ is a vector lattice.*

Problem 3.4: There is no obvious conjecture as to when either $\mathscr{K}(X,Y) \subseteq \mathscr{L}^r(X,Y)$ or $\mathscr{K}(X,Y) = \mathscr{K}^r(X,Y)$ for a specified *pair* (X,Y).

Note, in particular combining the following result of Godefroy with an example of Abramovich where $\mathscr{L}(X,Y)$ is a lattice shows that we need have neither X being isomorphic to an AL-space nor Y to an AM-space.

Proposition 3.5. *Let X and Y be Banach lattices with Y Dedekind complete and having the approximation property and suppose that $\mathscr{K}(X,Y) \subseteq \mathscr{L}^r(X,Y)$ then $\mathscr{K}(X,Y)$ is a lattice.*

Definition 3.6. We call $T \in \mathscr{L}(X,Y)$ *strongly non-regular* if T is not in the operator norm closure of the regular operators.

Arendt and Voigt, [10], showed that there are strongly non-regular operators in $\mathscr{L}(L_p)$ provided $1 < p < \infty$ and L_p is infinite dimensional. Their examples are necessarily not order bounded. In [72] Wickstead showed that it is possible even to take X and Y being unital AM-spaces and $T : X \to Y$ being order bounded but strongly non-regular.

Problem 3.7: Can compact operators be strongly non-regular? Note that all finite rank operators are regular, so this would imply that the range space failed to have the approximation property.

The space $\mathscr{L}^r(X,Y)$ of regular operators is not usually complete under the operator norm, but it is complete under the *regular norm*, $\|T\|_r = \inf\{\|U\| : U \geqslant \pm T\}$. Similarly, $\mathscr{K}^r(X,Y)$ is not usually complete under the regular norm, but it is under the k-norm, $\|T\|_k = \inf\{\|U\| : U \geqslant \pm T, U \in K(X,Y)_+\}$, see [18] and [17], where we also show that the regular and k-norms are not often even equivalent to each other.

Problem 3.8: Are there Banach lattices X and Y such that the regular and k-norms are equivalent on $\mathscr{K}^r(X,Y)$ without being equal?

We know that for regular operators in general, it need not be the case that $|T|^* = |T^*|$. To the best of our knowledge this question is still open:

Problem 3.9: If $T \in \mathscr{K}^r(X,Y)$ and T has a modulus, must $|T|^* = |T^*|$? What if $|T|$ itself is compact?

Note that Chen, [16], has given an example to show that $|T^*|$ can be compact even if $|T|$ is not.

Definition 3.10. The r-compact operators in $\mathscr{L}^r(X,Y)$ are those in the regular norm closure of the finite rank operators which will be denoted by $\mathscr{A}^r(X,Y)$.

They are very well behaved and always form a vector lattice. It is always the case that $\mathscr{A}^r(X,Y) \subseteq \mathscr{K}^r(X,Y)$ and in general they are different spaces. These have been studied in [8, 20, 57, 58, 59, 75]. One problem about them seems difficult.

PROBLEM 3.11: If T is r-compact then T^* is certainly r-compact. If $T \in \mathscr{L}(X,Y)$ and $T^* \in \mathscr{A}^r(Y^*,X^*)$ must we have $T \in \mathscr{A}^r(X,Y)$?

There is some evidence to conjecture a positive answer to the next question.

PROBLEM 3.12: Is it true that if X is a Banach lattice such that both X and X^* have an order continuous norm then $\mathscr{A}^r(X) = \mathscr{K}^r(X)$ if and only if X were atomic?

4. Order Structure of Spaces of Compact Operators

Unlike the bounded operator case, where we do not know exactly when $\mathscr{L}(X,Y)$ is a Banach lattice under the operator norm, the situation for compact operators is completely clear. In the following result the proof when Y is an AM-space is due originally to Krengel in [37], where he also obtained a partial proof for the case that X is an AL-space. A complete proof for that case is given in [64]. The converse is due to Cartwright and Lotz in [15].

Theorem 4.1. *The following conditions on a pair of Banach lattices (X,Y) are equivalent:*

(1) *Either X is an AL-space or Y is an AM-space.*

(2) *$\mathscr{K}(X,Y)$ is a Banach lattice under the operator norm.*

Furthermore, in this case the lattice operations are given by the Freudenthal–Kantorovich–Riesz formula.

However, the fact that the operator norm is a Banach lattice norm is crucial here. We saw earlier that $\mathscr{K}(X,Y)$ can be a lattice without either X being isomorphic to an AL-space or Y to an AM-space.

It is probably very hard to answer:

PROBLEM 4.2: For what pairs of Banach lattice X and Y is $\mathscr{K}(X,Y)$ a vector lattice?

What about the smaller space $\mathscr{K}^r(X,Y)$? The *compact domination property* seems to be relevant here.

DEFINITION 4.3. The pair of Banach lattices (X,Y) has the *compact domination property* if whenever $S,T : X \to Y$, $0 \leqslant S \leqslant T$ and $T \in \mathscr{K}(X,Y)$ then $S \in \mathscr{K}(X,Y)$.

Theorem 4.4 (Dodds–Fremlin [23] and Wickstead [71])**.** *If X and Y are Banach lattices then the following are equivalent:*

(1) *One of the following three (non-exclusive) conditions holds:*
- *(a) Both X^* and Y have an order continuous norm.*
- *(b) Y is an atomic Banach lattice with an order continuous norm.*
- *(c) X^* is an atomic Banach lattice with an order continuous norm.*

(2) *If $S,T : X \to Y$, $0 \leqslant S \leqslant T$ and T is compact then S is compact.*

If we ask for rather more than $\mathscr{K}^r(X,Y)$ being a lattice then our results are complete. The following theorem is true when P is taken to mean "Dedekind complete", "Dedekind σ-complete" or "countable interpolation property".

Theorem 4.5. *If X and Y are Banach lattices then the following are equivalent:*

(1) *Y has (P) and the pair (X,Y) has the compact domination property.*

(2) *$\mathscr{K}^r(X,Y)$ is a vector lattice with (P).*

But for just being a lattice the results are incomplete:

Theorem 4.6. *If the pair of Banach lattices (X,Y) has the compact domination property then $\mathscr{K}^r(X,Y)$ is a vector lattice.*

But this does not include the cases where behaviour is nicest, namely when X is an AL-space or Y an AM-space. So we ask:

PROBLEM 4.7: For what pairs of Banach lattices X and Y is $\mathscr{K}^r(X,Y)$ a lattice?

Presumably these should be rather easier questions:

PROBLEM 4.8: For what Banach lattices Y is $\mathscr{K}^r(X,Y)$ a lattice for all Banach lattices X?

PROBLEM 4.9: For what Banach lattices X is $\mathscr{K}^r(X,Y)$ a lattice for all Banach lattices Y?

DEFINITION 4.10. An ordered vector space X has the *Riesz separation property* if given $x_1,x_2,z_1,z_2 \in X$ with $x_1,x_2 \leqslant z_1,z_2$ there is $y \in X$ with $x_1,x_2 \leqslant y \leqslant z_1,z_2$. X is said to have the *Riesz decomposition property* if given $x,y_1,y_2 \in X$ with $0 \leqslant x \leqslant y_1+y_2$ there are $x_1,x_2 \in X$ such that $x = x_1+x_2$, $0 \leqslant x_1 \leqslant y_1$ and $0 \leqslant x_2 \leqslant y_2$.

These two properties are equivalent and are satisfied by any vector lattice.

Although we have examples where $\mathscr{L}^r(X,Y)$ has the RSP without being a lattice, we have no examples in $\mathscr{K}^r(X,Y)$.

PROBLEM 4.11: Either give an example of Banach lattices X and Y such that $\mathscr{K}^r(X,Y)$ has the RSP without being a lattice or prove that such an example cannot exist.

An interesting question of when the space of regular (or compact) operators is itself an AL-space or AM-space was raised by Wickstead in [74]; we present four results from this work.

Theorem 4.12 (Wickstead [74, Theorem 2.1])**.** *If X and Y are Banach lattices, neither of which is the zero space, with Y Dedekind complete then $\mathscr{L}^r(X,Y)$ is an AL-space under the regular norm if and only if X is an AM-space and Y is an AL-space.*

$\lhd$ See Wickstead [74, Theorem 2.1]. $\rhd$

Theorem 4.13. *If Y is a nonzero Dedekind complete Banach lattice then $\mathscr{L}^r(X,Y)$ is an AM-space under the regular norm for every AL-space X if and only if Y is an AM-space with a Fatou norm.*

$\lhd$ See Wickstead [74, Theorem 2.3]. $\rhd$

Theorem 4.14. *If X is a nonzero Banach lattices then $\mathscr{L}^r(X,Y)$ is an AM-space under the regular norm for every Dedekind complete AM-space Y if and only if X is an atomic AL-space.*

$\triangleleft$ See Wickstead [74, Theorem 2.4]. $\triangleright$

Theorem 4.15. *If X and Y are nonzero Banach lattices, then the following hold:*

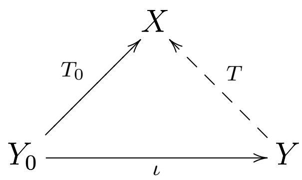
(1) *$\mathscr{K}^r(X,Y)$ is an AL-space under the k-norm if and only if X is an AM-space and Y is an AL-space.*

(2) *$\mathscr{K}^r(X,Y)$ is an AM-space under the k-norm if and only if X is an AL-space and Y is an AM-space.*

$\triangleleft$ See Wickstead [74, Theorem 2.5]. $\triangleright$

5. Injective Banach Lattices

Definition 5.1. A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ with $\|T_0\| = \|T\|$. This definition is illustrated by the commutative ($T_0 = T \circ \iota$) diagram:



Other equivalent conditions are presented in the next result, see Lotz [53].

Theorem 5.2. *For a Banach lattice X the following are equivalent:*

(1) *X is injective.*

(2) *If X is lattice isometrically embedded into a Banach lattice Y and T_0 is a positive linear operator from X to a Banach lattice Z then there exists a positive linear extension $T : Y \to Z$ with $\|T_0\| = \|T\|$.*

(3) *If X is lattice isometrically embedded into a Banach lattice Y then there exists a positive contractive projection from Y onto X.*

Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [9, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms.

Lotz [53] was the first who introduced this concept and proved among other things the following two results. But the first example of injective Banach lattice was indicated by Abramovich [1].

Theorem 5.3 (Abramovich, [1]; Lotz, [53])**.** *A Dedekind complete AM-space with unit is an injective Banach lattice.*

Taking into account the Kakutani–Kreĭn Representation Theorem one can state Theorem 5.3 equivalently: The Banach lattice of continuous functions $C(K)$ is injective, whenever K is an extremally disconnected Hausdorff compact topological space.

Theorem 5.4 (Lotz, [53])**.** *Every AL-space is an injective Banach lattice.*

Theorem 5.4 shows that there is an essential difference between injective Banach lattices and injective Banach spaces, since $C(K)$ with extremally disconnected compactum K is the only (up to isomorphism) injective object in the category of Banach spaces and linear contractions (see Goodner [26], Kelley [36], Nachbin [56]).

Definition 5.5. A separable Banach lattice X is said to be *separably-injective* if for every pair of separable Banach lattices $Y \subset Z$ and every positive linear map from Y to X, there exists a norm preserving positive linear extension from Z to Y.

In [13, Theorem 3] Buskes observed that every separably-injective Banach lattice is injective. Unlike the situation with separable Banach lattices, in the category of separable Banach spaces and bounded linear operators there is a unique (up to isomorphism) infinite dimensional injective object c_0, see Sobczyk [65] and Zippin [76]. More details concerning injective Banach lattices see in Cartwright [14], Gierz [25], Haydon [33], Lotz [53], Mangheni [54], Schaefer [63], Wickstead [73].

A geometric property which enables us to characterize injective Banach lattices was discovered by Cartwright [14].

Definition 5.6. A Banach lattice X has the *splitting property* if, given $x_1, x_2, y \in X_+$ with $\|x_1\| \leqslant 1$, $\|x_2\| \leqslant 1$, and $\|x_1 + x_2 + y\| \leqslant 2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant 1$, and $\|x_2 + y_2\| \leqslant 1$.

Definition 5.7. A Banach lattice X has the *Cartwright property* if, given $x_1, x_2, y \in X_+$ and $0 < r_1, r_2 \in \mathbb{R}$ with $\|x_1\| \leqslant r_1$, $\|x_2\| \leqslant r_2$, and $\|x_1 + x_2 + y\| \leqslant r_1 + r_2$, there exist $y_1, y_2 \in X_+$ such that $y_1 + y_2 = y$, $\|x_1 + y_1\| \leqslant r_1$, and $\|x_2 + y_2\| \leqslant r_2$.

Definition 5.8. A Banach lattice X has the *finite order intersection property* if, given $z \in X_+$, finite collections $x_1, \dots, x_n \in X_+$, $y_1, \dots, y_m \in X_+$, and strictly positive reals $r_1, \dots, r_n \in \mathbb{R}_+$, $s_1, \dots, s_m \in \mathbb{R}_+$ such that $\|x_i\| \leqslant r_i$, $\|y_j\| \leqslant s_j$, and $\|x_i + y_j + z\| \leqslant r_i + s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$, there exist $u, v \in X_+$ with $z = u + v$, $\|x_i + u\| \leqslant r_i$, and $\|y_j + v\| \leqslant s_j$ for all $i := 1, \dots, n$ and $j := 1, \dots, m$.

Theorem 5.9 (Cartwright, [14]). *For a Banach lattice the splitting property, the Cartwright property, and the finite order intersection property are equivalent.*

Theorem 5.10 (Cartwright, [14]). *A Banach lattice has the splitting property if and only if its second dual is injective.*

Gierz in [25, Corollaries 3.3 and 3.4] proved that every Banach lattice with the splitting property (and hence every injective Banach lattice) has the approximation property.

Cartwright [14, Corollary 3.8] proved that a Banach lattice is injective if and only if it has the Cartwright property and the property (P). Haydon demonstrated that the property (P) may be replaced with the intrinsic 'completeness' property.

Theorem 5.11 (Haydon, [33]). *A Banach lattice is injective if and only if it has the Cartwright, Fatou, and Levi properties.*

A crucial role in the structure theory of injective Banach lattices is played by the concept of an M-projection which, in addition to their structure as Banach lattices, determines important peculiar properties. The notion of an M-projection goes back to [6]; in a wider context of a general Banach space theory the concept is presented in [11] and [32].

Definition 5.12. A band projection π in a Banach lattice X is called an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^{\perp}x\|\}$ for all $x \in X$, where $\pi^{\perp} := I_X - \pi$. The set $\mathbb{M}(X)$ of all M-projections in X forms a Boolean subalgebra of $\mathbb{P}(X)$. The f-subalgebra of the center $\mathscr{Z}(X)$ generated by $\mathbb{M}(X)$ is called the *M-center* of X and denoted by $\mathscr{Z}_m(X)$. Clearly, $\mathscr{Z}_m(X) = \mathbb{R} \cdot I_X$ if and only if $\mathbb{M}(X) = \{\mathbb{0}, \mathbb{1}\}$.

Observe that $\mathbb{M}(X)$ is an order closed subalgebra of $\mathbb{P}(X)$ whenever X has the Fatou and Levi properties. In this event the relations $\mathbb{B} \simeq \mathbb{M}(X)$ and $\Lambda(\mathbb{B}) \simeq \mathscr{Z}_m(X)$ are equivalent. Note also that if X is an AL-space and Y is an AM-space then $\mathbb{M}(X) = \{0, I_X\}$ and $\mathbb{M}(Y) = \mathbb{P}(Y)$.

Theorem 5.13 (Haydon, [33]). *An injective Banach lattice X is an AL-space if and only if there is no M-projection in it other than zero and identity, i.e., $\mathbb{M}(X) = \{0, I_X\}$ (or, equivalently, if and only if its M-center is one-dimensional, i.e., $\mathscr{Z}_m(X) = \mathbb{R} \cdot I_X$).*

Definition 5.14. A real Banach lattice X is said to be *λ-injective*, if for every Banach lattice Y, closed sublattice $Y_0 \subset Y$, and positive $T_0 : Y_0 \to X$ there exists a positive extension $T : Y \to X$ with $\|T\| \leqslant \lambda\|T_0\|$.

It was proved in [51] that every finite-dimensional λ-injective Banach lattice is lattice isomorphic to $\big(\sum_{j\leqslant k}^{\oplus} l_1(n_j)\big)_{l_\infty}$, while it was shown in [54] that every order continuous λ-injective Banach lattice is lattice isomorphic to $L_1(\mu)$ space. But the general question is still open:

Problem 5.15: Is every λ-injective Banach lattice order isomorphic to 1-injective Banach lattice?

One of the intriguing problems, dating from the work [28], is the classification of the Banach space whose duals are isometric to an AL-space, see also [52]. We believe that the injective version of this problem deserves an independent study.

Problem 5.16: Classify and characterize the Banach spaces whose duals are injective Banach lattices.

6. Boolean Valued Analysis

The term *Boolean valued analysis,* coined by G. Takeuti (see [66]), signifies the technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, the classical Cantorian paradise in the shape of the *von Neumann universe* $\mathbb{V}$ and a specially-trimmed *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ are usually taken. Recall the following three basic principles of Boolean valued set theory.

Given a complete Boolean algebra $\mathbb{B}$, we can define the universe $\mathbb{V}^{(\mathbb{B})}$ of *$\mathbb{B}$-valued sets* [12, 45, 46]. To speak about $\mathbb{V}^{(\mathbb{B})}$ take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of set theory and replace the variables $u_1, \dots, u_n$ by $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. Then we obtain some statement about $x_1, \dots, x_n$. There is a natural way of assigning to each such statement some element $[\![\varphi(x_1,\dots,x_n)]\!]\in\mathbb{B}$ which acts as the *Boolean truth-value* of $\varphi(u_1,\dots,u_n)$ in the universe $\mathbb{V}^{(\mathbb{B})}$ and is defined by induction on the complexity of φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ (for instance, $[\![\varphi_1\vee\varphi_2]\!]:=[\![\varphi_1]\!]\vee[\![\varphi_2]\!]$ and $[\![\forall x\varphi(x)]\!]=\bigwedge\{[\![\varphi(u)]\!]:\, u\in\mathbb{V}^{(\mathbb{B})}\}$) and assigning the truth-values $[\![x\in y]\!]\in\mathbb{B}$ and $[\![x=y]\!]\in\mathbb{B}$, where $x,y\in\mathbb{V}^{(\mathbb{B})}$. We say that $\varphi(x_1,\dots,x_n)$ is *valid within* $\mathbb{V}^{(\mathbb{B})}$ if $[\![\varphi(x_1,\dots,x_n)]\!]=\mathbb{1}$. In this event, we also write $\mathbb{V}^{(\mathbb{B})}\models\varphi(x_1,\dots,x_n)$.

There is a smooth mathematical toolkit, the *ascending-and-descending technique* for revealing interplay between the interpretations of one and the same fact in the two universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

Definition 6.1. Given an arbitrary element X of the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $X{\downarrow}\in\mathbb{V}$ of X as $X{\downarrow}:=\{y\in\mathbb{V}^{(\mathbb{B})}:\ [\![y\in x]\!]=\mathbb{1}\}$.

Definition 6.2. Let $X\in\mathbb{V}$ and $X\subset\mathbb{V}^{(\mathbb{B})}$; i. e., let X be some set composed of $\mathbb{B}$-valued sets. There exists a unique $X{\uparrow}\in\mathbb{V}^{(\mathbb{B})}$ such that $[\![y\in X{\uparrow}]\!]=\bigvee\{[\![x=y]\!]:\, x\in X\}$ for all $y\in\mathbb{V}^{(\mathbb{B})}$. The element $X{\uparrow}$ is called the *ascent* of X.

If $X,Y,f,P\in\mathbb{V}^{(\mathbb{B})}$ are such that $[\![X\neq\varnothing]\!]=[\![X\neq\varnothing]\!]=\mathbb{1}$, $[\![f:X\to Y]\!]=\mathbb{1}$, and $[\![P\subset X\times Y]\!]=\mathbb{1}$ then $X{\downarrow}$ and $Y{\downarrow}$ are nonempty sets, $f{\downarrow}$ is a mapping from $X{\downarrow}$ to $Y{\downarrow}$, and $P{\downarrow}$ is a relation on $X{\downarrow}$. Similar assertion is true for ascents.

Definition 6.3. Given $X\in\mathbb{V}$, we define the *standard name* $X^{\wedge}\in\mathbb{V}^{(\mathbb{B})}$ of X by recursion on the well-founded relation $x\in X$: $X^{\wedge}$ is the ascent of $\{x^{\wedge}:\, x\in X\}$.

The standard name mapping $X\mapsto X^{\wedge}$ is an embedding of $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$. Moreover, the standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(2)}$, i. e., $\mathbb{V}\simeq\mathbb{V}^{(2)}\subset\mathbb{V}^{(\mathbb{B})}$, where $2:=\{\mathbb{O},\mathbb{1}\}\subset\mathbb{B}$.

A general scheme of applying the Boolean valued approach is as follows, see [46, 47]. Assume that $\mathbf{X}\subset\mathbb{V}$ and $\mathbb{X}\subset\mathbb{V}^{(\mathbb{B})}$ are two classes of mathematical objects, external and internal, respectively. Suppose we are able to prove the following

Boolean Valued Representation Result: Every external $X\in\mathbf{X}$ embeds into an Boolean valued model, becoming an internal object $\mathscr{X}\in\mathbb{X}$ within $\mathbb{V}^{(\mathbb{B})}$.

Boolean Valued Transfer Principle then tells us that every theorem about $\mathscr{X}$ within ZFC has its counterpart for the original object X interpreted as a Boolean valued object $\mathscr{X}$.

Boolean Valued Machinery enables us to perform some translation of theorems from $\mathscr{X}\in\mathbb{V}^{(\mathbb{B})}$ to $X\in\mathbb{V}$ making use of appropriate general operations (ascending-descending) and the following principles of Boolean valued analysis.

Theorem 6.4 (Transfer Principle). *For every theorem φ of ZFC, we have $[\![\varphi]\!]=\mathbb{1}$ (also in ZFC); i. e., φ is true within the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$.*

Theorem 6.5 (Maximum Principle). *Let $\varphi(x)$ be a formula of ZFC. Then (in ZFC) there is a $\mathbb{B}$-valued set x_0 satisfying $[\![(\exists\, x)\varphi(x)]\!]=[\![\varphi(x_0)]\!]$.*

Definition 6.6. A ZFC formula is called *restricted* provided that each of its quantifiers occurs in the form $(\forall\, x\in y)$ or $(\exists\, x\in y)$ (i. e. $(\forall\, x)(x\in y\to\cdots)$ or $(\exists\, x)(x\in y\wedge\cdots)$) or if it can be proved equivalent in ZFC to a formula of this kind.

Theorem 6.7 (Restricted Transfer Principle). *Let $\varphi(x_1, \dots, x_n)$ be a bounded formula of ZFC. Then (in ZFC) for all $x_1, \dots, x_n \in \mathbb{V}$ we have*

$$\varphi(x_1, \dots, x_n) \iff \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

Recall the well-known assertion of ZFC: *There exists a field of reals that is unique up to isomorphism.* Denote by $\mathbb{R}$ the field of reals (in the sense of $\mathbb{V}$). Successively applying the transfer and maximum principles, we find an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals $]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ satisfies the condition $[\![\, \mathscr{R}'$ is a field of reals $]\!] = \mathbb{1}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$. In other words, there exists an internal field of reals $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism.

Definition 6.8. We call $\mathscr{R}$ the *internal reals* in $\mathbb{V}^{(\mathbb{B})}$.

Consider another well-known assertion of ZFC: *If $\mathbb{P}$ is an Archimedean ordered field then there is an isomorphic embedding h of the field $\mathbb{P}$ into $\mathbb{R}$ such that the image $h(\mathbb{P})$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(\mathbb{P})$ is dense in $\mathbb{R}$.*

Note also that $\varphi(\cdot)$, presenting the conjunction of the axioms of an Archimedean ordered field, is bounded; therefore, $[\![\, \varphi(\mathbb{R}^\wedge)]\!] = \mathbb{1}$ by the Restricted Transfer Principle, i. e., $[\![\, \mathbb{R}^\wedge$ is an Archimedean ordered field $]\!] = \mathbb{1}$. "Pulling" the above assertion through the transfer principle, we conclude that $[\![\, \mathbb{R}^\wedge$ is isomorphic to a dense subfield of $\mathscr{R}]\!] = \mathbb{1}$. We further assume that $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$. It is easy to see that the elements $0^\wedge$ and $1^\wedge$ are the zero and unity of $\mathscr{R}$.

Definition 6.9. The *descent* $\mathbf{R} := \mathscr{R}{\downarrow} := (\mathrm{R}{\downarrow}, \oplus{\downarrow}, \odot{\downarrow}, \leqq{\downarrow}, 0, 1)$ of the algebraic structure $\mathscr{R} := (\mathrm{R}, \oplus, \odot, \leqq, 0, 1)$ is defined as the descent $\mathrm{R}{\downarrow}$ of the underlying set R equipped with the descended operations $\oplus{\downarrow}$ and $\odot{\downarrow}$ and order $\leqq{\downarrow}$ of the structure $\mathscr{R}$. For simplicity, we will denote the operations and order in $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\leqslant$.

The fundamental result of Boolean valued analysis is the Gordon Theorem which describes an interplay between $\mathbb{R}$, $\mathscr{R}$, and $\mathbf{R}$ and reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* In more detail:

Theorem 6.10 (Gordon, [27]). *Let $\mathscr{R}$ be a field of reals in $\mathbb{V}^{(\mathbb{B})}$ and $\mathbf{R} = \mathscr{R}{\downarrow}$. Then the following assertions hold:*

(1) *The internal field $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that*

$$[\![\, \mathbb{R}^\wedge \text{ is a dense subfield of the field } \mathscr{R} \,]\!] = \mathbb{1}.$$

(2) *The algebraic structure $\mathbf{R}$ (with the descended operations and order) is an universally complete vector lattice.*

(3) *There is a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}(\mathbf{R})$ such that*

$$\chi(b)x = \chi(b)y \iff b \leqslant [\![\, x = y \,]\!],$$
$$\chi(b)x \leqslant \chi(b)y \iff b \leqslant [\![\, x \leqslant y \,]\!]$$
$$(x, y \in \mathbf{R};\ b \in \mathbb{B}).$$

Definition 6.11. The *restricted descent* $\Lambda \subset \mathbf{R} = \mathscr{R}{\downarrow}$ of $\mathscr{R}{\downarrow}$ is the order ideal in $\mathbf{R}$ generated by $1^\wedge$ equipped with the order-unit norm $\|\cdot\|_\infty$:

$$\Lambda := \{x \in \mathbf{R} : (\exists C \in \mathbb{R}_+) \ -C1^\wedge \leqslant x \leqslant C1^\wedge\};$$
$$\|x\|_\infty := \inf\{0 < C \in \mathbb{R} : -C1^\wedge \leqslant x \leqslant C1^\wedge\} \quad (x \in \Lambda).$$

Write $\Lambda = \Lambda(\mathbb{B})$, since Λ is uniquely defined by $\mathbb{B}$. Clearly, Λ is a Dedekind complete AM-space with unit $1^\wedge$. By Kakutani–Kreĭn Representation Theorem $\Lambda \simeq C(K)$ with K being an extremally disconnected compact Hausdorff space.

Theorem 6.12 (Gordon's Theorem for complexes). *Each complex universally complete vector lattice is an interpretation of the complexes in an appropriate Boolean valued model. In more detail, if* $\mathscr{C}$ *is the field of complex numbers within* $\mathbb{V}^{(\mathbb{B})}$ *then* $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$.

7. Boolean Valued Banach Lattices

In this section we present some Boolean valued representation results for Banach lattices needed in the sequel. Assume that X is a Banach lattice and $\mathscr{B}$ is a complete subalgebra of a complete Boolean algebra $\mathbb{B}(X)$ consisting of projection bands and denote by $\mathscr{B}'$ the corresponding Boolean algebra of band projections. If a Boolean algebra $\mathbb{B}$ is isomorphic to $\mathscr{B}'$ then we will identify the Boolean algebras $\mathscr{B}'$ and $\mathbb{B}$, writing $\mathbb{B} \subset \mathbb{P}(X)$. We also will identify $\mathbb{P}(\Lambda)$ and $\mathbb{B}$.

Definition 7.1. If $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$ and $(x_\xi)_{\xi\in\Xi}$ is a family in X, then there is at most one element $x \in X$ with $b_\xi x_\xi = b_\xi x$ for all $\xi \in \Xi$. This element x, if existing, is called the *mixing* of (x_ξ) by (b_ξ). Clearly, $x = o\text{-}\sum_{\xi\in\Xi} b_\xi x_\xi$. A Banach lattice X is said to be $\mathbb{B}$-*cyclic* or $\mathbb{B}$-*complete* if the mixing of every family in the unit ball $U(X)$ of X by each partition of unity in $\mathbb{B}$ (with the same index set) exists in $U(X)$.

A Banach lattice $(X, \|\cdot\|)$ is $\mathbb{B}$-cyclic with respect to a complete Boolean algebra $\mathbb{B}$ of band projections on X if and only if there exists a $\Lambda(\mathbb{B})$-valued norm $|\!\cdot\!|$ on X such that $(X, |\!\cdot\!|)$ is a Banach-Kantorovich space, $|x| \leqslant |y|$ implies $|\!x\!| \leqslant |\!y\!|$ for all $x, y \in X$, and $\|x\| = \||\!x\!|\|_\infty$ $(x \in X)$, see Kusraev and Kutateladze [47, Theorems 5.8.11 and 5.9.1].

Definition 7.2. Let X and Y be Banach spaces with $\mathbb{B} \subset \mathscr{L}(X)$ and $\mathbb{B} \subset \mathscr{L}(Y)$. An operator $T : X \to Y$ is called $\mathbb{B}$-*linear*, if it is linear and commutes with all projections from $\mathbb{B}$, i. e., if $b{\circ}T = T{\circ}b$. (Here, of course, we mean $\varphi_Y(b){\circ}T = T{\circ}\varphi_X(b)$ with $\varphi_X : \mathbb{B} \to \mathbb{B}(X)$ and $\varphi_Y : \mathbb{B} \to \mathbb{B}(Y)$ being Boolean isomorphism.) A bijective $\mathbb{B}$-linear operator is called a $\mathbb{B}$-*isomorphism* and an isometric $\mathbb{B}$-*isomorphism* is called a $\mathbb{B}$-*isometry.* A $\mathbb{B}$-isometric lattice homomorphism is referred to as *lattice* $\mathbb{B}$-*isometry.*

Let $\mathscr{L}_{\mathbb{B}}(X, Y)$ stands for the set of all bounded $\mathbb{B}$-linear operators from X into Y. Clearly $\mathscr{L}_{\mathbb{B}}(X, Y)$ is a $\mathbb{B}$-cyclic Banach space whenever Y is.

Definition 7.3. Denote by $X^\# := \mathscr{L}_{\mathbb{B}}(X, \Lambda)$, where $\Lambda = \Lambda(\mathbb{B})$, the $\mathbb{B}$-*dual* to X.

Now we are able to answer the question: *What kind of category is produced by applying the descending procedure to the category of Banach lattices within* $\mathbb{V}^{(\mathbb{B})}$*?* The answer is given in the following two results.

Let $(\mathscr{X}, \|\cdot\|)$ be a Banach lattice within $\mathbb{V}^{(\mathbb{B})}$. Define the mapping N from $\mathscr{X}{\downarrow}$ to $\mathbf{R}_+ := \mathscr{R}{\downarrow}_+$ as the descent $N(\cdot) := (\|\cdot\|){\downarrow}$ of the norm $\|\cdot\|$. Then $\mathscr{X}{\downarrow}$ (with the descended operations and order) is a vector lattice (and even an $\mathbf{R}$-module) and N is an $\mathbf{R}$-valued norm on $\mathscr{X}{\downarrow}$ (i. e., $N(x) = 0 \iff x = 0$, $N(x+y) \leqslant N(x) + N(y)$, $N(\lambda x) = \lambda N(x)$ for all $x, y \in \mathscr{X}{\downarrow}$ and $\lambda \in \mathbf{R}_+$).

Definition 7.4. The *bounded descent* $\mathscr{X}{\Downarrow}$ of $\mathscr{X}$ is defined as the set

$$\mathscr{X}{\Downarrow} := \{x \in \mathscr{X}{\downarrow} : \ N(x) \in \Lambda\}$$

equipped with the descended operations, order relation and *mixed norm*:

$$|||x||| := \|N(x)\|_\infty \quad (x \in \mathscr{X}{\Downarrow}).$$

Theorem 7.5. *A restricted descent of a Banach lattice from the model* $\mathbb{V}^{(\mathbb{B})}$ *is a* $\mathbb{B}$*-cyclic Banach lattice. Conversely, if* X *is a* $\mathbb{B}$*-cyclic Banach lattice, then in the model* $\mathbb{V}^{(\mathbb{B})}$ *there exists up to the lattice isometry a unique Banach lattice* $\mathscr{X}$ *whose restricted descent* $\mathscr{X}{\Downarrow}$ *is isometrically* $\mathbb{B}$*-isomorphic to* X*. Moreover,* $\mathbb{B} = \mathbb{M}(X)$ *if and only if* $[\![$*there is no* M*-projection in* $\mathscr{X}$ *other than* 0 *and* $I_{\mathscr{X}}]\!] = \mathbb{1}$.

$\lhd$ See Kusraev and Kutateladze [47, Theorem 5.9.1]. $\rhd$

Definition 7.6. The elements $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ in Theorem 7.5 and $\mathscr{T} \in \mathbb{V}^{(\mathbb{B})}$ in Theorem 7.7 below are said to be the *Boolean valued representations of* X and T, respectively.

Denote by $\mathscr{L}^r_{\mathbb{B}}(X, Y)$ the space of all regular $\mathbb{B}$-linear operators from X to Y equipped with the *regular norm* $\|T\|_r := \inf\{\|S\| : S \in \mathscr{L}_{\mathbb{B}}(X, Y), \pm T \leqslant S\}$. Let $\mathscr{X}$ and $\mathscr{Y}$ be the Boolean valued representations of $\mathbb{B}$-cyclic Banach lattices X and Y, respectively, while $\mathscr{L}^r(\mathscr{X}, \mathscr{Y})$ stands for the space of all regular operators from $\mathscr{X}$ to $\mathscr{Y}$ with the regular norm within $\mathbb{V}^{(\mathbb{B})}$. The following result states that $\mathscr{L}^r(\mathscr{X}, \mathscr{Y})$ is the *Boolean valued representation of* $\mathscr{L}^r_{\mathbb{B}}(X, Y)$.

Theorem 7.7. *Assume that* X *and* Y *are* $\mathbb{B}$*-cyclic Banach lattices, while* $\mathscr{X}$ *and* $\mathscr{Y}$ *are their respective Boolean valued representations. The space* $\mathscr{L}^r_{\mathbb{B}}(X, Y)$ *is order* $\mathbb{B}$*-isometric to the bounded descent* $\mathscr{L}^r(\mathscr{X}, \mathscr{Y}){\Downarrow}$ *of* $\mathscr{L}^r(\mathscr{X}, \mathscr{Y})$*. The* $\mathbb{B}$*-isometry is set up by assigning to any* $T \in \mathscr{L}^r_{\mathbb{B}}(X, Y)$ *the element* $\mathscr{T} := T{\uparrow}$ *of* $\mathbb{V}^{(\mathbb{B})}$ *is uniquely determined from the formulas* $[\![\, \mathscr{T} : \mathscr{X} \to \mathscr{Y} \,]\!] = \mathbb{1}$ *and* $[\![\, \mathscr{T}x = Tx \,]\!] = \mathbb{1}$ $(x \in X)$.

$\lhd$ Observe that $\mathscr{L}_{\mathbb{B}}(X, Y)$ and $\mathscr{L}(\mathscr{X}, \mathscr{Y}){\Downarrow}$ are $\mathbb{B}$-isometric by [38, Theorem 8.3.6]. Since $T(X_+){\uparrow} = T{\uparrow}(X_+{\uparrow}) = \mathscr{T}(\mathscr{X}_+)$, it follows that $T(X_+) \subset Y_+$ if and only if $[\![\mathscr{T}(\mathscr{X}_+) \subset \mathscr{Y}_+]\!] = \mathbb{1}$. This means that the bijection $T \leftrightarrow \mathscr{T} = T{\uparrow}$ preserves positivity and hence is an order $\mathbb{B}$-isomorphism between $\mathscr{L}^r_{\mathbb{B}}(X, Y)$ and $\mathscr{L}^r(\mathscr{X}, \mathscr{Y}){\Downarrow}$. Since for $S \in \mathscr{L}^r_{\mathbb{B}}(X, Y)$ and $\mathscr{S} := S{\uparrow}$ the relations $\pm T \leqslant S$ and $[\![\pm\mathscr{T} \leqslant \mathscr{S}]\!] = \mathbb{1}$ are equivalent, we have $[\![\|\mathscr{T}\|_r = |T|_r]\!] = \mathbb{1}$, where $|T|_r = \inf\{|S| : S \in \mathscr{L}^r_{\mathbb{B}}(X, Y), \pm T \leqslant S\}$ and $|S| := \sup\{|Sx| : |x| \leqslant \mathbb{1}\}$. Thus, it remains to prove that $\|T\|_r = \||T|_r\|_\infty$ $(T \in \mathscr{L}^r_{\mathbb{B}}(X, Y))$.

If $\pm T \leqslant S$ then $\||T|\|_\infty \leqslant \||S|\|_\infty = \|S\|$ and hence $\|T\|_r \geqslant \||T|_r\|_\infty$. To prove the reverse inequality take an arbitrary $0 < \varepsilon \in \mathbb{R}$ and choose a partition of unity $(\pi_\xi)_{\xi\in\Xi}$ in $\mathbb{B}$ and a family $(S_\xi)_{\xi\in\Xi}$ in $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ such that $S_\xi \geqslant \pm T$ and $\pi_\xi|S_\xi| \leqslant (1+\varepsilon)|T|_r$ for all $\xi \in \Xi$. Define an operator $S \in \mathscr{L}^r_{\mathbb{B}}(X,Y)$ by $Sx := \operatorname{mix}_{\xi\in\Xi} \pi_\xi S_\xi x$ $(x \in X)$, where the mixing exists in Y, since $|S_\xi x| \leqslant (1+\varepsilon)|T|_r|x|$ and hence $(S_\xi x)$ is norm bounded in Y. Moreover, $Sx = \sum_\xi \pi_\xi S_\xi x$ in the sense of Λ-valued norm on Y. Therefore, $S \geqslant \pm T$ and $|S| \leqslant (1+\varepsilon)|T|_r$, whence $\|T\|_r \leqslant \|S\| = \||S|\|_\infty \leqslant (1+\varepsilon)\||T|_r\|_\infty$. $\rhd$

Theorem 7.8. *Let X be a $\mathbb{B}$-cyclic Banach lattice and let $\mathscr{X}$ be its Boolean valued representation in $\mathbb{V}^{(\mathbb{B})}$. Then the following hold:*

(1) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is Dedekind complete" if and only if X is Dedekind complete.*

(2) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is Fatou (Levi)" if and only if X is Fatou (Levi).*

(3) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is order continuous" if and only if X is order $\mathbb{B}$-continuous.*

(4) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is a KB-space" if and only if X is order $\mathbb{B}$-continuous and Levi.*

(5) $\mathbb{V}^{(\mathbb{B})} \models$ *"$\mathscr{X}$ is an AM-space" if and only if X is an AM-space.*

$\lhd$ See Kusraev and Kutateladze [47, Theorems 5.9.6 and 5.12.1 (2)]. $\rhd$

Now, we describe a Boolean valued analysis approach to the theory of injective Banach lattices developed in [39, 41, 42]. First we clarify what the Boolean valued representation of an injective Banach lattice is, see [42, Theorem 4.1].

Theorem 7.9. *Suppose that X is a $\mathbb{B}$-cyclic Banach lattice and $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is its Boolean valued representation. Then the following assertions hold:*

(1) *X is injective if and only if $[\![\mathscr{X}$ is injective$]\!] = \mathbb{1}$.*

(2) *X is injective and $\mathbb{B} \simeq \mathbb{M}(X)$ if and only if $[\![\mathscr{X}$ is injective and $\mathbb{M}(\mathscr{X}) = \{0, I_{\mathscr{X}}\}]\!] = \mathbb{1}$.*

$\lhd$ See Kusraev and Kutateladze [47, Theorem 5.12.1 (1, 3)]. $\rhd$

Theorem 7.10 (Haydon, [33]). *Let X is an injective Banach space. Then X is an AL-space if and only if $\mathbb{M}(X) = \{0, I_X\}$.*

Now, putting together Theorems 7.5, 7.9, and 7.10 we arrive at our main representation theorem for injectives, see [39, Theorem 1] and [42, Theorem 4.4].

Theorem 7.11. *A bounded descent $\mathscr{X}{\Downarrow}$ of an AL-space $\mathscr{X}$ from $\mathbb{V}^{(\mathbb{B})}$ is an injective Banach lattice with $\mathbb{B} \simeq \mathbb{M}(\mathscr{X}{\Downarrow})$. Conversely, if X is an injective Banach lattice and $\mathbb{B} \simeq \mathbb{M}(X)$, then there exist an AL-space $\mathscr{X}$ in $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is lattice $\mathbb{B}$-isometric to X; in symbols, $X \simeq_{\mathbb{B}} \mathscr{X}{\Downarrow}$.*

Theorem 7.11 implies the *transfer principles* from *AL*-spaces to injective Banach spaces which can be stated as follows:

(1) Every injective Banach lattice embeds into an appropriate Boolean valued model, becoming an *AL*-space (Theorem 7.11).

(2) Each theorem about the *AL*-space within Zermelo–Fraenkel set theory with choice has its counterpart for the original injective Banach lattice interpreted as a Boolean valued *AL*-space (Boolean valued Transfer Principe, Theorem 6.4).

(3) Translation of theorems from AL-spaces to injective Banach lattices is carried out by general operations and principles of Boolean valued analysis (outlined at the beginning of Sections 6).

The following important representation result (see [42, Corollary 4.5] and [39, Theorem 2]) which do not involve the concept of Boolean valued model can deduce immediately from Theorem 7.11. Before stating this result, recall some definitions.

Definition 7.12. A positive operator $T : X \to Y$ between vector lattices is said to: (1) be a *Maharam operator* whenever it is an order continuous and order interval preserving, i. e., $T([0,x]) \subset [0,Tx]$ for all $x \in X_+$; (2) have the *Levi property* if $\sup x_\alpha$ exists in Y for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y; (3) be *strictly positive* if $Tx = 0$ implies $x = 0$ for all $x \in X_+$.

If $Y = \Lambda$ and T is strictly positive then $L^1(T)$ denotes the domain of T endowed with the norm $|||x||| = \|T(|x|)\|_\infty$ $(x \in L^1(T))$, see Definition 7.4.

Theorem 7.13. *If T is a strictly positive Maharam operator with the Levi property taking values in a Dedekind complete AM-space Λ with unit, then $(L^1(\Phi), |||\cdot|||)$ is an injective Banach lattice with $\mathbb{M}(L^1(\Phi)) \simeq \mathbb{P}(\Lambda)$.*

Conversely, any injective Banach lattice X is lattice $\mathbb{B}$-isometric to $(L^1(T), |||\cdot|||)$ for some strictly positive Maharam operator T with the Levi property taking values in a Dedekind complete AM-space Λ with unit, where $\mathbb{B} = \mathbb{M}(L^1(T)) \simeq \mathbb{P}(\Lambda)$.

Haydon proved three representation theorems for injective Banach lattices, see [33, Theorems 5C, 6H, and 7B]. These results may be also deduced from the above representation theorem (see Theorems 7.11 and 7.13 and [42, Remark 4.13]). An alternative approach relies upon Gutman's theory of bundle representation of lattice normed spaces developed in [29, 30].

As is seen from Theorem 7.13, an arbitrary injective Banach lattice X has a mixed L-M-structure. Thus, the dual X' and the $\mathbb{B}$-dual $X^{\#}$ should have, in a sense, an M-L-structure. Hence a natural question arises:

Problem 7.14: What kind of duality theory is there for injectives?

Definition 7.15. An *orthogonally additive convex modular* [34, § 3.3] on a vector lattice X is an operator $\Theta : X \to \Lambda$ satisfying (for all $x, y \in X$ and $a \in [0, \mathbb{1}]$): (1) $\Theta(x) = 0 \Longleftrightarrow x = 0$; (2) $|x| \leqslant |y| \Longrightarrow \Theta(x) \leqslant \Theta(y)$; (3) $\Theta(ax + (\mathbb{1} - a)y) \leqslant a\Theta(x) + (ax + (\mathbb{1} - a)\Theta(y)$; (4) $|x| \wedge |y| = 0 \Longrightarrow \Theta(x + y) = \Theta(x) + \Theta(y)$.

Definition 7.16. Say that an orthogonally additive convex modular $\Theta : X \to \Lambda$ *factors* through injective Banach lattice L, if $\Theta = \Phi \circ \theta$ for a strictly positive Maharam operator $\Phi : L \to \Lambda$ with the Levi property and an orthogonally additive (nonlinear) embedding $\theta : X \to L$.

Problem 7.17: Find conditions under which an orthogonally additive convex modular admits factorization through injective Banach lattice.

Definition 7.18. An *Orlicz $\mathbb{B}$-lattice* is a $\mathbb{B}$-cyclic Banach lattice X (cf. [34, § 3.3] if there is an orthogonally additive convex modular $\Theta : X \to \Lambda = \Lambda(\mathbb{B})$ with $\|x\| = \inf\{\alpha > 0 : \ \Theta(x/\alpha) \leqslant \mathbb{1}\}$ $(x \in X)$.

Problem 7.19: Prove a representation theorem for Orlicz $\mathbb{B}$-lattices making use of the above representation of injective Banach lattices (Theorem 7.13).

8. Operators on Injective Banach Latices

By Boolean valued transfer principle all the theorems in Sections 2–4 are true within each Boolean valued model of set theory. Now we are going to produce new results by externalization of these internal facts. Below $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{T}$ stand for Boolean valued representations of X, Y and T, respectively.

We consider first the question under which conditions the space of regular $\mathbb{B}$-linear operators between $\mathbb{B}$-cyclic Banach lattices is itself an injective Banach lattice or an AM-space.

Theorem 8.1. *Let X and Y be $\mathbb{B}$-cyclic Banach lattices with Y Dedekind complete. Then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an injective Banach lattice under the regular norm with $\mathbb{B} \simeq \mathbb{M}(\mathscr{L}^r_{\mathbb{B}}(X,Y))$ if and only if X is an AM-space and Y is an injective Banach lattice with $\mathbb{B} = \mathbb{M}(Y)$.*

$\lhd$ This is a Boolean valued interpretation of Theorem 4.12. In view of Theorems 7.7 and 7.11 $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an injective Banach lattice under the regular norm with $\mathbb{B}(\mathscr{L}^r_{\mathbb{B}}(X,Y))$ isomorphic to $\mathbb{B}$ if and only if $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AL-space under the regular norm within $\mathbb{V}^{(\mathbb{B})}$. Theorem 4.12 (applicable by Theorem 7.8 (1)) tells us that the latter is equivalent to saying that $\mathscr{X}$ is an AM-space and $\mathscr{Y}$ is an AL-space. It remains to refer again to Theorems 7.8 (5) and 7.11. $\rhd$

Theorem 8.2. *Let Y be a nonzero $\mathbb{B}$-cyclic Dedekind complete Banach lattices. Then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an AM-space under the regular norm with $\mathbb{M}(\mathscr{L}^r_{\mathbb{B}}(X,Y)) \simeq \mathbb{B}$ for every injective Banach lattice X with $\mathbb{B} = \mathbb{M}(X)$ if and only if Y is an AM-space with a Fatou norm.*

$\lhd$ The proof is similar to that of Theorem 8.1: Theorem 4.13 is true within $\mathbb{V}^{(\mathbb{B})}$ and hence $\mathscr{L}^r(\mathscr{X},\mathscr{Y})$ is an AM-space under the regular norm for every AL-space $\mathscr{X}$ if and only if $\mathscr{Y}$ is an AM-space with a Fatou norm. Moreover, Y has the Fatou norm if and only if $[\![\mathscr{Y}$ has the Fatou norm$]\!] = \mathbb{1}$, see [47, Theorem 5.9.6 (2)]. Now, combining Theorems 7.7 and 7.8 completes the proof. $\rhd$

Definition 8.3. A positive element x of a $\mathbb{B}$-cyclic Banach lattice X is said to be $\mathbb{B}$-*indecomposable* or a $\mathbb{B}$-*atom* if for any pair of disjoint elements $x, y \in X_+$ with $y + z \leqslant x$ there exists a projection $\pi \in \mathbb{B}$ such that $\pi y = 0$ and $\pi^{\perp} z = 0$, while X is called $\mathbb{B}$-*atomic* if the only element of X disjoint from every $\mathbb{B}$-atom is the zero element.

Theorem 8.4. *If X is a $\mathbb{B}$-cyclic Banach lattices then $\mathscr{L}^r_{\mathbb{B}}(X,Y)$ is an AM-space under the regular norm for every Dedekind complete $\mathbb{B}$-cyclyc AM-space Y if and only if X is a $\mathbb{B}$-atomic injective Banach lattice with $\mathbb{B} \simeq \mathbb{M}(X)$.*

$\lhd$ The proof is similar to that used above involving Theorem 4.14. $\rhd$

It is easy to observe that a $\mathbb{B}$-cyclic Banach lattice X is atomic with respect to its natural module structure over the ring $\mathscr{Z}_m(X)$, see Definition 5.12. Representation and classification of $\mathbb{B}$-atomic injective Banach lattices can be found in Kusraev [43].

Combining the notions of mixing (Definition 7.1) and compactness yields the following concept of mix-compactness (or cyclical compactness) and the corresponding class of linear operators.

Definition 8.5. Denote by $\mathrm{Prt}(\mathbb{B})$ (respectively, $\mathrm{Prt}_\sigma(\mathbb{B})$) the set of all partitions (respectively, countable partitions) of unity in $\mathbb{B}$. A set U in X is said to be mix-*complete* if, for all $(\pi_\xi)_{\xi\in\Xi} \in \mathrm{Prt}(\mathbb{B})$ and $(u_\xi)_{\xi\in\Xi} \subset U$, there is $u \in U$ such that $u = \mathrm{mix}_{\xi\in\Xi}\,\pi_\xi u_\xi$. Suppose that X is a $\mathbb{B}$-cyclic Banach lattice, $(x_n)_{n\in\mathbb{N}} \subset X$, and $x \in X$. Say that a *sequence* $(x_n)_{n\in\mathbb{N}}$ $\mathbb{B}$-*approximates* x if, for each $k \in \mathbb{N}$, we have $\inf\{\sup_{n\geqslant k}\|\pi_n(x_n - x)\| :\ (\pi_n)_{n\geqslant k} \in \mathrm{Prt}_\sigma(\mathbb{B})\} = 0$. Call a set $K \subset X$ mix-*compact* if K is mix-complete and for every sequence $(x_n)_{n\in\mathbb{N}} \subset K$ there is $x \in K$ such that $(x_n)_{n\in\mathbb{N}}$ $\mathbb{B}$-approximates x.

It can easily be checked that whenever $\|\cdot\|$ is defines as $\|x\| = \||x|\|_\infty$ $(x \in X)$ with a $\Lambda(\mathbb{B})$-valued norm $|\cdot|$, then a sequence $(x_n)_{n\in\mathbb{N}}$ in X $\mathbb{B}$-approximates x if and only if $\inf_{n\geqslant k}|x_n - x| = 0$ for all $k \in \mathbb{N}$.

Definition 8.6. An operator from a Banach space into a $\mathbb{B}$-cyclic Banach lattice (space) is called *cyclically compact* or *mix-compact* if the image of any bounded subset is contained in a cyclically compact set.

It is clear that in case $E = \mathbb{R}$ mix-compactness is equivalent to compactness in the norm topology. Note also that the concept of mix-compactness in Gutman and Lisovskaya [31] coincides with that of cyclically compactness introduced by Kusraev [38], see [31, Theorem 3.4] and [47, Proposition 2.12.C.5].

Given $\mathbb{B}$-cyclic Banach lattices X and Y, denote by $\mathscr{K}_\mathbb{B}(X,Y)$ the space of $\mathbb{B}$-linear mix-compact operators from X to Y and let $\mathscr{K}_\mathbb{B}^r(X,Y)$ stands for the linear span of positive $\mathbb{B}$-linear mix-compact operators in $\mathscr{K}_\mathbb{B}(X,Y)$, see [38, 8.5.5]. The latter is a Banach lattice under the k-norm defined as

$$\|T\|_k := \inf\{\|S\| :\ \pm T \leqslant S \in \mathscr{K}_\mathbb{B}^r(X,Y)\}.$$

Clearly, $\mathscr{K}(X,Y) = \mathscr{K}_\mathbb{B}(X,Y)$ and $\mathscr{K}^r(X,Y) = \mathscr{K}_\mathbb{B}^r(X,Y)$, if $\mathbb{B} = \{0, \mathbb{1}\}$, cp. [74].

Theorem 8.7. *Let X and Y be $\mathbb{B}$-cyclic Banach lattices. Then $\mathscr{K}_\mathbb{B}^r(X,Y)$ is an injective Banach lattice under the k-norm with $\mathbb{M}(\mathscr{L}_\mathbb{B}^r(X,Y)) \simeq \mathbb{B}$ if and only if X is an AM-space and Y is an injective Banach lattice with $\mathbb{M}(Y) \simeq \mathbb{B}$.*

$\lhd$ The proof runs along the same lines interpreting Theorem 4.15 within Boolean valued model. We have only to observe that an operator $T \in \mathscr{L}_\mathbb{B}^r(X,Y)$ is mix-compact if and only if $[\![\, \mathscr{T} = T\!\uparrow$ is a compact linear operator from $\mathscr{X}$ into $\mathscr{Y}\,]\!] = \mathbb{1}$, see [38, Proposition 8.5.5 (1)]. Thus the $\mathbb{B}$-isometry between $\mathscr{L}_\mathbb{B}^r(X,Y)$ and $\mathscr{L}^r(\mathscr{X},\mathscr{Y})\!\Downarrow$ induces a $\mathbb{B}$-isometry between $\mathscr{K}_\mathbb{B}^r(X,Y)$ and $\mathscr{K}^r(\mathscr{X},\mathscr{Y})\!\Downarrow$. $\rhd$

Theorem 8.8. *The following conditions on a $\mathbb{B}$-cyclic Banach lattice X are equivalent:*

(1) *X is isomorphic to an injective Banach lattice.*

(2) *For every $\mathbb{B}$-cyclic Banach lattices Y, every mix-compact $\mathbb{B}$-linear operator from X into Y is regular.*

(3) *For every $\mathbb{B}$-cyclic Banach lattice Y, $\mathscr{K}_\mathbb{B}(X,Y)$ is a vector lattice.*

$\lhd$ This is a Boolean valued interpretation of Theorem 3.2. $\rhd$

Theorem 8.9. *The following conditions on a pair of $\mathbb{B}$-cyclic Banach lattices (X,Y) are equivalent:*

(1) *There exist M-projections π_1 and π_2 such that $\pi_1 X$ is an injective Banach lattice with $\mathbb{M}(\pi_1 X) \simeq \pi_1\mathbb{B} := [0, \pi_1]$, $\pi_2 Y$ is an AM-space, and $\pi_1 \vee \pi_2 = \mathbb{1}$.*

(2) *$\mathscr{K}_{\mathbb{B}}(X, Y)$ is a Banach lattice under the operator norm.*
Furthermore, in this case the lattice operations are given by the Freudenthal–Kantorovich–Riesz formula.

$\lhd$ This is a Boolean valued interpretation of Theorem 4.1. $\rhd$

Definition 8.10. Say that a downward directed set $A \subset X$ is $\mathbb{B}$-*convergent to zero* if for every $0 < \varepsilon \in \mathbb{R}$ there exists a partition of unity $(\pi_a)_{a\in A}$ in $\mathbb{B}$ such that $\|\pi_a a\| \leqslant \varepsilon$ for all $a \in A$. The norm in X is said to be order $\mathbb{B}$-*continuous* if every downward directed set $A \subset X$ with $\inf A = 0$ is $\mathbb{B}$-convergent to zero.

Problem 8.11: Characterize $\mathbb{B}$-cyclic Banach lattices X with $\mathbb{B}$-atomic order $\mathbb{B}$-continuous $\mathbb{B}$-dual $X^{\#}$. In view of Theorem 7.5 it is sufficient to settle the case $\mathbb{B} = \{\mathbb{O}, \mathbb{1}\}$: Characterize Banach lattices X with atomic order continuous dual X'.

In connection with Problem 8.11 it should be noted that there exist non-atomic Banach lattices with atomic duals, see Lacey and Wojtaszczyk [48].

Problem 8.12: Consider mix-compact versions of Problems 3.4 and 3.7.

Interpreting Theorem 4.4 in an appropriate Boolean valued model and making use of the observation made in the proof of Theorem 8.7 yields the following result.

Theorem 8.13. *If X and Y are $\mathbb{B}$-cyclic Banach lattices then the following two assertions are equivalent:*

(1) *One of the following three (non-exclusive) conditions holds:*

(a) *Both $X^{\#}$ and Y have an order $\mathbb{B}$-continuous norm.*

(b) *Y is an $\mathbb{B}$-atomic Banach lattice with an order $\mathbb{B}$-continuous norm.*

(c) *$X^{\#}$ is an $\mathbb{B}$-atomic Banach lattice with an order $\mathbb{B}$-continuous norm.*

(2) *If $S, T \in \mathscr{L}_{\mathbb{B}}(X, Y)$, $0 \leqslant S \leqslant T$ and T is mix-compact then S is mix-compact.*

In conclusion we consider the Boolean valued interpretation of a portion of the theory of cone absolutely summing operators.

Definition 8.14. Let X be a Banach lattice and Y be a $\mathbb{B}$-cyclic Banach space. Denote by $\mathscr{P}_{\text{fin}}(X)$ the collection of all finite subsets of X. For $T \in \mathscr{L}(X, Y)$ define

$$\sigma(T) := \sigma_{\mathbb{B}}(T) := \sup\Bigg\{ \inf_{(\pi_k)\in \text{Prt}_\sigma(\mathbb{B})} \sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k T x_i\| : \{x_1, \dots, x_n\} \in \mathscr{P}_{\text{fin}}(X),\ \Bigg\| \sum_{i=1}^{n} |x_i| \Bigg\| \leqslant 1 \Bigg\}.$$

An operator $T \in \mathscr{L}(X, Y)$ is said to be cone $\mathbb{B}$-*summing* if $\sigma(T) < \infty$. Thus, T is cone $\mathbb{B}$-summing if and only if there exists a positive constant C such that for any finite collection $x_1, \dots, x_n \in X$ there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ with

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k T x_i\| \leqslant C \Bigg\| \sum_{i=1}^{n} |x_i| \Bigg\|;$$

moreover, in this event $\sigma(T) = \inf\{C\}$.

Denote by $\mathscr{S}_{\mathbb{B}}(X,Y)$ the set of all cone $\mathbb{B}$-summing operators. The class $\mathscr{S}_{\mathbb{B}}(X,Y)$ was introduced in Kusraev [39], see also Kusraev and Kutateladze [47, 5.13.1]. Observe that if $\mathbb{B} = \{0, I_Y\}$ then $\mathscr{S}(X,Y) := \mathscr{S}_{\mathbb{B}}(X,Y)$ is the space of cone absolutely summing operators, see Schaefer [62, Ch. 4, § 3, Proposition 3.3 (d)] or (which is the same) 1-concave operators, see Diestel, Jarchow, and Tonge [22, p. 330]. Cone absolutely summing operators were introduced by Levin [49] and later independently by Schlotterbeck, see [62, Ch. 4].

Theorem 8.15. *Let X and Y be nonzero Banach lattices. The following are equivalent:*

(1) *$\mathscr{S}(X,Y)$ is an AL-space.*

(2) *X is an AM-space and Y is an AL-space.*

$\lhd$ This result is due to Schlotterbeck, see Schaefer [62, Ch. 4, Proposition 4.5]. $\rhd$

Theorem 8.16. *Let X be a nonzero Banach lattice and Y be a $\mathbb{B}$-cyclic Banach lattice. The following are equivalent:*

(1) *$\mathscr{S}_{\mathbb{B}}(X,Y)$ is an injective Banach lattice with $\mathbb{M}(\mathscr{S}_{\mathbb{B}}(X,Y))$ isomorphic to $\mathbb{B}$.*

(2) *X is an AM-space and Y is an injective Banach lattice with $\mathbb{M}(Y)$ isomorphic to $\mathbb{B}$.*

$\lhd$ There is an order preserving $\mathbb{B}$-isometry from $\mathscr{S}_{\mathbb{B}}(X,Y)$ onto the restricted descent $\mathscr{S}(\mathscr{X},\mathscr{Y})\Downarrow$, see [47, Theorem 5.13.6]. Now, the proof can be carried out in similar lines by Boolean valued interpretation of Theorem 8.15, see [44, Theorem 4.10]. $\rhd$

A linear operator is cone $\mathbb{B}$-summing precisely when it factors through injective Banach lattice, see [47, Theorem 5.13.8]. At the same time for $p \in \mathscr{Z}(\mathbb{L})$ one can define the p-convexification $\mathbb{L}^{(p)}$ of a Banach lattice $\mathbb{L}$ by means of *generalized functional calculus*, see [34, 68]). This observation motivates the following problems.

Problem 8.17: Characterize operators factorable through the p-convexification $\mathbb{L}^{(p)}$ of an injective Banach lattice $\mathbb{L}$ with $I \leqslant p \in \mathscr{Z}(\mathbb{L})$ or $I \leqslant p \in \mathscr{Z}_m(\mathbb{L})$, $I := I_{\mathbb{L}}$.

Problem 8.18: Introduce and explore some $\mathbb{B}$-versions of p-summing concept and its variations (such as lattice p-summing, positive p-summing etc.) in $\mathbb{B}$-cyclic Banach lattices, cf. [22].

Problem 8.19: Adopt real and complex interpolation methods to the variable parameter scale of Banach lattices $\{\mathbb{L}^{(p)} : I \leqslant p \in \mathscr{Z}(\mathbb{L})\}$, cf. [21, Sect. 3.7.8].

It is proved in [40] that Kaplansky–Hilbert lattices and injective Banach lattices may be produced from each other by means of the convexification and concavification procedures. Thus, one more natural question arises:

Problem 8.20: Characterize operators factorable through Kaplansky–Hilbert lattices. Under what conditions every operator in $\mathscr{L}(X,Y)$ (with X and Y being Banach lattices) factors through Kaplansky–Hilbert lattice?

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